Theory of superconductivity in the presence of electron-hole pairing in semimetals at $T=0$

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The possibility of simultaneous Cooper pairing of the electrons inside each of the bands (S) and of electron-hole pairing (D) is investigated in the isotropic model of a semimetal with overlapping bands, with the Coulomb interaction and the structural instability taken into consideration. A solution is found for equal concentrations of electrons and holes, which describes the metastable SD phase at $T=0$. The mutual influence of the S and D pairings on each other is studied for the case of a doped semimetal. It is shown that under certain conditions D pairing leads to an increase of the superconductor critical temperature. The regions of existence of the S,D, and SD phases are determined for $T=0$ (depending on the ratio of the coupling constants), and the nature of the phase transitions between them is also determined.

1. INTRODUCTION

At the present time the search for new mechanisms for the purpose of increasing the critical temperature T_c of superconductors is a very pressing problem. One of the possibilities in this project consists in the utilization of the strong dependence of T_c on the density of states N(0) of the normal state, which is predicted by the BCS theory.[1] In the isotropic model of a semimetal with overlapping bands, proposed in[2], a realignment of the electronic spectrum appears upon taking the Coulomb repulsion into account, and this is accompanied by the formation of an anomalous density of states at the edge of the forbidden gap. The influence of this realignment on superconductivity (in the presence of an attraction between the electrons inside each of the bands) was investigated in[3–5].

The possibility in principle of increasing T_c in a system with electron-hole pairing (D-pairing) was pointed out in[3]. However, subsequent numerical calculations[5] have not led to the anticipated effect (in the absence of hybridization). It will be shown in Secs. 4 and 5 of the present article that D-pairing favors superconducting pairing (S-pairing) only in the presence of a certain restriction on the choice of the coupling constants. A realignment of the crystal lattice, which leads to an additional D-pairing mechanism in semimetals,[6] will play the essential role in our investigation.

In addition to refining the results of[3,4] for the case of weak S-pairing in a doped semimetal, we also investigate the more general case when the binding energies for S- and D-pairings are generally of the same order of magnitude. This permits us to construct a general picture describing the coexistence of these two types of pairing at T = 0 (Sec. 5).

The electromagnetic properties of a semimetal in the SD-phase in the absence of doping are considered in the recent articles by Lo and Wong.[7] However, the authors did not demonstrate that it was possible for two order parameters to simultaneously exist in the system. Section 3 of the present article is devoted to this question. Here the fundamental result is that, under the assumptions made in[7], the minimum energy corresponds to a pure phase (S or D). A solution for the mixed SD-phase in a metastable state also exists.

The results of the present work carry over, almost without any changes, to the model of a single-band metal with an electron spectrum of the type $\epsilon(\mathbf{p}) = -\epsilon(\mathbf{p}+\mathbf{q})$;[8] in particular our results pertain to quasi one-dimensional systems.[9] For such systems the deviation of the number of electrons per center from unity corresponds to a discrepancy between the electron and hole concentrations ($\delta n \neq 0$) in a semimetal. With the structural instability[6] taken into account, the coupling constants for the S- and D-pairings do not agree.

2. FORMULATION OF THE PROBLEM

Let us write the initial Hamiltonian (without taking the retardation of the electron-phonon interaction into account) in the form

$$\hat{H}=\sum_i \hat{\psi}_i^+(\mathbf{x})\,\varepsilon_i(\hat{\mathbf{p}})\,\psi_i(\mathbf{x})\,d\mathbf{x}+\frac{1}{2}\sum_{ij}\lambda_{ij}\int(\hat{\psi}_i^+(\mathbf{x})\psi_i(\mathbf{x}))(\hat{\psi}_j^+(\mathbf{x})\hat{\psi}_j(\mathbf{x}))\,d\mathbf{x}$$
$$+\gamma\int\hat{\psi}_1^+(\mathbf{x})\,u(\mathbf{x})\,\hat{\psi}_2(\mathbf{x})\,d\mathbf{x}+\gamma^*\int\hat{\psi}_2(\mathbf{x})\,u(\mathbf{x})\,\psi_1(\mathbf{x})\,d\mathbf{x}, \quad (1)$$

where i labels the band (i = 1, 2). The last two terms describe (see[6]) the interaction of the electrons with a static deformation $u(\mathbf{x})$ of the crystal lattice—the strength of this interaction being characterized by the coupling constant γ (for the sake of brevity we have omitted the energy associated with elastic deformations of the lattice). The spectrum of the electrons is assumed to be isotropic:

$$\varepsilon_{1,2}(p)=\delta\mu\pm\varepsilon,\quad \varepsilon=p^2/2m-\varepsilon_F, \quad (2)$$

where $\delta\mu$ denotes the shift of the Fermi level in each band due to doping. Neglecting the local impurity levels in the D-phase,[10] one can regard the difference δn between the electron and hole concentrations as given. This condition enables us to determine the quantity $\delta\mu$.

The definition of the Green's functions of the system has the conventional form:

$$\mathcal{G}_{ij}^{\alpha\beta}(x-x')=\langle T\psi_{i\alpha}(x)\psi_{j\beta}^+(x')\rangle,$$
$$\mathcal{F}_{ij}^{\alpha\beta}(x-x')=\langle T\psi_{i\alpha}(x)\psi_{j\beta}(x')\rangle. \quad (3)$$

Following[3], we study here the case of singlet S- and D-pairings, that is, we shall assume that $G^{\alpha\beta} = G\delta_{\alpha\beta}$ and $F^{\alpha\beta} = i\sigma_{\alpha\beta}^y F$.

After separating out the spin dependence and Fourier transforming, the system of equations for the functions (3) in the Matsubara representation at T = 0 can be written in the form

$$\begin{pmatrix} i\omega-\varepsilon_1 & -\Delta & \Sigma_{11} & \Sigma_2 \\ -\Delta^* & i\omega-\varepsilon_2 & \Sigma_2 & \Sigma_{22} \\ \Sigma_{11}^* & \Sigma_2^* & i\omega+\varepsilon_1 & \Delta^* \\ \Sigma_2^* & \Sigma_{22}^* & \Delta & i\omega+\varepsilon_2 \end{pmatrix}\begin{pmatrix} \mathcal{G}_{11}(\mathbf{p},\omega) \\ \mathcal{G}_{21}(\mathbf{p},\omega) \\ \mathcal{F}_{11}^+(\mathbf{p},\omega) \\ \mathcal{F}_{21}^+(\mathbf{p},\omega) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4)$$

where

$$\Sigma_{ij}=-\lambda_{ij}\mathscr{F}(x,x), \quad \Delta=\tilde{\lambda}_{21}\mathscr{G}_{21}(x,x), \quad \Sigma_2\equiv\Sigma_{21}$$

and $\omega = \pi T(2n + 1)$ (n is an integer). The interaction constants λ_{11} and λ_{22} (we shall assume them to be equal) contain both the electron-phonon interaction and the intraband Coulomb interaction (which is weakened, owing to the factor $\ln(\omega_c/\omega_D) \gg 1$), and these constants are assumed to be negative. For interband transitions the corresponding interactions are included in the constant λ_{21}. Here the case $\lambda_{21} < 0$ is possible, which corresponds to an attraction between electrons from different bands. Later we shall see that S-pairing of the electrons from different bands is possible for any sign of λ_{21}; however, the case $\lambda_{21} < 0$ is more favorable for the existence of superconductivity.

The coupling constant $\tilde{\lambda}_{21}$, which is responsible for D-pairing, is given by[6]

$$\tilde{\lambda}_{21}=\lambda_{21}+4|\gamma|^2/\omega_D. \tag{5}$$

The second term in this expression, corresponding to an electron-hole attraction, is associated with a structural instability. D-pairing appears if $\tilde{\lambda}_{21} > 0$. It is assumed below that this condition is satisfied. However, in this connection the quantity λ_{21} may turn out to be negative.

We immediately note that the quantities Σ_{11} and Σ_{22} can be regarded as real without any loss of generality. This follows directly from the invariance of the Hamiltonian (1) under the gauge transformation $\psi_j \to \psi_j' \exp(i\varphi_j)$, $\gamma \to \gamma' \exp[i(\varphi_1-\varphi_2)]$. In this connection, according to Eqs. (3) and (4) we have $\Sigma_{jj} \to \Sigma_{jj}' \exp(2i\varphi_j)$. By choosing the quantities Σ_{ii} to be real, we thereby automatically fix the phases of the parameters Σ_2 and Δ.

Under the assumption that $|\Sigma_{11}| = |\Sigma_{22}|$, one can easily show with the aid of Eq. (4) that the following two distinct cases exist (Δ and Σ_2 are real):

a) $\Sigma_{11}=-\Sigma_{22}\equiv\Sigma_a$, $\Sigma_2\equiv0$;

b) $\Sigma_{11}=\Sigma_{22}\equiv\Sigma_s$, $\Sigma_2\neq0$.

In what follows we shall call the solution $\Sigma_a(\Sigma_s)$ the antisymmetric (symmetric) solution.[1)] We emphasize that this terminology is purely a convention, since it is possible only for a quite definite choice of the phases of the order parameters. Another choice of the phases, in which the parameters Δ and Σ_2 are pure imaginary (but $\mathrm{Im}\,\Sigma_{ii} = 0$ as before), is also possible. This case can be reduced to the case indicated above by making a gauge transformation (upon changing the signs of the quantities Σ_2 and Σ_{22}).

We shall see below that in the weak-coupling limit under consideration ($\lambda_{21} \to 0$) the parameter Σ_2 in case b) is small ($\Sigma_2 \ll \Sigma$) over a wide range of values of the coupling constants. Therefore, the difference between cases a) and b) essentially consists of only a different choice for the phases of the quantities Σ_{11} and Σ_{22}. We emphasize that in both cases the anomalous function $F_{21}(x, x')$ is by no means small and, generally speaking, is of the order of $F_{11}(x, x')$.

Let us write the determinant of the system (4) in the form

$$\mathscr{D}=(\omega^2+\omega_+^2)(\omega^2+\omega_-^2),$$

where $\omega_\pm(\mathbf{p})$ denotes the energy of the elementary excitations in the SD-phase,

$$\omega_\pm^2=(E\pm\delta\tilde{\mu})^2+\tilde{\Sigma}^2,\quad E^2=\varepsilon^2+\tilde{\Delta}^2. \tag{6}$$

We have the following result for the antisymmetric solution:

$$\tilde{\Delta}_a^2=\Delta^2(1+\Sigma^2/\delta\mu^2),\quad \tilde{\Sigma}_a^2=\Sigma^2(1-\Delta^2/\delta\mu^2),\quad \delta\tilde{\mu}=\delta\mu. \tag{7a}$$

And for the symmetric solution we obtain

$$\tilde{\Delta}_s=(\Delta\delta\mu+\Sigma\Sigma_2)/\delta\tilde{\mu},\quad \tilde{\Sigma}_s=(\Sigma\delta\mu-\Sigma_2\Delta)/\delta\tilde{\mu},\quad \delta\tilde{\mu}^2=\delta\mu^2+\Sigma_2^2. \tag{7b}$$

In concluding this section, we write down the equation for the determination of $\delta\mu$ under the assumption that the difference δn between the electron and hole concentrations is given:

$$\delta n=2\int[1-n_1(\mathbf{p})-n_2(\mathbf{p})]\frac{d\mathbf{p}}{(2\pi)^3}; \tag{8}$$

here $n_i(\mathbf{p})$ is the electron distribution function in the i-th band; it can be determined with the aid of the functions G_{ii}. In what follows it will be convenient for us to use $n \equiv \delta\mu_0$ instead of δn, where n denotes the shift of the Fermi level in the normal phase ($\Delta = \Sigma = 0$). According to Eqs. (3) and (8) the relation between these quantities has the form

$$\delta n=4N(0)n. \tag{9}$$

3. THE CASE OF EQUAL CONCENTRATIONS OF ELECTRONS AND HOLES (n = 0)

First let us consider the antisymmetric case ($\Sigma_2 \equiv 0$) for T = 0. Performing the integration with respect to ω in expression (8) for δn, we find

$$n=\frac{1}{2}\int_0^\infty d\varepsilon\left(\frac{E+\delta\mu}{\omega_+}-\frac{E-\delta\mu}{\omega_-}\right)+\left(\frac{\Delta\Sigma}{\delta\mu}\right)^2\int_0^\infty\frac{d\varepsilon}{2E}\left(\frac{1}{\omega_-}-\frac{1}{\omega_+}\right)\quad (T=0). \tag{10}$$

The function $n(\delta\mu)$ is an odd function; therefore it is sufficient to examine the case $\delta\mu \geq 0$. In addition, both of the integrals appearing here are everywhere nonnegative. Therefore, the condition for strict equality of the electron and hole concentrations (n = 0) can be satisfied only in the following two limiting cases: 1) $\delta\mu = 0$, Σ arbitrary; 2) $\delta\mu \leq \Delta$, $\Sigma = 0$. The second solution describes the equilibrium D-phase; therefore we shall not discuss it.

For $\delta\mu = 0$ it is not difficult to write down the equation which describes the nontrivial solution $\Sigma \neq 0$ and $\Delta \neq 0$ in the antisymmetric case (T = 0):

$$g_0\equiv\ln z_0=\frac{1-z^2}{2z}\ln\left|\frac{1+z}{1-z}\right|,\quad z=\frac{\Delta}{\Sigma},$$
$$\ln\left(\frac{\Sigma_0}{\Sigma}\right)^2=\ln|1-z^2|+z\ln\left|\frac{1+z}{1-z}\right|. \tag{11}$$

The quantities Σ_0 and Λ_0 introduced here correspond to the values of the order parameters in the pure S- and D-phases (in the absence of doping), respectively, for T = 0:

$$\Sigma_0=2\omega_D\exp[-1/|\lambda_{11}|N(0)],$$
$$\Delta_0=2\omega_c\exp[-1/\tilde{\lambda}_{21}N(0)]. \tag{12}$$

Later on we shall see that use of the variables Σ_0 and Λ_0 instead of the coupling constants λ_{11} and $\tilde{\lambda}_{21}$ is also very convenient in considering the phase diagrams even when $n \neq 0$.

In connection with a variation of z in the interval $(0, \infty)$ the right-hand side of the first equation in (11) decreases monotonically within the range from 1 to −1. Therefore, a nontrivial solution of the system of equations (11) exists provided the inequality

$$|\ln(\Delta_0/\Sigma_0)|<1.$$

is satisfied. We have the following results at the ends of the interval: $\Sigma = 0$ for $\Sigma_0 = e\Lambda_0$ and $\Delta = 0$ for $\Lambda_0 = e\Sigma_0$. One can also easily see that the parameter Σ decreases monotonically with increasing values of the coupling constant $|\lambda_{11}|$, i.e., Σ_0; the parameter Δ behaves in a similar fashion as a function of Δ_0. Such an anomalous behavior of the solutions suggests that a mixed SD-phase, existing in a metastable state (see Fig. 1), is realized in the region contained between the straight lines $\Sigma_0 = e\Lambda_0$ and $\Lambda_0 = e\Sigma_0$ on the (Λ_0, Σ_0) plane.

One can easily verify this by calculating the free energy corresponding to the solution (11). In the general case (for $\Sigma_2 = 0$) we obtain

$$\delta F_{SD}=F_{SD}-F_N=-2\int_C\left(\frac{|\Delta|^2}{\tilde{\lambda}_{21}^2}d\tilde{\lambda}_{21}+\frac{|\Sigma|^2}{\lambda_{11}^2}d\lambda_{11}\right), \qquad (13)$$

where F_N denotes the free energy of the normal phase (for $\lambda_{11} = \tilde{\lambda}_{21} = 0$). The contour C connects the given point (Δ_0, Σ_0) with the origin of coordinates. In the case under consideration the contour of integration must lie completely inside the region of existence of the solutions, i.e., inside the cross-hatched region shown in Fig. 1. It is convenient to perform the integration in expression (13) along the straight line z = const; with the aid of expressions (11) we easily find

$$\delta F_{SD}=f(z)\,\delta F_S=f(z^{-1})\,\delta F_D,$$
$$\ln f(z)=\ln\left|\frac{1+z^2}{1-z^2}\right|-z\ln\left|\frac{1+z}{1-z}\right|. \qquad (14)$$

The quantities δF_S and δF_D represent the free energies of the pure S- and D-phases, measured from the energy of the semimetal in its normal state:

$$\delta F_S=F_S-F_N=-N(0)\Sigma_0^2,$$
$$\delta F_D=F_D-F_N=-N(0)\Delta_0^2. \qquad (15)$$

The function f(z) in (14) decreases monotonically upon variation of z in the interval $(0, \infty)$, where $f(0) = 1$ and $f(\infty) = e^{-2}$. Hence one can conclude that the SD-phase which has been found is energetically unfavorable in comparison with the pure superconducting and the pure dielectric phases. The dependence of the free energies δF_{SD}, δF_S, and δF_D on the quantity $g_0 = \ln(\Delta_0/\Sigma_0)$ is shown in Fig. 2.

At first glance the presence of the metastable SD-phase in the antisymmetric case indicates the possible existence in the system of hysteresis phenomena such as the "supercooling" of one of the (S or D) phases. In actual fact the transition from the S-phase into the D-phase (and conversely) will occur on the line $\Lambda_0 = \Sigma_0$. The point is that for n = 0 the system of equations in the symmetric case also has a nontrivial solution, if we set $\delta\mu = \Sigma_2 = 0$ in Eqs. (23) (see below). This solution is

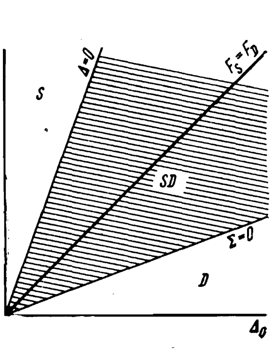

FIG. 1.

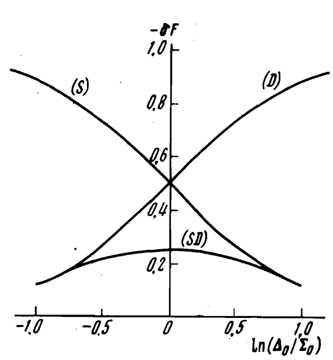

FIG. 2.

realized on the line corresponding to equilibrium of the pure S- and D-phases and has the form $\Delta^2 + \Sigma^2 = \Sigma_0^2$. The energy of this state obviously coincides with the energies of the pure S- and D-phases.

Thus, in the absence of doping either the pure S-phase $(\Sigma_0 > \Lambda_0)$ or the pure D-phase $(\Delta_0 > \Sigma_0)$ is realized in the system at T = 0. A phase transition of the second kind, corresponding to the formation of the symmetric solution $\Sigma^2 + \Delta^2 = \Sigma_0^2$, takes place on the line $\Delta_0 = \Sigma_0$. As a consequence of its degeneracy (with respect to one of the quantities Σ or Δ), this solution is apparently unstable with respect to small perturbations. However, we cannot solve this problem within the framework of the approximation we are using, in which we are neglecting the interband transitions.

4. THE CASE OF DOPED SEMIMETALS

A. The antisymmetric solution ($\Sigma_2 \equiv 0$). For T = 0 one can write down the initial system of equations in the form

$$[|\lambda_{11}|N(0)]^{-1}=J_1(\omega_D)+\left(\delta\mu-\frac{\Delta^2}{\delta\mu}\right)J_2,$$
$$[\tilde{\lambda}_{21}N(0)]^{-1}=J_1(\omega_c)-\frac{\Sigma^2}{\delta\mu}J_2, \quad n=J_3+\frac{\Sigma^2\Delta^2}{\delta\mu^2}J_2, \qquad (16)$$

where the following notation has been introduced:

$$J_1(\tilde{\omega})=\frac{1}{2}\int_0^{\tilde{\omega}}\frac{d\varepsilon}{E}\left(\frac{E+\delta\mu}{\omega_+}+\frac{E-\delta\mu}{\omega_-}\right),$$
$$J_2=\frac{1}{2}\int_0^{\infty}\frac{d\varepsilon}{E}\left(\frac{1}{\omega_-}-\frac{1}{\omega_+}\right), \qquad (17)$$
$$J_3=\frac{1}{2}\int_0^{\infty}d\varepsilon\left(\frac{E+\delta\mu}{\omega_+}-\frac{E-\delta\mu}{\omega_-}\right).$$

In the general case these quantities can be expressed in terms of elliptic integrals.

We shall solve the system (16) for $\Sigma \to 0$, i.e., in the limit of weak S-pairing. For $\Sigma = 0$ we obtain the following results from Eqs. (16) and (17):

$$n=J_3=(\delta\mu^2-\Delta^2)^{1/2}, \quad \Sigma=0. \qquad (18)$$

In view of the fact that n is an odd function of $\delta\mu$, in what follows we shall always assume that $n > 0$. The integral J_2 diverges logarithmically for small values of Σ; therefore, the first equation in (16), which describes the S-pairing, has no solutions in the limit $\Sigma = 0$ $(\delta\mu > 0)$. Physically this means that in the presence of an excess of electrons (or holes) the D-phase is always unstable with respect to the creation of Cooper pairs.

Corrections $\sim\Sigma\ln\Sigma$ appear in the integrals J_1 and J_3 for small values of Σ; therefore, according to (16) one can neglect the influence (to the first-order approximation in Σ) of the S-pairing on the D-pairing and on Eq. (18) for $n(\delta\mu)$. Hence we find

$$\Delta^2=\Delta_0(\Delta_0-2n), \quad \Delta_0\geqslant 2n, \qquad (19)$$

where Δ_0 is given by expression (12).

The integral J_2 is represented by an elliptic integral of the first kind, whose asymptotic form as $\Sigma \to 0$ is well known:

$$J_2\approx\frac{1}{(\delta\mu^2-\Delta^2)^{1/2}}\ln\frac{4(\delta\mu^2-\Delta^2)}{|\Sigma|[\delta\mu+(\delta\mu^2-\Delta^2)^{1/2}]}, \quad \Sigma\to 0. \qquad (20)$$

From this we obtain the antisymmetric solution $\tilde{\Sigma}_a$ with the aid of Eqs. (16), (18), and (19):

$$\tilde{\Sigma}_a = \frac{4n^2}{\Delta_0}\exp\left[-\frac{\Delta_0 - n}{n} g_0\right], \quad \Delta_0 > 2n, \quad g_0 > 0, \tag{21}$$

where $g_0 = \ln(\Delta_0/\Sigma_0)$. The derived expression is valid provided $\tilde{\Sigma}_a \ll 4n^2/\Delta_0$, i.e., provided that the argument of the exponential is large in absolute magnitude ("weak" coupling).

According to Eqs. (7a), (18), and (19) the quantities Σ_a and $\tilde{\Sigma}_a$ are related by the equation

$$\Sigma_a = \tilde{\Sigma}_a(\Delta_0 - n)/n.$$

The superconducting properties of the SD-phase evidently determine the quantity $\tilde{\Sigma}$, which appears in the definition (6) of the system's spectrum. At the end of this section we calculate the superconducting transition temperature T_c of the SD-phase and show that T_c and $\tilde{\Sigma}$ are related by the usual BCS relationship: $\tilde{\Sigma} = \pi T_c/\gamma$ ($\gamma = 1.78$).

The concentration of excess electrons $n = \Delta_0/2$ is critical for D-pairing,[11] that is, $\tilde{\Delta} = 0$ for $n \geq \Delta_0/2$. It follows from (21) that we have $\tilde{\Sigma}_a = \Sigma_0$ for $\Delta_0 = 2n$, just as one would expect. In Sec. 5 we give an exact expansion for $\tilde{\Sigma}$ near the line $\Delta = 0$, and this coincides with the result cited above for $\Sigma_0 \ll \Delta_0 \approx 2n$.

For small concentrations of the carriers ($n \ll \Delta_0$) the superconductivity in the SD-phase is exponentially weak, that is, $\Sigma \sim \exp(-g_0\Delta_0/n)$. This agrees with the conclusions of Sec. 3 concerning the impossibility of an equilibrium SD-phase in the absence of doping.

According to Eq. (21) $\tilde{\Sigma}_a$ is a monotonically decreasing function of Δ_0 for $\Delta_0 > 2n$, i.e., in the antisymmetric case D-pairing impedes the formation of superconductivity in the system. We shall show below that this is not so for the symmetric solution Σ_S, namely, a region exists in the (Δ_0, Σ_0) plane where D-pairing leads to an enhancement of superconductivity.

It is not difficult to calculate the energy of the SD-phase which has been discovered. In order to do so, in the general formula (13) we choose the integration contour C to consist of two straight line segments such that $\lambda_{11} \equiv 0$ on the first segment and $\tilde{\lambda}_{21}$ = const on the second. For such a choice of the contour, it is not necessary to know the variation of the parameter Δ due to Σ. Using expressions (19)—(21) for Δ and Σ and performing the indicated integration, we find

$$F_{SD}^{(a)} = F_D - \frac{n\Sigma_a^2}{\Delta_0 - n}N(0), \quad F_D = -N(0)(\Delta_0 - 2n)^2. \tag{22}$$

The first term in the expression for F_{SD} arises upon integration over the segment with $\lambda_{11} = 0$. Hence it is obvious that $F_{SD}^{(a)} < F_D$, that is, the formation of the SD-phase is energetically favored in comparison with the pure dielectric phase, a fact which was mentioned at the beginning.

B. The symmetric solution. For T = 0 the system of equations for Σ, Σ_2, Δ, and $\delta\mu$ has the following form:

$$\begin{gathered}[|\lambda_{11}|N(0)]^{-1}\Sigma = \Sigma J_1(\omega_D) + \delta\mu\tilde{\Sigma}J_2,\\ [\lambda_{21}N(0)]^{-1}\Sigma_2 = -(\Sigma_2/\delta\tilde{\mu})J_3 + (\delta\mu/\delta\tilde{\mu})\tilde{\Sigma}\tilde{\Delta}J_2,\\ [\tilde{\lambda}_{21}N(0)]^{-1}\Delta = \Delta J_1(\omega_c) - \Sigma_2\tilde{\Sigma}J_2,\\ n = (\delta\mu/\delta\tilde{\mu})[J_3 + (\Sigma^2 - \tilde{\Sigma}^2)J_2].\end{gathered} \tag{23}$$

Being guided by the limiting case of a normal semimetal, we require that the function $n(\delta\mu)$ be odd. Then it follows from the structure of the written equations that the function $\Sigma_2(\delta\mu)$ is also odd. We assume, just as before, that $\delta\mu > 0$ (i.e., $n > 0$).

The integrals J_2 and J_3, which do not have any divergences for large values of ϵ, appear in the equations for Σ_2. This implies that the diagrams which describe the scattering of pairs of electrons from different bands do not have any singularities. As mentioned at the beginning, the appearance of anomalous averages F_{21} is completely due to the presence of the nonvanishing anomalous Green's functions F_{11}, F_{22}, and G_{21}.

In the weak-coupling limit ($N(0)\lambda_{21} \ll 1$) it follows from Eqs. (23) that $\Sigma_2 \sim \lambda_{21}$ in the general case. Therefore, at first glance it appears that one can set $\Sigma_2 = 0$ in the remaining equations. This assertion is valid everywhere, with the exception of the case of extremely small values of Σ. In fact, by setting $\Sigma_2 = 0$ on the right-hand side of the second equation in (23), we obtain the following result in the limit $\Sigma \ll \Delta \sim \delta\mu$:

$$\Sigma_2/\Sigma \sim N(0)\lambda_{21}\ln(\Delta/\Sigma),$$

i.e., the smallness of the coupling constant λ_{21} may be compensated by the large quantity $\ln(\Delta/\Sigma) \gg 1$. Turning to the definition (7b) for the physical gap $\tilde{\Sigma}$, we see that under these conditions the quantity Σ_2 gives a contribution of the same order of magnitude as Σ. We note that the region in which it is essential to take the quantity Σ_2 into account is very narrow in comparison with the domain of existence of the SD-phase (see Fig. 3).

In the limit of weak S-pairing ($|\tilde{\Sigma}_S| \ll 4n^2/\Delta_0$) the system of Eqs. (23) can be reduced with the aid of Eqs. (18)—(20) to the following simple form:

$$\begin{gathered}g_0\Sigma = \frac{\Delta_0 - n}{n}\tilde{\Sigma}_s\ln\left(\frac{4n^2}{|\tilde{\Sigma}_s|\Delta_0}\right), \quad g_0 > 0,\\ \Sigma_2\left(\frac{1}{N(0)\lambda_{21}}\right) = \frac{[\Delta_0(\Delta_0 - 2n)]^{1/2}}{n}\tilde{\Sigma}_s\ln\left(\frac{4n^2}{|\tilde{\Sigma}_s|\Delta_0}\right).\end{gathered} \tag{24}$$

In the derivation of these equations we have assumed $\delta\tilde{\mu} \approx \delta\mu$ and $\tilde{\Delta} \approx \Delta$. Using the definition (7b) for $\tilde{\Sigma}_S$, we obtain the following result in the approximation under consideration:

$$\tilde{\Sigma}_s \approx \Sigma - \frac{[\Delta_0(\Delta_0 - 2n)]^{1/2}}{\Delta_0 - n}\Sigma_2.$$

The final expression for $\tilde{\Sigma}_S$ is obtained by substitution of the values of Σ and Σ_2 from (24) into this formula:

$$\begin{gathered}|\tilde{\Sigma}_s| = \frac{4n^2}{\Delta_0}\exp\left[-\frac{n}{\Delta_0 - n}g^*\right], \quad |\tilde{\Sigma}_s| \ll \frac{4n^2}{\Delta_0},\\ g^* = g_0\left[1 - g_0\lambda_{21}N(0)\frac{\Delta_0(\Delta_0 - 2n)}{(\Delta_0 - n)^2}\right]^{-1}.\end{gathered} \tag{25}$$

For $\Delta_0 = 2n$, $|\tilde{\Sigma}_S| = \Sigma_0$ independently of the value of λ_{21}, just as in the antisymmetric case. The contribution of Σ_2 to $|\tilde{\Sigma}_S|$ reduces to a redefinition of the factor g_0, which plays the role of the inverse "constant" characterizing the strength of the effective interaction of the electrons inside each band. The conditions under which one can neglect this contribution coincide with the conditions obtained above. Although the region in which it is essential to take Σ_2 into account is extremely narrow

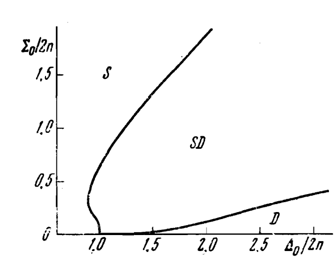

FIG. 3.

($g_0^{-1} \ll \lambda_{21}N(0) \ll 1$), it is interesting to note the different behavior of $|\tilde{\Sigma}_S|$ in this region depending on the sign of λ_{21}. For negative values of λ_{21} ($\tilde{\lambda}_{21}$ is always positive) expression (25) indicates an increase of $|\tilde{\Sigma}_S|$ for increasing values of $|\lambda_{21}|$. The opposite situation arises for $\lambda_{21} > 0$; in this case the expression for g^* has a pole associated with a definite value of Σ_0, that is, $|\tilde{\Sigma}_S| = 0$. Referring to the (Δ_0, Σ_0) plane, this corresponds to the line shown in Fig. 3. Below this line $\tilde{\Sigma}_S = 0$ everywhere, i.e., the pure D-phase is realized.[2)]

Neglecting the term associated with λ_{21} in (25), it is easy to see that the quantity $\tilde{\Sigma}_S$ has, in contrast to $\tilde{\Sigma}_a$, a maximum as a function of Δ_0 or n, and this maximum is located inside the SD-phase provided that $g_0 > 1$.

Regarded as a function of the concentration n, the superconducting gap $|\tilde{\Sigma}_S|$ has a maximum at the point

$$\frac{2n_{max}}{\Delta_0} = 2+\frac{g_0}{2}-\left[g_0\left(2+\frac{g_0}{4}\right)\right]^{1/2}. \qquad (26)$$

It is clear from this expression that the maximum falls in the physical region ($2n \le \Delta_0$) when the condition $g_0 \ge 1$ is satisfied, that is $\Sigma_0 \le e\Delta_0$. With the aid of Eqs. (25) and (26) one can rigorously show that at its maximum the quantity $\tilde{\Sigma}_S$ exceeds the value of the gap Σ_0 in the S-phase. Here we consider the limiting case of large values $g_0 \gg 1$:

$$2n_{max}/\Delta_0 \approx 4/g_0 \ll 1, \quad 8 \ll g_0 \ll (\lambda_{21}N(0))^{-1},$$
$$(\tilde{\Sigma}_s/\Sigma_0)_{max} \approx (4/eg_0)^2 e^{g_0} \gg 1.$$

Thus, the increase of $\tilde{\Sigma}_S$ (and also the increase of T_c) in the SD-phase relative to the value of Σ_0 can become arbitrarily large as the value of Σ_0 decreases. Taking the dependence of g^* on λ_{21} (see Eq. (25)) into account leads to a saturation of this growth (for $\lambda_{21} > 0$) at a value $\lambda_{21}N(0) \sim g_0^{-1}$.

In principle the nature of the dependence of $\tilde{\Sigma}_S$ on Δ_0 for a fixed value of n has a similar character. Expression (25) gives the following result near the critical concentration $n = \Delta_0/2$:

$$(\partial\tilde{\Sigma}_s/\partial\Delta_0^2)_{\Delta_0=2n} = -(\Sigma_0/2n^2)\left[1-g_0+2g_0^2\lambda_{21}N(0)\right], \quad g_0 \gg 1. \qquad (27)$$

This formula is a special case (for small values of φ) of the more general expression (39) which describes the variation of $|\tilde{\Sigma}_S|$ near the line $\Delta = 0$. The initial growth of $|\tilde{\Sigma}_S|$ with respect to Δ_0 for $\lambda_{21} > 0$ occurs upon fulfillment of the condition

$$2g_0\lambda_{21}N(0) < 1. \qquad (28)$$

The origin of the maximum in the dependence of $\tilde{\Sigma}_S$ on n is related to competition between two phenomena. There is, on the one hand, the appearance of a peak in the density of states near the Fermi surface (which arises as a consequence of the D-pairing) which favors the increase in the value of $|\tilde{\Sigma}_S|$. On the other hand, the quantities $\tilde{\Sigma}_S$ and T_c are determined by the concentration n of excess electrons via the preexponential factor $\sim n^2$ appearing in expression (25).[3)]

In concluding this section, let us calculate the superconducting transition temperature of the SD-phase under the assumption that it is small in comparison with the transition temperature in the dielectric phase. For $T = T_c$ we may confine our attention to the terms which are linear in Σ in the expressions for F_{ik}. In this connection, in view of the assumption made concerning the smallness of the superconducting transition temperature, we may use the value of Δ at $T = 0$ (see Eq. (19)) for this parameter. Here we confine our investigation to the case of the symmetric solution ($\Sigma_2 \neq 0$). When $T = T_c$ the system of equations for Σ and Σ_2 is analogous to (23) and the result coincides with expression (25) in the limit $T_c \ll n^2/\Delta_0$, provided that the substitution $\tilde{\Sigma} = \pi T = \pi T_c/\gamma$ (γ denotes the Euler constant) is made in it.

Thus, we arrive at the conclusion that in the SD-phase the quantity $\tilde{\Sigma}$, which is related to the critical temperature T_c by the usual BCS formula, has the physical meaning of the superconducting order parameter.

5. THE PHASE DIAGRAM AT ABSOLUTE ZERO

We still have to consider one important question: how far into the (Δ_0, Σ_0) plane does the mixed SD-phase discovered above extend for an arbitrary ratio of the "coupling constants" Δ_0 and Σ_0.

In order to answer this question let us first of all determine the upper boundary of the region of existence of the SD-phase (the line $\Delta = 0$) associated with an increase of the parameter Σ_0. It is immediately necessary to stipulate that the line $\Delta = 0$ is not always an equilibrium line. This line would correspond to equilibrium only in the case when the transition from the SD-phase into the S-phase occurs via a second-order phase transition. If the indicated transition is a phase transition of first order (i.e., if it is accompanied by an abrupt change in the values of the parameters Δ and Σ), the line $\Delta = 0$ describes the lower boundary of absolute instability of the S-phase with respect to D-pairing. The question of the nature of the transition can be solved by investigating the solution for Δ near the line $\Delta = 0$.

The utilization of Eqs. (16) and (23) in connection with expansions of all quantities in powers of Δ leads to very cumbersome calculations; therefore in practice it is more convenient to use the system of equations which is obtained after integration of the functions G and F with respect to ϵ. This system of equations is also useful for investigating the case $T \neq 0$.

A. The antisymmetric case ($\Sigma_2 = 0$). Let us write down the initial system (16) in the form

$$[|\lambda_{11}|N(0)]^{-1} = \pi T\sum_\omega \frac{1}{\tilde{\omega}}\,\mathrm{Re}\,\frac{1}{\Omega}\left[\tilde{\omega}-i\delta\tilde{\mu}+i\frac{\Delta^2}{\delta\mu}\right],$$
$$[\tilde{\lambda}_{21}N(0)]^{-1} = \pi T\sum_\omega \frac{1}{\tilde{\omega}}\,\mathrm{Re}\,\frac{1}{\Omega}\left[\tilde{\omega}+i\frac{\Sigma^2}{\delta\mu}\right], \qquad (29)$$
$$n = \delta\mu+\pi T\sum_\omega \frac{1}{\tilde{\omega}}\,\mathrm{Im}\,\frac{1}{\Omega}\left[\tilde{\omega}(\tilde{\omega}-i\delta\mu)+\frac{\Sigma^2\Delta^2}{\delta\mu^2}\right],$$

where

$$\tilde{\omega} = (\omega_n^2+\tilde{\Sigma}^2)^{1/2}, \quad \Omega = [\tilde{\Delta}^2+(\tilde{\omega}-i\delta\tilde{\mu})^2]^{1/2}$$

and Re $\Omega > 0$. In the first two equations a cut-off is assumed at frequencies $\omega_n \approx \omega_D$ and $\omega_n \approx \omega_c$, respectively.

To the zero-order approximation in Δ the first and third equations in (29) describe the S-phase with the parameter $\tilde{\Sigma} = \Sigma_0$ and $\delta\mu = n$, and the second equation describes the curve $\Delta = 0$. For $T = 0$ the latter has the form

$$g_0 = \mathrm{ch}\,\varphi \ln \mathrm{cth}(\varphi/2), \quad \Sigma_0 = n\,\mathrm{sh}\,\varphi. \qquad (30)$$

The function $\Sigma_0(\Delta_0)$ increases monotonically on the line $\Delta = 0$, as is shown in Fig. 4. Such behavior indicates that the presence of superconductivity in the system inhibits electron-hole pairing, that is, the latter is possible for large values of the coupling constant $\tilde{\lambda}_{21}$.

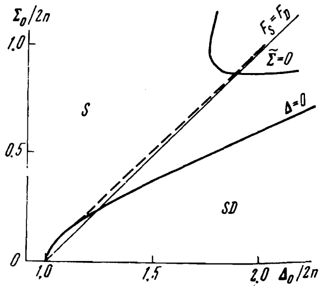

FIG. 4.

In limiting cases the dependence of Δ_0 on Σ_0 takes the form

$$\ln\left(\frac{\Delta_0}{2n}\right)=\frac{1}{2}\left(\frac{\Sigma_0}{n}\right)^2\ln\left(\frac{2n\sqrt{e}}{\Sigma_0}\right),\quad \Sigma_0\ll n, \tag{31}$$
$$\ln\left(\frac{\Delta_0}{\Sigma_0}\right)=1+\frac{1}{3}\left(\frac{n}{\Sigma_0}\right)^2,\quad \Sigma_0\gg n.$$

The doping becomes unimportant in the region of large values of Σ_0 and Δ_0, and the line $\Delta = 0$ coincides with the straight line $\Delta_0 = e\Sigma_0$, which describes the lower boundary of metastability of the S-phase (see Sec. 3). In this region the line $\Delta = 0$ lies below the straight line $\Sigma_0 = \Delta_0 - 2n$, on which the S-phase and the D-phase exist in equilibrium.

On the other hand, the opposite disposition of the indicated curves occurs for small values of Σ_0; therefore, on the initial segment the curve $\Sigma_0(\Delta_0)$ describes the line corresponding to phase transitions of the second kind between the S- and the SD-phases.[4)]

In order to clarify the question, to what values of Σ_0 does the curve $\Delta = 0$ correspond to equilibrium, let us investigate the behavior of the solutions near it. To second-order in Δ we obtain the following results from Eqs. (29) (for T = 0):

$$\delta\mu=n\left\{1+\frac{1}{2}\left(\frac{\Delta}{n}\right)^2\left[1-\operatorname{ch}\varphi\operatorname{th}^2\varphi\ln\operatorname{cth}\frac{\varphi}{2}\right]\right\},$$
$$\ln\frac{\Sigma_0}{\Sigma}=\frac{1}{2}\left(\frac{\Delta}{n}\right)^2\left(\ln\operatorname{cth}\frac{\varphi}{2}\right)\Big/\operatorname{ch}\varphi>0. \tag{32}$$

The last inequality reflects the fact noted earlier that, in the antisymmetric case the superconducting parameter Σ always decreases ($\Sigma < \Sigma_0$) in connection with the transition into the SD-phase.

Finally, expanding the second expression in (29) in powers of Δ^2 and using the expression for $\delta\mu$ and Σ, we find

$$\delta\left\{\ln\frac{\Delta_0}{\Sigma_0}-\operatorname{ch}\varphi\ln\operatorname{cth}\frac{\varphi}{2}\right\}=\left(\frac{\Delta}{2n}\right)^2\left[1-3\operatorname{ch}\varphi\operatorname{th}^2\varphi\ln\operatorname{cth}\frac{\varphi}{2}-2\operatorname{th}^2\varphi(1-\operatorname{sh}^2\varphi)\ln^2\operatorname{cth}\frac{\varphi}{2}\right]. \tag{33}$$

The symbol $\delta\{\ldots\}$ indicates that the increment of the expression inside the curly brackets associated with small deviations of the point (Δ_0, Σ_0) from the curve $\Delta = 0$ is to be taken. It is convenient to create an arbitrary deviation from the curve by varying the quantity Δ_0 under the condition Σ_0 = const. In this case we have

$$\delta\{\ldots\}=\delta\Delta_0/\Delta_0.$$

In the limiting cases of small and large values of Σ_0, expression (33) takes the following simple form:

$$\frac{\delta\Delta_0}{\Delta_0}=\begin{cases}(\Delta/2n)^2, & \Sigma_0\ll n,\\ -\frac{2}{3}(\Delta/\Sigma_0)^2, & \Sigma_0\gg n.\end{cases} \tag{34}$$

The change in the sign of the right-hand side of expression (33) obviously indicates a change in the nature of the phase transition on the line $\Delta = 0$ for $\Sigma_0 \sim n$. The line of phase transitions of the second kind ($\Delta = 0$) ends at the point where the right-hand side of expression (33) vanishes. The corresponding values of Δ_0 and Σ_0 are given by

$$\Delta_0/2n=1.104,\quad \Sigma_0/2n=0.144.$$

The line corresponding to phase transitions of the first kind from the SD-phase into the S-phase is determined by the equation $F_{SD} = F_S$. We recall that $F_{SD} < F_D$; therefore the line of equilibrium always lies above the line corresponding to equilibrium of the pure S- and D-phases. The line corresponding to the equilibrium $F_{SD} = F_S$ was calculated by using the approximate expression (22) for F_{SD}; it is indicated on Fig. 4 by the dashed line. The curve begins at the point where the line corresponding to phase transitions of the second kind ($\Delta = 0$) ends, and for large values of Σ_0 and Δ_0 it asymptotically approaches the straight line $\Delta_0 - 2n = \Sigma_0$ from above.

We have already seen in section 4 that in the presence of doping ($n \neq 0$) the system (16) doesn't admit solutions corresponding to nucleation of the S-phase into the D-phase (the line $\Sigma = 0$). This question requires some interpretation since in Sec. 2 it was shown that for n = 0 and T = 0 the line $\Sigma = 0$ exists on the straight line $\Sigma_0 = e\Lambda_0$. The point is that the shift of the Fermi level doesn't vanish ($\delta\mu = \Delta_0$) as n tends to zero (for $T \equiv 0$)—in contrast to the case when it is assumed from the very beginning that the concentrations of electrons and holes are equal to each other ($n = 0$, $\delta\mu = 0$). In the first case the excess electrons are found at the edge of the forbidden band ($\epsilon \geq \Delta_0$); therefore, S-pairing is possible at T = 0 for any arbitrary value of n. However, even at an arbitrarily small temperature $T \sim n^2/\Delta_0 \ll \Delta_0$ the quantity $\delta\mu$ rapidly tends to zero, and the system goes over into the pure D-phase ($\Sigma = 0$). Due to lack of space we shall not cite here the corresponding expressions for Σ and $\delta\mu$ in this temperature regime.

The system (16) admits the following class of metastable solutions on the line $\tilde\Sigma = 0$ (that is, $\delta\mu = \Delta$, $\Sigma \neq 0$):

$$\Sigma=\Sigma_0\cos\alpha,\quad \Delta=\Sigma_0\sin\alpha,\quad n=\Sigma_0\alpha\cos\alpha,\quad g_0=\alpha\operatorname{ctg}\alpha.$$

The parameter α varies over the range from 0 to $\pi/2$. The last two equations describe a curve on the (Δ_0, Σ_0) plane with asymptotes $\Delta_0 = \Sigma_0$ (for $\alpha = \pi/2$) and $\Delta_0 = e\Sigma_0$ (for $\alpha = 0$). The minimum value of Δ_0 on the curve $\tilde\Sigma = 0$ (see Fig. 4) is reached at the point $\Delta_0/2n = 1.75$, $\Sigma_0/2n = 0.961$. For the antisymmetric case under consideration, the line $\tilde\Sigma = 0$ determines the boundary of metastability of the SD-phase. We shall not investigate this question in more detail since we shall see below that the equilibrium SD-phase, corresponding to the symmetric solution, is realized in the region $\Sigma_0 \sim \Delta_0$.

B. The symmetric case. The initial system (23) for the quantities Δ, Σ_2, Σ and $\delta\mu$ can be represented in the form

$$[\lambda_{21}N(0)]^{-1}\Delta=\pi T\sum_\omega\frac{1}{\tilde\omega}\operatorname{Re}\frac{1}{\Omega}[\tilde\omega\Delta+i\tilde\Sigma\Sigma_2],$$
$$[\lambda_{21}N(0)]^{-1}\Sigma_2=\pi T\sum_\omega\frac{1}{\tilde\omega\delta\tilde\mu}\operatorname{Im}\frac{1}{\Omega}[-\tilde\omega(\tilde\omega-i\delta\tilde\mu)\Sigma_2+\tilde\Sigma\tilde\Delta\delta\mu],$$
$$[|\lambda_{11}|N(0)]^{-1}\Sigma=\pi T\sum_\omega\frac{1}{\tilde\omega}\operatorname{Re}\frac{1}{\Omega}[\tilde\omega\Sigma-i\tilde\Sigma\delta\mu],$$

$$n=\delta\mu+\pi T\sum_{\omega}\frac{\delta\mu}{\tilde\omega\delta\tilde\mu}\,\mathrm{Im}\frac{1}{\Omega}[\tilde\omega(\tilde\omega-i\delta\tilde\mu)+\Sigma^2-\bar{\tilde\Sigma}^2], \quad (35)$$

where the quantity Ω is defined above (see Eq. (29)).

It follows from Eqs. (35) that for $\Delta = 0$ one simultaneously has $\Sigma_2 = 0$. In the linear approximation in Δ the relation between Σ_2 and Δ has the form

$$\Sigma_2=\lambda_{21}N(0)\tilde\Delta\,\mathrm{th}\,\varphi\ln\mathrm{cth}(\varphi/2)\quad(\Sigma_0=n\,\mathrm{sh}\,\varphi),$$
$$\tilde\Delta=\Delta[1-\lambda_{21}N(0)\,\mathrm{sh}\,\varphi\,\mathrm{th}\,\varphi\ln\mathrm{cth}(\varphi/2)]^{-1}. \quad (36)$$

It follows from the last equation that the quantities Δ and $\tilde\Delta$ differ very little since $\lambda_{21}N(0) \ll 1$ and the quantity $\sinh\varphi\tanh\varphi\ln\coth(\varphi/2)$ is bounded. In the higher-order approximations in Δ we can no longer assume $\Delta \equiv \tilde\Delta$ (i.e., neglect Σ_2) since in the SD-phase it becomes essential to take Σ_2 into account for small values of Σ_0 (see the discussion of Eqs. (23)).

With what has been said taken into account and assuming $\Sigma_2 = 0$, from the first equation in (35) we immediately find the equation for the line $\Delta = 0$:

$$g_0\,\mathrm{ch}\,\varphi=\ln\mathrm{cth}\,(\varphi/2). \quad (37)$$

The dependence of Σ_0 on Δ_0 for n = const is shown on Fig. 3 (the boundary separating the S- and SD-phases). The characteristic feature of this dependence (in contrast to the antisymmetric solution) is the region of double valuedness in the interval $0.9 \le \Delta_0/2n \le 1$; the minimum value $\Delta_0 = 1.80n$ on the curve $\Delta = 0$ is reached at $\Sigma_0 = 0.66n$. The decrease of the function $\Delta_0(\Sigma_0)$ on the segment $0 < \Sigma_0 < 0.66n$ indicates that the intraband S-pairing favors interband D-pairing. We shall show below that in this range of values of Σ_0 the D-pairing, in turn, favors superconductivity.

It is clear from Eq. (37) that for $\Delta_0 \gg n$ the dependence of Σ_0 on Δ_0 is linear in character: $\Delta_0 \approx \Sigma_0$. It is easy to see that the line corresponding to equilibrium of the pure S- and D-phases ($\Sigma_0 = \Delta_0 - 2n$) always lies below the line $\Delta = 0$ (see Fig. 3). This enables us to conclude that a phase transition of the second kind from the S-phase into the SD-phase occurs on the line $\Delta = 0$. One can rigorously verify this by calculating Δ near the line $\Delta = 0$. After very cumbersome calculations the solution of the system of equations (35) correct to terms $\sim\Delta^3$ leads to the following result:

$$\frac{\delta\Delta_0}{\Delta_0}=\left(\frac{\Delta}{2n}\right)^2\frac{1}{\mathrm{ch}^4\varphi}\Big[1+5\,\mathrm{sh}\,\varphi\,\mathrm{th}^2\varphi\ln\mathrm{cth}\frac{\varphi}{2}-2(2\,\mathrm{th}^2\varphi-\mathrm{sh}^2\varphi)\ln^2\mathrm{cth}\frac{\varphi}{2}\Big]. \quad (38)$$

The right-hand side of this equation is positive for all values of φ, that is, the solution $\Delta \neq 0$ on the (Δ_0, Σ_0) plane only exists to the right of the line $\Delta = 0$. This proves the assertion made above.

Thus, one can conclude that the symmetric solution (in contrast to the antisymmetric solution) for the mixed SD-phase exists over a significantly wider range of values of the parameters Δ_0 and Σ_0. The antisymmetric solution, as has already been mentioned in section 4, becomes energetically more favorable for sufficiently small values of Σ_0 upon fulfillment of the condition $\lambda_{21}N(0)g_0 \sim 1$.

Finally, we present the result of a calculation of $\tilde\Sigma_S$ in the SD-phase near the line $\Delta = 0$:

$$\ln\frac{\Sigma_0}{|\tilde\Sigma_s|}=\frac{1}{2}\left(\frac{\tilde\Delta}{n}\right)^2\frac{1}{\mathrm{ch}^2\varphi}\Big[1-\mathrm{ch}^{-1}\varphi\ln\mathrm{cth}\frac{\varphi}{2}+2\lambda_{21}N(0)\ln^2\mathrm{cth}\frac{\varphi}{2}\Big]. \quad (39)$$

The term containing λ_{21} becomes important for small values of $\varphi \ll 1$. If $\varphi \gtrsim 1$, one can neglect this term. In this case the right-hand side of expression (39) changes sign at the point $\ln\coth(\varphi/2) = \cosh\varphi$; returning to Eq. (37) we see that it is precisely at this point on the curve $\Delta = 0$ that the quantity Δ_0 assumes its minimum value $\Delta_0 = 1.8n$ for $\Sigma_0 = 0.66n$. The increase of $\tilde\Sigma_S$ in the SD-phase ($\Delta \neq 0$) corresponds to values of $\Sigma_0 < 0.66n$. If $\Sigma_0 > 0.66n$ then Σ_S decreases as a function of Δ_0.

For small values of φ it is necessary to take the term proportional to λ_{21} into consideration in (39) since $\ln\coth(\varphi/2) \gg 1$. In this region we may assume $\cosh\varphi \approx 1$ and, according to Eq. (37), $\ln\coth(\varphi/2) \approx g_0$. From here it is seen that there is again a decrease of $\tilde\Sigma_S$ in the SD-phase for sufficiently small values of Σ_0 and for $\lambda_{21} > 0$ (repulsion of the electrons from different bands); for the case of attraction ($\lambda_{21} < 0$, but $\tilde\lambda_{21} > 0$) the value of $\tilde\Sigma_S$ always increases near the line $\Delta = 0$ for $\Sigma_0 < 0.66n$. The condition for the growth of $\tilde\Sigma_S$ in the SD-phase associated with small values of Σ_0 and $\lambda_{21} > 0$ coincides with the result obtained earlier in section 4 (see (28)).

Thus, within the framework of the simple model adopted here we have been able to show the feasibility of the simultaneous existence in the system of superconducting and electron-hole pairings over a wide range of values of the coupling constants λ_{11} and λ_{21} for T = 0. Unfortunately, only the case of weak S-pairing ($T_c \ll \Delta_0$) yields to analytic investigation when $T \neq 0$. A calculation of the phase diagram in the (T, n) plane would certainly be of interest. The whole series of questions requires special consideration: the influence of anisotropy and local impurity levels in the D-phase, and also the role of hybridization and interband transitions.

The authors thank the participants in the seminar on superconductivity, directed by V. L. Ginzburg, for a discussion of the work.

1) Below we shall see that for the present choice of phases the determinant $D(\epsilon)$ of the system (4) is an even function: $D(\epsilon) = D(-\epsilon)$. The existence of the antisymmetric solution was pointed out in[4]. However, the general expression for $D(\epsilon)$ given in that article is incorrect, since it contains terms linear in ϵ.

2) More precisely, the mixed SD-phase corresponding to the antisymmetric solution is realized. The phase transition line (for a phase transition of the first kind) can be calculated by equating the energies: $F_{SD}^{(a)} = F_{SD}^{(s)}$.

3) The presence of a maximum in the dependence of T_c on n is correct provided the condition $\omega_D \gg \Delta$ is satisfied. In the opposite limiting case, $\Delta \ll \omega_D \ll (\delta\mu^2 - \Delta^2)^{1/2}$, the value of T_c (and Σ_s) is given by the usual BCS formula:

$$T_c\sim\omega_D\exp\left[-\frac{(\delta\mu^2-\Delta^2)^{1/2}}{\delta\mu}\frac{1}{N(0)|\lambda_{11}|}\right],$$

that is, it corresponds to an increase of T_c in the dielectric phase.

4) This agrees with the conclusion reached in [11] that a transition occurs at the point $\Delta_0 = 2n$ (for $\Sigma_0 = 0$) from the normal state into the exciton state by means of a phase transition of the second kind.

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