Passage of weak Gaussian beam through a nonlinear medium

V. N. Lugovoĭ

P. N. Lebedev Physics Institute
(Submitted March 15, 1973)
Zh. Eksp. Teor. Fiz. **65**, 886–893 (September 1973)

Passage of a monochromatic Gaussian beam through a medium with an arbitrary local nonlinearity is considered. Nonlinearity of the medium is taken into account in the first approximation of perturbation theory. Asymptotic expressions for the beam field at large distances from the entrance plane of the medium are obtained and it is shown that even in the given (lower) approximation of perturbation-theory propagation of a beam in a medium with arbitrary nonlinearity is accompanied by deviation of the field in it from a Gaussian distribution. The nature of the deviation in this case depends essentially on the form of nonlinearity of the medium and thus permits an analysis of the form to be carried out, nonlinear absorption being included. The cause of the discrepancy between the results of Refs. 6 and 7 on the propagation of focused beams in a medium with Kerr nonlinearity is elucidated.

Great interest has recently been shown in the problem of the propagation of light beams in nonlinear media (see[1]). In this, the greatest attention has been paid to beams with Gaussian initial distribution, since, usually, the intensity profile of coherent laser beams is, to a great degree of accuracy, Gaussian. The greater part of the results has been obtained in this case by the numerical solution of the corresponding problems. The applicability of the analytical calculations is limited to only a small region near the initial plane of the nonlinear medium[2]. At the same time, the problem of the passage of a relatively weak light beam through a medium, when the nonlinearity of this medium can be regarded as a perturbation only in the first approximation, has thus far not been considered. Such an approximation is similar to the Born approximation in quantum-mechanical collision theory, and it allows us to investigate, in particular, the beam field at large distances from the layer of the nonlinear medium traversed by this beam. In the present paper the passage of an arbitrary, including a focused, monochromatic Gaussian, beam through a layer of a medium with arbitrary local nonlinearity, including, in general, nonlinear absorption, is investigated in this approximation. From the expressions obtained below follows, in particular, that the passage of a beam through a layer of a nonlinear medium is accompanied by a deviation of the field distribution in the layer from the Gaussian distribution, and that this deviation depends essentially on the specific form of the nonlinearity of the medium. Therefore, the analysis of the deviation in question can be used to determine the form of the nonlinearity of the medium and the form of the nonlinear absorption in it.

Let the layer of the medium under consideration occupy the region of space $0 \le z \le l$, and let us write the permittivity ϵ of this medium in the form of an arbitrary function of the intensity of the oscillations of the field in the beam:

$$\varepsilon=\varepsilon_0[1+\nu(|E|^2)],\quad \nu(|E|^2)=\sum_{m=1}^{\infty} n_{2m}|E|^{2m}. \tag{1}$$

Here E is the complex amplitude of the electric-field oscillations and is connected with the field's true value $\mathscr{E}$ by the relation

$$\mathscr{E}={}^1\!/_2E\exp(ikz-i\omega t)+\text{c.c.},$$

ω is the frequency of the oscillations of the field in the beam, $k = 2\pi/\lambda$, $\lambda = 2\pi c/\omega\sqrt{\epsilon_0}$ is the wavelength of the light, $\epsilon_0 > 0$, and the coefficients n_{2m} are, generally speaking, complex quantities. The expansion of the function $\nu(|E|^2)$ in the form of the series (1) is convenient in that the decisive role in this expansion is usually played by one or a small number of terms. The term of the series with the number $m = k - 1$ may correspond to a k-photon process in the medium and, in particular, the imaginary part of this term may be determined by the k-photon absorption. Below, we shall, for simplicity, assume that the refractive index of the medium at $z > l$ is equal to $n \equiv \sqrt{\epsilon_0}$. We shall also assume that the beam under consideration is incident at the boundary $z = 0$ from the half-space $z < 0$ and that the diameter of the focus of the unperturbed (by the nonlinearity $\nu(|E|^2)$) beam is much larger than the wavelength λ. If the Born approximation is applicable, then the last condition obviously remains valid for the perturbed beam. In this case the propagation of light in the nonlinear medium is described by the following equation for the complex field amplitude E (see, for example,[1]):

$$\Delta_\perp E+2ik\frac{\partial E}{\partial z}+k^2\nu(|E|^2)E=0, \tag{2}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$. We shall consider Eq. (2) with the boundary condition

$$E|_{z=0}=E_0\exp\left[-\left(\frac{1}{2a^2}+\frac{ik}{2R}\right)r_\perp^2\right],\quad r_\perp=\sqrt{x^2+y^2}, \tag{3}$$

which corresponds to a Gaussian beam incident at the boundary $z = 0$ and having in this plane an intensity-distribution radius a and, in the corresponding linear medium (with a refractive index $n \equiv \sqrt{\epsilon_0}$), a point $z = R$ of geometrical convergence of the rays; the value $1/R = 0$ corresponds to an incident beam of parallel rays, the value $R > 0$ to a converging, and the value $R < 0$ to a diverging beam.

Let us represent the solution to Eq. (2) in the form of an expansion in powers of ν:

$$E=E^{(0)}+E^{(1)}+E^{(2)}+\ldots \tag{4}$$

Then we obtain for the functions $E^{(f)}$ of the successive approximations the following equations:

$$\Delta_\perp E^{(0)}+2ik\frac{\partial E^{(0)}}{\partial z}=0,$$
$$\Delta_\perp E^{(1)}+2ik\frac{\partial E^{(1)}}{\partial z}+k^2\nu(|E^{(0)}|^2)E^{(0)}=0,\ldots \tag{5}$$

The solution of the equation for $E^{(0)}$ with the boundary condition (3) is known (see, for example,[1]) and can be written in the form

$$E^{(0)}=\frac{v_0}{1+i\zeta/l_0}\exp\left(-\frac{1}{2a_0^2}\frac{1}{1+i\zeta/l_0}r_\perp^2\right),\quad \zeta=z-z_0, \tag{6}$$

where

$$l_0=ka_0^2=\frac{R^2}{ka^2}\left[1+\frac{R^2}{(ka^2)^2}\right]^{-1},\quad z_0=R\left[1+\frac{R^2}{(ka^2)^2}\right]^{-1},$$
$$v_0=E_0\left(1-i\frac{ka^2}{R}\right). \tag{7}$$

The above-introduced symbols have the following meaning: a_0 is the minimum radius of the unperturbed beam, $z = z_0$ is the cross section corresponding to this radius, and v_0 is the value of the complex amplitude of the axial field at this cross section. Under the condition $|R| \ll ka^2$, i.e., at large Fresnel numbers for the beam in the plane $z = 0$, the cross section $z = z_0$ practically coincides with the cross section $z = R$. Thus, the problem reduces to the problem of finding the functions $E^{(f)}$ of the successive approximations with the boundary conditions

$$E^{(f)}|_{z=0}=0,\quad f=1,2\ldots \tag{8}$$

Below we shall consider only the equation for the function $E^{(1)}$ of the first approximation. According to (5) and (6), the solution $E^{(1)}$ can be written in the form of the sum:

$$E^{(1)}=k^2\sum_{m=1}^{\infty}n_{2m}E_m^{(1)}, \tag{9}$$

the addends $E_m^{(1)}$ of which satisfy the following equations:

$$\Delta_\perp E_m^{(1)}+2ik\frac{\partial E_m^{(1)}}{\partial z}=-\Phi_m(\zeta)\exp\{\gamma_m(\zeta)r_\perp^2\}, \tag{10}$$

where

$$\Phi_m(\zeta)=\frac{|v_0|^{2m}v_0}{(1+i\zeta/l_0)(1+\zeta^2/l_0^2)^m},$$
$$\gamma_m(\zeta)=-\frac{1}{2a_0^2}\frac{2m+1-i\zeta/l_0}{1+\zeta^2/l_0^2},\quad \zeta=z-z_0. \tag{11}$$

Let us consider Eq. (10) for arbitrary m with the boundary condition $E_m^{(1)}|_{z=0} = 0$, which, according to (9), allows us to determine the function $E^{(1)}$ satisfying the required condition (8). The required solution $E_m^{(1)}$ can be written in the form of a convolution of the Green function of the Eq. (10) with the function on the right-hand side of the equation. For $z > l$, we obtain

$$E_m^{(1)}(r)=\int_{-\infty}^{\infty}d\xi\,d\eta\int_{-z_0}^{l-z_0}d\zeta\frac{\Phi(\zeta)}{4\pi(z-\zeta-z_0)}$$
$$\times\exp\left\{\frac{ik[(x-\xi)^2+(y-\eta)^2]}{2(z-\zeta-z_0)}+\gamma(\zeta)(\xi^2+\eta^2)\right\}. \tag{12}$$

Setting $(z - \zeta - z_0)^{-1} \approx z^{-1}$ in the pre-exponential factor of the integrand and $(z - \zeta - z_0)^{-1} \approx z^{-1}(1 + (\zeta + z_0)/z)$ in the exponent, and neglecting further in the exponent the terms

$$\frac{ik}{2z}(\xi^2+\eta^2),\quad -\frac{ik(\zeta+z_0)}{2z^2}(2x\xi+2y\eta+\xi^2+\eta^2),$$

we arrive at an expression that is valid for $z \to \infty$, $|x/z| \lesssim C_1$ = const, and $|y/z| \lesssim C_2$ = const. Going over in this expression from the Cartesian variables ξ, η to the polar variables ρ, φ and performing the integration over φ, we find

$$E_m^{(1)}(r)=\frac{1}{2z}\exp\left(\frac{ikr_\perp^2}{2z}+\frac{ikz_0}{2}n_\perp^2\right)\cdot$$
$$\times\int_{-z_0}^{l-z_0}d\zeta\,\Phi_m(\zeta)\exp\left(\frac{ikn_\perp^2}{2}\zeta\right)\int_0^\infty\rho J_0(kn_\perp\rho)\exp(\gamma_m(\zeta)\rho^2)d\rho, \tag{13}$$

where $n_\perp = ((x^2 + y^2)/z^2)^{1/2} > 0$ and $J_0(z)$ is a Bessel function. The subsequent integration over ρ, using the relation $kl_0 = k^2a_0^2$ and the expressions (11) for the functions $\Phi_m(\zeta)$ and $\gamma_m(\zeta)$, yields

$$E_m^{(1)}(r)=\frac{A_m}{z}\exp\left(\frac{ikr_\perp^2}{2z}+\frac{ikz_0}{2}n_\perp^2\right)\cdot$$
$$\times\int_{L_1}^{L_2}du\frac{\exp\{-\alpha[(2m+1)u+i]/[u+i(2m+1)]\}}{(1+iu)(1+2m-iu)(1+u^2)^{m-1}}. \tag{14}$$

Here

$$A_m=\frac{1}{2}a_0^2l_0|v_0|^{2m}v_0,\quad \alpha=\frac{k^2a_0^2}{2}n_\perp^2,\quad L_1=-\frac{z_0}{l_0},\quad L_2=\frac{l-z_0}{l_0}, \tag{15}$$

the quantity α can also be written in the form $\alpha = \frac{1}{2}(\theta/\theta_d)^2$, where θ is the viewing angle and $\theta_d = 1/ka_0$ is the angle of divergence of the unperturbed beam due to diffraction.

To evaluate the integral (14), we make the change of variable

$$u=i\frac{(2m+1)w+2m}{2m-w}, \tag{16}$$

as a result of which the expression (14) assumes the form

$$E_m^{(1)}(r)=\frac{A_m}{z}T_m(\alpha)\exp\left(\frac{ikr_\perp^2}{2z}+\frac{ikz_0}{2}n_\perp^2-\alpha\right), \tag{17}$$

where

$$T_m(\alpha)=\frac{i(-1)^m}{2^{2m-1}(m+1)^m m^{m-1}}\int_{\Gamma_m}\frac{(w-2m)^{2m-2}e^{-\alpha w}}{w^m(w+2)^{m-1}}dw. \tag{18}$$

The integration contour Γ_m is shown in Fig. 1; according to (16), the initial and end points w_1 and w_2 of this contour are determined by the equality $w_j = 2m(L_j - i)[L_j + i(2m + 1)]^{-1}$, where $j = 1, 2$.

Let us now write out, on the basis of (4), (6), and (7) and in the approximation in ν under consideration, the asymptotic (for $z \to \infty$ and $n_\perp \lesssim C$) expression for the resultant field E of a beam that has traversed the considered layer of the nonlinear medium. Adding the corresponding asymptotic expression for $E^{(0)}$ obtained from (6) to the expression (17) for $E_m^{(1)}$, we find

$$E=\frac{l_0v_0}{iz}\left[1+\frac{i}{2}n_{2m}kl_0|v_0|^{2m}T_m(\alpha)\right]\exp\left(\frac{ikr_\perp^2}{2z}+i\alpha\frac{z_0}{l_0}-\alpha\right). \tag{19}$$

Here we bear in mind that only one of the functions $E_m^{(1)}$ corresponding to a definite (although arbitrary) number m makes the dominant contribution to (19). If, however, several of the functions $E_m^{(1)}$ are simultaneously important, then the expression inside the square brackets in (19) should be summed over m. As can be seen from (19), the field distribution in a beam that has traversed a layer of a nonlinear medium differs from the unperturbed distribution by the factor in the square brackets, with the result that the function $T_m(\alpha)$ determines the nature of the deviation of the perturbed-beam field distribution from the Gaussian initial distribution.

Let us compute $T_m(\alpha)$ for $L_1 \to -\infty$ and $L_2 \to \infty$, i.e., for the case when the focal region of the beam is located deep inside the layer under consideration, so that its length l_0 is much smaller than the width l of the layer. In this case the integration contour Γ_m closes into the complete circle shown in Fig. 1, and the integral (18) is determined by the residue of the integrand at the point $w = 0$. Computing this residue for different m, we find

$$T_1(\alpha)=\frac{\pi}{2},\quad T_2(\alpha)=\frac{\pi}{9}(\alpha+1),\quad T_3(\alpha)=\frac{9\pi}{256}\left(\frac{\alpha^2}{2}+\frac{5\alpha}{3}+\frac{19}{12}\right),\ldots, \tag{20}$$

$$T_m(\alpha)=\frac{\pi\Gamma(2m-1)}{2^{2m-2}(m+1)^m\,\Gamma(m-1)}$$
$$\times\sum_{s=0}^{m-1}\sum_{n=0}^{s}(2\alpha)^n\frac{m^s\Gamma(m+s-n-1)}{\Gamma(m+s)\Gamma(m-s)\Gamma(n+1)\Gamma(s-n+1)}.$$

Here $\Gamma(z)$ is the gamma function. According to the first of these expressions, the two-photon ($m = 1$) process leads, under the conditions being considered, only to the appearance of a constant (α-independent) factor in the field distribution of the unperturbed beam, with the result that this process for $L_1 \to -\infty$, $L_2 \to \infty$, and large z does not, in the approximation in ν under consideration, distort the Gaussian field distribution. At the same time, three-photon, four-photon, etc., processes lead under these same conditions to deviations from the Gaussian distribution, and the nature of these deviations is determined by the order $k = m + 1$ of the process under consideration. According to (20), the quantity $T_m(\alpha)$ is a polynomial of degree $m - 1$ in α. Thus, the order of the multiphoton process in the medium determines the degree of this polynomial. It is also worth noting that the quantity $T_m(\alpha)$ is positive for all m and all values of $\alpha > 0$.

Let us now write out the condition of applicability of the expressions obtained, a condition which is connected with the use of the first approximation in ν. In general, such a condition follows from the requirement that the inequality $|E^{(1)}| \ll |E^{(0)}|$, which in turn yields for any m[1)]

$$^1/_2 k l_0 |n_{2m} v_0{}^{2m} T_m(0)| \ll 1. \tag{21}$$

be fulfilled in the region of space that makes the dominant contribution to the integrals (12) and (13). Besides this, notice that at large values of α, the integral over ρ in (13) contains the function $J_0(kn_\perp\rho)$, which undergoes many oscillations in the interval $0 \le \rho \lesssim a_0/\sqrt{m}$ that determines the dominant contribution to (13). A similar situation arises in the subsequent integration over ζ. The exponential smallness of the considered integral for $\alpha \to \infty$ (as can be seen from (17): $|E_m^{(1)}| \propto T_m(\alpha)e^{-\alpha}$) is connected with this. The result is that for large α, the requirement for accuracy of description of the integrand becomes substantially more rigorous than the requirement defined by the condition (21). Therefore, under the present condition and at large α, the obtained expressions, generally speaking, become invalid in the same way as (see[3]) the Born approximation can lose its validity in quantum-mechanical collision theory.

Let us now discuss in greater detail the case $m = 1$, including, as is well known, the case of the Kerr nonlinearity. In this case the expression (18) gets simplified, and it is easy to derive an explicit expression for $T_1(\alpha)$ for arbitrary L_1 and L_2. Making in (18), for convenience, the change of variable $w = -t/\alpha$, we find for $m = 1$

$$T_1(\alpha)=-\frac{i}{4}\int\limits_\Gamma\frac{e^t}{t}dt=-\frac{i}{4}\left(\int\limits_{\gamma_2}\frac{e^t}{t}dt-\int\limits_{\gamma_1}\frac{e^t}{t}dt\right)=\frac{i}{4}[\mathrm{Ei}(z_1)-\mathrm{Ei}(z_2)], \tag{22}$$

where the integration contours Γ, γ_1, and γ_2, and the cut in the t plane are shown in Fig. 2; Ei(z) is an exponential integral function; $z_j = -\alpha w_j = 2\alpha(i - L_j)[L_j + 3i]^{-1}$ ($j = 1, 2$)[2)]. The expression (22) for $T_1(\alpha)$ solves the formulated problem. We see that generally (for arbitrary L_1 and L_2) the deviation from the Gaussian distribution of the field distribution in a

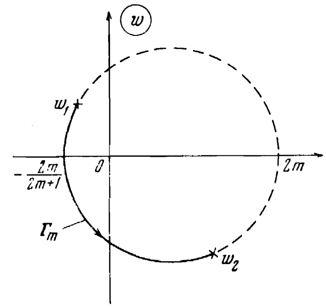

FIG. 1

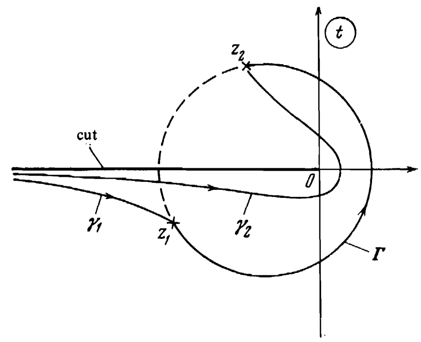

FIG. 2

beam that has traversed a layer of a nonlinear medium arises also in the $m = 1$ case. The function $T_1(\alpha)$ determining this deviation is, according to (22), complex. Let us now consider in greater detail the condition (21) for $m = 1$, i.e., the condition of applicability of the expression (22). We shall assume in this connection that $L_2 - L_1 \gtrsim 1$ and $|L_2| \lesssim 1$ (the last conditions imply that the width l of the layer of the nonlinear medium is greater than, or of the order of, the length l_0 of the focal region of the beam and that the focal region is located inside, or not far from, this layer). Using the expansion for Ei(z) given in the footnote 2), we can easily convince ourselves that the inequality (21) can be rewritten in the form $P \ll c\sqrt{\epsilon_0}/4|n_2|k^2$, where P is the power of the incident beam, and c is the velocity of light in vacuum. For a real n_2 (or for $n_2' \gg n_2''$), this condition assumes the form

$$P \ll {}^1/_2 P_c^{(1)}, \tag{23}$$

where $P_c^{(1)} \approx c\sqrt{\epsilon_0}/2n_2k^2$ is the critical power above which a multifocus structure is formed in the beam under consideration[1,5,6].

In conclusion, let us consider the question of applicability of the transformation that a thin lens can produce in the problem under consideration. This transformation consists in the change of variables

$$z'=\frac{zF}{z+F},\quad \mathbf{r}_\perp'=\frac{\mathbf{r}_\perp F}{z+F},$$
$$E'(\mathbf{r}_\perp', z')=\frac{F}{F-z'}E\left(\frac{\mathbf{r}_\perp' F}{F-z'},\frac{z'F}{F-z'}\right)\exp\left(\frac{ikr_\perp'^2}{2}\frac{1}{z'-F}\right), \tag{24}$$

under which Eq. (2) is, as is well known, invariant when $\nu = 0$. The substitution (24) transforms the initial boundary condition at $z = 0$ in the same way it would be transformed if the initial beam passed through a thin lens of focal length F located at $z = 0$. Talanov points out in his paper[7] that the invariance of Eq. (2) under the transformation (24) is also preserved when

$\nu(|E|^2) = n_2|E|^2$ (where n_2 is, in general, complex), and investigates on this basis the propagation in the medium under consideration of a beam that has passed through a thin lens. He obtains for the space behind the focal plane of the lens, i.e., for $z' > F$, and for $P > P_c^{(1)}$ a solution that is qualitatively different from the solution obtained earlier[6] by other authors. Let us now consider the cause of this discrepancy. We shall show that the solution presented in[7] for $z' > F$ is incorrect, since the transformation (24), when applied to an infinite nonlinear medium (as is done in[7]), leads to a discontinuity in the transformed solution at $z' = F$.

For this purpose, let us on the basis of the above-obtained expressions consider the limiting values of E' at $z' = F \pm 0$. As a preliminary, let us write out that solution to Eq. (2) in an infinite nonlinear medium which satisfies the condition (3) at $z = 0$. Above we considered this solution (E) for $z > 0$. We obtain the continuation $\overline{E}$ of this solution to $z < 0$ with the aid of the relation $\overline{E}(r_\perp, z, R) = E^*(r_\perp, -z, -R)$ (we assume, for simplicity, that E_0 is real). Further, taking account of (7), we find on the basis of (19) for $z \to -\infty$ that

$$\overline{E} = \frac{l_0 v_0}{iz}\left[1 - \frac{i}{2} n_2^* k l_0 |v_0|^2 T_1^*(\alpha)\right] \exp\left(\frac{ikr_\perp^2}{2z} + i\alpha\frac{z_0}{l_0} - \alpha\right). \quad (25)$$

Here the quantity $T_1(\alpha)$ corresponds to an arbitrary value of L_2 and to the value $L_1 = \infty$, i.e.,

$$T_1(\alpha) = \frac{i}{4}\left\{\mathrm{Ei}(-2\alpha) - \mathrm{Ei}\left[2\alpha\frac{i - L_2}{L_2 + 3i}\right]\right\};$$

the asterisk denotes complex conjugation. Applying now the transformation (24) to the solution under consideration and computing the limits $E'|_{z'=F-0}$ and $E'|_{z'=F+0}$, we obtain

$$E'|_{z'=F-0} = \frac{l_0 v_0}{iF}\left[1 + \frac{i}{2} n_2 k l_0 |v_0|^2 T_1\left(\frac{k^2 a_0^2}{2F^2} r_\perp'^2\right)\right]\mathscr{A},$$

$$E'|_{z'=F+0} = \frac{l_0 v_0}{iF}\left[1 - \frac{i}{2} n_2^* k l_0 |v_0|^2 T_1^*\left(\frac{k^2 a_0^2}{2F^2} r_\perp'^2\right)\right]\mathscr{A}, \quad (26)$$

$$\mathscr{A} = \exp\left\{\frac{ikr_\perp'^2}{2F} + \left(i\frac{z_0}{l_0} - 1\right)\frac{k^2 a_0^2}{2F^2} r_\perp'^2\right\}.$$

Evidently, these limits coincide only in a linear medium (when $n_2 = 0$). They differ in a nonlinear medium owing to the difference in the factors in the square brackets. In general, these factors depend on $r_\perp'$, and their phases have opposite signs. Thus, the discontinuity of the transformed solution in the plane $z' = F$ can convert a diverging beam into a converging one with the formation at $z' > F$ of a "mirror" (with respect to the $z' = F$ plane) type of picture. This precisely explains the appearance at $z' > F$ of the point of convergence described in[7] for the $P > P_c^{(1)}$, $\nu(|E|^2) = n_2|E|^2$ case, a convergence point which is uniquely connected with the corresponding convergence point at $z' < F$. The fact, however, is that the (mirror-type) convergence point at $z' > F$ described in[7] does not exist, since the correctly obtained results of[6] lead to a different picture.

The author thanks A. M. Prokhorov for discussions in the course of the work.

---

1) Notice that the existence itself of a finite region of space making the dominant contribution to the integrals (12) and (13) obviously implies that the order in which the passage to the limit $L_1 \to \infty$, and $z \to \infty$ is carried out is of no importance. In consequence, the investigation carried out in this paper also pertains to the case of an infinite (semi-infinite) nonlinear medium.

2) It is convenient for us to determine the exponential integral function when there is a cut along the negative real semiaxis, as shown in Fig. 2; for then we can continuously and independently vary L_1 and L_2 from $-\infty$ to ∞. The function Ei(z) determined in this way coincides with the corresponding function determined in [4] when $-\pi < \arg z < \pi/2$ and differs from the latter by a term $2\pi i$ when $\pi/2 < \arg z < \pi$. For $|z| < 1$, Ei(z) can be expanded in the series

$$\mathrm{Ei}(z) = i\pi + C + \ln z + z + \frac{z^2}{2\cdot 2!} + \frac{z^3}{3\cdot 3!} + \ldots,$$

where C = 0.577... is the Euler-Mascheroni constant; the multivaluedness of this function is allowed for by the multivaluedness of the term ln z, and therefore the restriction on arg z in this equality can be dropped (we cite the expansion in question here, since there is an error in this expansion in [4]).

1 V. N. Lugovoĭ and A. M. Prokhorov, Usp. Fiz. Nauk 111, 203 (1973) [Sov. Phys.-Uspekhi 16, No. 5 (1974)].
2 V. N. Lugovoĭ, Dokl. Akad. Nauk SSSR 176, 58 (1967) [Sov. Phys.-Doklady 12, 866 (1968)].
3 L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Fizmatgiz, 1963 (Eng. Transl., Addison-Wesley, Reading, Mass., 1958).
4 A. Kratzer and W. Franz, Transzendente Funktionen, Akademishe Verlagsgesellschaft Geest and Portig, Leipzig, 1963 (Russ. Transl., IIL, 1963).
5 A. L. Dyshko, V. N. Lugovoĭ, and A. M. Prokhorov, ZhETF Pis. Red. 6, 655 (1967) [JETP Lett. 6, 146 (1967)]; Zh. Eksp. Teor. Fiz. 61, 2305 (1971) [Sov. Phys.-JETP 34, 1235 (1972)].
6 A. L. Dyshko, V. N. Lugovoĭ, and A. M. Prokhorov, Dokl. Akad. Nauk SSSR 188, 792 (1969) [Sov. Phys.-Doklady 14, 976 (1970)].
7 V. I. Talanov, ZhETF Pis. Red. 11, 303 (1970) [JETP Lett. 11, 199 (1970)].

Translated by A. K. Agyei
92