Nonlinear helical perturbations of a plasma in the tokamak

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Kink instability of a current-carrying highly-conducting column in a strong magnetic field is considered in the general case without assuming the linear approximation. It is shown that helical perturbations should grow into "bubbles," i.e., plasma-free helical plaits that penetrate below the surface of the plasma column. The relation between this process and the so-called disruptive instability is discussed.

1. INTRODUCTION

Kink instability of a current-carrying plasma filament in a longitudinal magnetic field was one of the first instabilities to be investigated theoretically[1,2], and its stabilization by a strong longitudinal field is the basis of systems of the Tokamak type[3]. The longitudinal field is made strong enough to reach the so-called Kruskal-Shafranov limit. In other words, the quantity $q = rB_0/RB_\theta$ is chosen to be larger than unity (B_0 is the longitudinal field, B_θ is the azimuthal field, R is the major radius of the torus, and r is the distance from the magnetic axis). Although usually q is chosen with a large margin and the principal mode of the helical instability m = 1 is stabilized, weaker higher modes can develop in the filament, especially on its boundary. In particular, it has been suggested[4,5] that the very unpleasant so-called "disruptive" instability, which is manifest in the form of downward spikes on the loop-voltage curves and corresponds to a small but rapid jump of the self-induction of the plasma filament, meaning its sharp expansion, is due precisely to the development of helical instabilities. An analysis of the helical instability in the quasilinear approximation[6] did not yield voltage spikes of the required sign, at least in the simple variant without allowance for the small toroidality effect. The question whether this is a defect of only the quasilinear approximation or is due to the erroneous idea that the disruptive instability is connected with the helical instability still remains open. It is clear that this question can be answered only after a more detailed investigation of nonlinear helical perturbations, without the use of the quasilinear approximation. The investigation of helical perturbations of large amplitude is of interest also from the point of view of the estimate of the danger of the kink instability, when it is predicted by the linear theory. The present paper is devoted to a study of the evolution of nonlinear helical perturbations of a current-carrying plasma filament in a strong longitudinal magnetic field.

2. FUNDAMENTAL EQUATIONS

As is well known, kink instability is not very sensitive to the plasma pressure (since it is caused by the longitudinal current and not by the plasma pressure) or to the toroidal bending of the filament. We therefore start out immediately with a certain simplified model, namely we consider a straight plasma filament of radius a situated inside an ideally conducting jacket of radius b. We assume the plasma to be an ideally conducting gas with negligibly small gas kinetic pressure. The region between the plasma and the jacket is assumed to be a vacuum. The filament is assumed to be of unlimited length, but it is assumed that all the perturbations have a longitudinal period $L = 2\pi R$, thus imitating a toroidal filament of large radius R. We assume that the longitudinal field B_0 is much stronger than the azimuthal field B_θ produced by the longitudinal current. This is precisely the situation in Tokamaks. As is well known, the greatest interest in the case of helical instability lies in perturbations that vary slowly along the force lines. Consequently it suffices to consider perturbations that vary slowly along the longitudinal coordinate z of the cylindrical coordinate system r, θ, z. In such perturbations the z-component of the velocity is small, so that we can assume approximately that the motion in each cross section is planar, i.e., it suffices to consider only the transverse velocity component $v_\perp$. Using the condition $B_\perp/B_0 \ll 1$, we attempt to simplify the equation of motion of ideal magnetohydrodynamics:

$$\rho\frac{d\mathbf{v}}{dt}=\frac{1}{4\pi}[\operatorname{rot}\mathbf{B}\,\mathbf{B}]=-\nabla_\perp\frac{B^2}{8\pi}+\frac{B^2}{4\pi}(\mathbf{h}\nabla)\mathbf{h}, \qquad (1)^*$$

where $\nabla_\perp = \nabla - \mathbf{h}(\mathbf{h}\cdot\nabla)$, $\mathbf{h} = \mathbf{B}/B$, and ρ is the density.

We assume now that the longitudinal field is almost homogeneous, i.e., its small deviation B_z' from homogeneity is of the order of $B_\perp^2/B_0$. Accordingly, the quantity B^2 in (1) can be expressed in the form $B_0^2 + 2B_0B_z' + B_\perp^2$. We neglecting $B_z'^2$ and retain only quantities of first order in $B_\perp/B_0$; thus, with the same accuracy, we can take $\Delta_\perp$ in (1) to mean simply the gradient in the plane z = const. The quantity h, accurate to small quantities of first order in $B_\perp/B_0$, is equal to $\mathbf{h} = \mathbf{e}_z + \mathbf{B}_\perp/B_0$, and the quantity ∂/∂z should also be regarded as a small quantity of order $(B_\perp/B_0)$. Accordingly, neglecting the small quantities or order $B_\perp/B_0$ and higher and assuming B_0 = const, we rewrite (1) in the form

$$\rho\frac{d\mathbf{v}}{dt}=-\nabla_\perp\left(\frac{2B_0B_z'+B_\perp^2}{8\pi}\right)+\frac{B_0}{4\pi}\frac{\partial}{\partial z}\mathbf{B}_\perp+\frac{1}{4\pi}(\mathbf{B}_\perp\nabla)\mathbf{B}_\perp \qquad (2)$$

It is clearly seen from this equation that the flow v can be regarded as planar, since all the forces in the right-hand side of (2) lie in the plane z = const.

We consider now the equation describing the freezing-in of the magnetic field in the plasma

$$\partial\mathbf{B}/\partial t=\operatorname{rot}[\mathbf{vB}]. \qquad (3)$$

Putting $\mathbf{B} = B_0\mathbf{e}_z + \mathbf{B}_\perp$ and assuming $\mathbf{v} = \mathbf{v}_\perp$, we can transform (3) into

$$\partial\mathbf{B}/\partial t=-B_0\mathbf{e}_z\operatorname{div}\mathbf{v}+B_0\partial\mathbf{v}/\partial z+\operatorname{rot}[\mathbf{vB}_\perp]. \qquad (4)$$

From the z-component of this equation we get

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$$\mathrm{div}\,\mathbf{v} = -B_0^{-1}\partial B_z'/\partial t,$$

but the quantity in the right-hand side is of the order of $B_\perp^2/B_0^2$, i.e., we should neglect it. Consequently, the transverse flow of the plasma is incompressible

$$\mathrm{div}\,\mathbf{v} = 0. \tag{5}$$

Further, it is necessary to neglect the small quantity B_z' in the equation $\mathrm{div}\,\mathbf{B} = \mathrm{div}\,\mathbf{B}_\perp + \partial B_z'/\partial z = 0$, so that we get

$$\mathrm{div}\,\mathbf{B}_\perp = 0. \tag{6}$$

We proceed now to consider the helical flows, i.e., we assume that all the components of the vector quantities v_r, v_θ, B_r, B_θ, and ρ considered by us depend only on the two variables r and $\theta - \alpha z$. In other words, we put $\partial/\partial z = -\alpha\partial'/\partial\theta$, where the prime indicates that we differentiate only the vector components but not the unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_r$. Then Eqs. (2) and (4) become

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla_\perp\left(\frac{2B_0B_z' + B_\perp^2}{8\pi}\right) - \frac{B_0}{4\pi}\alpha\frac{\partial'\mathbf{B}_\perp}{\partial\theta} + \frac{1}{4\pi}(\mathbf{B}_\perp\nabla)\mathbf{B}_\perp, \tag{7}$$

$$\partial\mathbf{B}_\perp/\partial t = -\alpha B_0\partial'\mathbf{v}/\partial\theta + \mathrm{rot}\,[\mathbf{v}\mathbf{B}_\perp]. \tag{8}$$

We introduce the auxiliary transverse field

$$\mathbf{B}_* = \mathbf{B}_\perp - \alpha r B_0\mathbf{e}_\theta. \tag{9}$$

Then, as can be easily verified, Eq. (8) with allowance for $\mathrm{div}\,\mathbf{v} = 0$, can be expressed in the form

$$\partial\mathbf{B}_*/\partial t = \mathrm{rot}\,[\mathbf{v}\mathbf{B}_*], \tag{10}$$

i.e., it takes the form of the condition for the freezing-in of the two-dimensional field $\mathbf{B}_*$ in a plasma that executes planar motion with velocity $\mathbf{v}$. Taking (6) into account, we have

$$\mathrm{div}\,\mathbf{B}_* = 0, \tag{11}$$

from which it follows that we can introduce the stream function

$$\mathbf{B}_* = [\mathbf{e}_z\nabla\psi] \tag{12}$$

and Eq. (10) then takes the form

$$d\psi/dt = \partial\psi/\partial t + \mathbf{v}\nabla\psi = 0, \tag{13}$$

i.e., it shows that the flux ψ is transported together with the plasma. In particular, if we agree to assume that $\psi = 0$ on the unperturbed plasma surface, then this condition remains in force for all helical motions of the plasma.

This same condition $\psi = \mathrm{const}$ on the boundary can be obtained also from the relation $\mathbf{B}\cdot\mathbf{n} = 0$, where $\mathbf{n}$ is the outward normal to the plasma boundary. In fact, taking helical symmetry into account, we have accurate to small first-order quantities

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{\nabla_\perp F + \mathbf{e}_z\partial F/\partial z}{|\nabla_\perp F|} = \mathbf{n}_\perp - \alpha r n_\theta\mathbf{e}_z,$$

where $F = \mathrm{const}$ is the equation of the boundary and $\mathbf{n}_\perp$ is the transverse component of the normal. This yields $\mathbf{n}\cdot\mathbf{B} = \mathbf{n}_\perp\cdot\mathbf{B}_\perp - \alpha r B_0 n_\theta = \mathbf{B}_*\cdot\mathbf{n}_\perp = 0$. In other words, $\mathbf{B}_*$ is tangent to the plasma boundary, and consequently $\psi = \mathrm{const}$ on the boundary.

We turn now to Eq. (7). We note first that

$$\frac{\partial'\mathbf{B}_\perp}{\partial\theta} = \frac{\partial'\mathbf{B}_*}{\partial\theta} = -\frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2}\mathbf{e}_r + \frac{\partial^2\psi}{\partial r\,\partial\theta}\mathbf{e}_\theta.$$

Taking this into account, we can easily verify that Eq. (7) can be reduced to the form

$$\rho\frac{d\mathbf{v}}{dt} + \nabla P = \frac{1}{4\pi}(\mathbf{B}_*\nabla)\mathbf{B}_*, \tag{14}$$

where the quantity

$$P = \frac{1}{8\pi}(2B_0B_z' + B_\perp^2 + 4\alpha^2r^2B_0^2) + \frac{\alpha B_0}{2\pi}\psi$$

assumes the role of pressure.

Thus, the problem of plasma motion has been reduced to that of two-dimensional flow of an ideally conducting incompressible liquid with a frozen-in field $\mathbf{B}_*$. The fundamental equations for the internal part of the filament are (10), (11), (5), and (14).

We proceed now to the vacuum region outside the filament. Here, too, B_z' is negligibly small, so that $\mathrm{div}\,\mathbf{B}_\perp = 0$. Accordingly, outside of the plasma we can also introduce the auxiliary field $\mathbf{B}_*$ and the stream function ψ:

$$\mathbf{B}_* = \mathbf{B}_\perp - \alpha r B_0\mathbf{e}_\theta,\quad \mathbf{B}_* = [\mathbf{e}_z\nabla\psi]. \tag{15}$$

Since in vacuum we have $\mathrm{curl}\,\mathbf{B}_\perp = 0$, it follows that

$$\mathrm{rot}_z\mathbf{B}_* = \Delta\psi = -2\alpha B_0. \tag{16}$$

Since $\mathbf{B}\cdot\mathbf{n} = 0$ on the outer boundary of the plasma, we again have $\psi = \mathrm{const}$ on the plasma boundary. Without loss of generality, we can put $\psi = 0$ on the plasma boundary, in such a way that ψ is continuous on the plasma boundary. On the conducting jacket, $\mathbf{B}$ likewise has no normal component and $\psi = \psi_b = \mathrm{const}$ at $r = b$ (but ψ_b can depend on the time).

We consider two variants of a boundary-value problem for ψ. If we assume that the jacket is ideally conducting and closed, then

a) $$\psi_b(t) = \psi_b(0), \tag{17}$$

i.e., ψ_b on the jacket should retain its initial value at all time, corresponding to a constant magnetic flux on the outside of the plasma.

In Tokamak experiments, however, it is usually the total current flowing in the plasma which is kept constant. Then the magnetic flux in the region between the plasma and the jacket is not constant, it can be introduced from the outside through a cut in the jacket. The condition for the constancy of the current I signifies that

$$\oint \mathbf{B}_{\perp e}\,dl = \frac{4\pi}{c}I = \mathrm{const},$$

where the integral is taken over any contour enclosing the plasma filament. If the contour is taken to be the plasma boundary, then we obtain a boundary condition for the derivative $\partial\psi/\partial n$ with respect to the outer normal to the plasma boundary:

b) $$\oint_p \frac{\partial\psi}{\partial n}dl = \oint_p \mathbf{B}_\perp dl - \oint_p \alpha r B_0\mathbf{e}_\theta dl = \frac{4\pi}{c}I - 2\alpha S_p B_0 = \frac{4\pi}{c}I - 2\pi\alpha a^2 B_0, \tag{18}$$

where S_p is the area of the plasma cross section, equal to πa^2 for all incompressible deformations, and the symbol p under the integral sign denotes integration on the plasma boundary.

The same condition for the contour along the jacket boundary takes the form

b′) $$\oint_b \frac{\partial\psi}{\partial n}dl = \frac{4\pi}{c}I - 2\pi\alpha b^2 B_0. \tag{19}$$

If we put $I = caB_a/2$, then conditions (18) and (19) are expressed in terms of the initial value of the azimuthal field B_a on the filament boundary.

We note that the magnetic flux Φ_θ of the poloidal field outside the filament per unit filament length can vary with time at a constant current I. It can be expressed in terms of the quantity ψ_b, the value of the stream function ψ on the jacket. To this end we consider a surface with helical symmetry, which "partitions off" the space between the plasma and the jacket. If we choose a cylinder of length $L_0 = 2\pi/\alpha$, then on moving along the filament the surface is rotated in azimuth through an angle 2α over this length. We consider now the flux $\Phi^* = \int \mathbf{B}\cdot\mathbf{n}dS$ through this surface. Since $\mathbf{n} = \mathbf{n}_\perp - \alpha r n_\theta \mathbf{e}_z$ on the helical surface, this flux is equal to

$$\Phi^* = L_0 \int \mathbf{B}\mathbf{n}dl = L_0 \int \mathbf{B}.\mathbf{n}_\perp\, dl = L_0\psi_b,$$

where the line integral is taken along a line from the plasma to the jacket in the cross section. On the other hand, $\Phi^* = L_0\Phi_\theta - B_0\pi(b^2 - a^2)$, since the longitudinal magnetic field passes through the helical surface, and the projection of this surface on the plane $z = \text{const}$ is equal to $\pi(b^2 - a^2) = \text{const}$. We thus obtain

$$\Phi_\theta = \psi_b + {}^1/_2\alpha B_0(b^2 - a^2). \tag{20}$$

In order to make the system of equations closed, it remains for us to find the boundary condition for the pressure P on the plasma boundary. Just as in all incompressible motions, this condition can be expressed accurate to a constant. To find P on the boundary, we use the condition that the magnetic pressures be equal on the plasma boundary

$$2B_0B_{zi}' + B_{\perp i}^2 = 2B_0B_{ze}' + B_{\perp e}^2, \tag{21}$$

where the subscripts i and e pertain respectively to the internal and external regions. We recognize that the field outside is potential, so that $B = \nabla\varphi$, where φ is the potential of the magnetic field, which we should regard as a function of r and of $\theta - \alpha z$. Consequently, $B_z' = \partial\varphi/\partial z = -\alpha\partial\varphi/\partial\theta = -\alpha r B_\theta$. Substituting this expression in (21) and expressing $B_{\perp e}^2$ in terms of B_*^2, we obtain

$$2B_0B_{zi}' + B_{\perp i}^2 = -2\alpha r B_0 B_\theta + B_\perp^2 = B_{*e}^2 - \alpha^2 r^2 B_0^2.$$

But since $\psi = 0$ on the plasma boundary, we get in accordance with (14)

$$P = \frac{1}{8\pi}(\nabla\psi_e)^2, \tag{22}$$

where both quantities are taken on the plasma boundary, P inside and $(\nabla\psi_e)^2$ outside the plasma.

Thus, the problem of nonlinear helical perturbations of the plasma has been reduced to a study of two-dimensional motions of an ideally conducting incompressible liquid, described by Eqs. (10), (11), (5), (14), and (16) with boundary conditions (17), (18), and (22) and with the condition $\psi = 0$ on the plasma boundary.

3. ENERGY INTEGRAL

We shall show that we can obtain from this system of equations an expression for the total energy $\mathscr{E}$, which is conserved as the plasma moves. The expressions for $\mathscr{E}$ are somewhat different in the cases $\Phi_\theta = \text{const}$ and $I = \text{const}$. Accordingly, we denote them by $\mathscr{E}_\Phi$ and $\mathscr{E}_I$. Multiplying (14) by $\mathbf{v}$ and (10) by $\mathbf{B}_*/4\pi$ and adding, we obtain

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \frac{B_*^2}{8\pi}\right) - \frac{v^2}{2}\frac{\partial\rho}{\partial t} + \rho(\mathbf{v}\nabla)\frac{v^2}{2} + \operatorname{div}(\mathbf{v}P) - \frac{\mathbf{v}}{4\pi}(\mathbf{B}_*\nabla)\mathbf{B}_* - \mathbf{B}_*\operatorname{rot}[\mathbf{v}\mathbf{B}_*] = 0. \tag{23}$$

We have used here the incompressibility condition, from which it follows that $\partial\rho/\partial t = -\mathbf{v}\cdot\nabla\rho$. Using in addition the conditions $\operatorname{div}\mathbf{v} = 0$ and $\operatorname{div}\mathbf{B}_* = 0$, we can reduce (23) to the form

$$\frac{\partial}{\partial t}\left(\rho\frac{v^2}{2} + \frac{B_*^2}{8\pi}\right) + \operatorname{div}\left\{\rho\mathbf{v}\frac{v^2}{2} + \mathbf{v}P + \mathbf{v}\frac{B_*^2}{8\pi} + \frac{1}{4\pi}\mathbf{B}_*(\mathbf{v}\mathbf{B}_*)\right\} = 0. \tag{24}$$

We consider now the derivative of the integral over the plasma cross section S_i

$$\frac{\partial}{\partial t}\int_{S_i}\left(\rho\frac{v^2}{2} + \frac{B_*^2}{8\pi}\right)dS = \oint_p v_n\left(\rho\frac{v^2}{2} + \frac{B_{*i}^2}{8\pi}\right)dl - \oint_p v_n\left(\rho\frac{v^2}{2} + P + \frac{B_{*i}^2}{8\pi}\right)dl,$$

where the first integral on the right-hand side is due to the differentiation of the integration limits, while the second is obtained with the aid of (24). We see that

$$\frac{\partial}{\partial t}\int_{S_i}\left(\rho\frac{v^2}{2} + \frac{B_*^2}{8\pi}\right)dS = -\oint_p Pv_n\,dl = -\oint_p\frac{B_{*e}^2}{8\pi}v_n\,dl, \tag{25}$$

which is perfectly natural, for the term on the right-hand side corresponds to the work performed by the plasma. Since according to (22) we have

$$P = \frac{1}{8\pi}(\nabla\psi_e)^2 = \frac{1}{8\pi}B_{*e}^2$$

on the plasma boundary, this work can be expressed in terms of the change of the field energy in the vacuum outside the plasma. To this end we transform the quantity

$$\frac{\partial}{\partial t}\int_{S_e}B_{*e}^2dS = -\oint_p B_{*e}^2v_n dl + 2\int_{S_e}\nabla\psi\,\nabla\dot\psi dS, \tag{26}$$

where $\dot\psi = \partial\psi/\partial t$ and the first term on the right is due to the differentiation of the integration region.

We consider the second integral in (26) for the case $\Phi_\theta = \text{const}$, i.e., $\psi_b = \text{const}$. We take into account relations (16) and (17) as well as the condition $\psi = 0$ on the plasma boundary, from which it follows that $(d\psi/dt)_p = (\partial\psi/\partial t + v_n\partial\psi/\partial n)_p = 0$. Recognizing, in addition, that on the plasma boundary the normal $\mathbf{n}$, which is directed out of the plasma, is the inward normal with respect to the outer region, we have

$$\int_{S_e}\nabla\psi\,\nabla\dot\psi\,dS = \oint_b\dot\psi\frac{\partial\psi}{\partial n}dl - \oint_p\dot\psi\frac{\partial\psi}{\partial n}dl - \int_{S_e}\Delta\psi\dot\psi\,dS = \oint_p v_n\left(\frac{\partial\psi}{\partial n}\right)^2dl + 2\alpha B_0\frac{\partial}{\partial t}\int_{S_e}\psi dS. \tag{27}$$

Substituting the resultant expression in (26) and using (25) as well as the condition $(\partial\psi/\partial n)^2 = (\nabla\psi)^2$ on the plasma boundary, we obtain the energy conservation law $\partial\mathscr{E}_\Phi/\partial t = 0$, where

$$\mathscr{E}_\Phi = \int_{S_i}\rho\frac{v^2}{2}dS + \int_{S_i+S_e}\frac{B_*^2}{8\pi}dS + \frac{\alpha B_0}{2\pi}\int_{S_e}\psi dS. \tag{28}$$

The last two terms in this expression play the role of the potential energy.

For the case $I = \text{const}$, the expression for the energy is even simpler. Namely, integrating by parts and then taking into account the boundary condition (19), we obtain

$$\int_{S_e}\nabla\psi\,\nabla\dot\psi\,dS = \oint_b\psi\frac{\partial\dot\psi}{\partial n}dl - \oint_{S_e}\psi\Delta\dot\psi\,dS = \psi_b\frac{\partial}{\partial t}\oint_b\frac{\partial\psi}{\partial n}dl = 0. \tag{29}$$

Accordingly, we obtain with the aid of (26) and (25)

$$\mathscr{E}_I = \int_{S_i}\left(\rho\frac{v^2}{2} + \frac{B_*^2}{8\pi}\right)dS - \int_{S_e}\frac{B_*^2}{8\pi}dS. \tag{30}$$

It can be readily shown that (28) coincides, apart from a constant, with the quantity

$$\mathscr{E}_\Phi' = \int_{S_i} \rho \frac{v^2}{2} dS + \int_{S_i+S_e} \frac{B_\perp^2}{8\pi} dS, \tag{31}$$

which is to be expected at constant B_0 and at a frozen-in azimuthal magnetic field. In fact, taking (15) into account, we can replace B_*^2 in (28) by

$$B_*^2 = B_\perp^2 - 2\alpha r B_0 \frac{\partial\psi}{\partial r} - \alpha^2 r^2 B_0^2 = B_\perp^2 - 2\alpha B_0 \operatorname{div}(\mathbf{r}\psi) + 4\alpha B_0 \psi - \alpha^2 r^2 B_0^2. \tag{31'}$$

The integral of the second term of (31′) then reduces to a surface integral and is independent of the time if $\psi_b = \text{const}$, the integral of the last term is simply equal to a constant, and the integral of the third term over the outer region cancels out the last term in (28). The quantity $\int_{S_i} \psi\, dS$, as can be easily shown with the aid of (13), is independent of the time. Thus, $\mathscr{E}_\Phi = \mathscr{E}_\Phi' + \text{const}$.

As to expression (30) for $\mathscr{E}_I$, it is not so simply connected with the energy of the external and internal magnetic fields, for at constant I the energy of the outer circuit also comes into play and leads, in particular, to a negative sign of the integral of $B_*^2/8\pi$ over the outer region in expression (30).

4. "BUBBLES" IN A PLASMA

In the expressions (28), (30), and (31) for the energy, we can easily separate the terms corresponding to the potential energy, which we shall designate W. Obviously, our system will evolve in a direction of minimum W. If this minimum is reached not in the fundamental cylindrical symmetrical state, then the plasma in the initial state will be unstable either to linear perturbations if W has a maximum in the initial state, or to finite-amplitude perturbations if the absolute minimum of W is separated from the local minimum of the linearly stable initial state by a potential barrier. In any case, it is desirable above all to know the absolute minimum of W, since the equilibrium near this minimum should certainly be stable.

We start the analysis with the case of constant I, i.e., we use the expression

$$8\pi W_I = \int_{S_i} B_*^2\, dS - \int_{S_e} B_*^2\, dS. \tag{32}$$

In our approximation, i.e., neglecting small terms of order B^2/B_0^2, the deformation of the magnetic field inside the plasma is always connected with an increase of the energy. It is therefore desirable to start the analysis from the most favorable case, when B_* inside the plasma is equal to zero. This means that the current inside the plasma is homogneous, and the magnetic field is

$$B_\theta = \alpha r B_0. \tag{33}$$

A linear distribution of the field corresponds to a margin factor $q = 1/\alpha R$ which is constant along the radius. If the perturbations are m-th order symmetry with respect to the angle θ, then all the functions should be periodic with period 2π with respect to the variable $m(\theta - \alpha z) = m\theta - m\alpha z$. On the other hand, since we deal with a toroidal filament with large radius R, we should regard the perturbations as periodic in z with period $2\pi R$, so that $m\alpha = l/R$, where l is an integer. Thus,

$$\alpha = l/mR \tag{34}$$

and consequently the condition (33) is satisfied when

$$q = m/l. \tag{35}$$

It is easily seen that under this condition we are dealing with flute perturbations of the plasma; these perturbations are constant along the force lines, and all the force lines have one and the same pitch along the filament.

Thus, under conditions (33)–(35) the potential energy is determined only by the vacuum region outside the plasma. We need to find the minimum of this expression under incompressible deformations of the plasma and under the conditions $\Delta\psi = -2\alpha B_0$, $\psi_b = \text{const}$, and $\psi_p = 0$.

Obviously, equilibrium (and a stable one at that) should obtain at the minimum of W. But P is constant at equilibrium, i.e., according to the boundary conditions the quantity $(\nabla\psi)^2$ should be constant outside the plasma. The simplest configuration of this type is realized when the outer boundary of the plasma is a cylinder that is concentric with the jacket. We denote the radius of this cylinder by a_*. If a_* is larger than the initial plasma radius a, then vacuum helical "braids," which look like "bubbles" in cross section, must of necessity exist inside the plasma. Let us examine the properties of these bubbles. Let the bubble be a circle of radius ρ_0. Since $\Delta\psi = -2\alpha B_0$ inside the bubble and $\psi = 0$ on its boundary, we have

$$\psi = {}^1/_2 \alpha B_0 (\rho_0^2 - \rho^2). \tag{36}$$

Let us find the bubble energy W_{ρ_0}. Recognizing that $(\nabla\psi)^2 = \operatorname{div}(\psi\nabla\psi) + 2\alpha B_0\psi$ and $\psi = 0$ the plasma boundary, we obtain

$$W_{\rho_0} = -\frac{1}{8\pi}\int (\nabla\psi)^2\, dS = -\frac{\alpha B_0}{4\pi}\int \psi\, dS = -\frac{\alpha^2 B_0^2}{16}\rho_0^4. \tag{37}$$

We see therefore that the bubble energy is the same for constant I and constant Φ_θ.

It is easy to see that when the shape of a bubble of given cross section $S_{\rho_0} = \pi\rho_0^2$ changes, the value of ψ decreases and the bubble energy increases. Consequently, each bubble should be circular. From the same considerations it follows that when the bubbles coalesce the energy decreases, i.e., the number of bubbles should be minimal. At a symmetry of order m, the minimum number of bubbles is equal either to m if the bubbles exist separately, or to unity if they coalesce. Since the total area of N bubbles is equal to $N\pi\rho_0^2 = \pi(a_*^2 - a^2)$, it follows that their energy is equal to

$$W_N = N W_{\rho_0} = -\frac{\alpha^2 B_0^2}{16N}(a_*^2 - a^2)^2. \tag{38}$$

When the bubbles coalesce into one, this energy decreases by a factor N. In our approximation, the bubbles do not interact with one another at all prior to coalescence, and can be arbitrarily disposed within the limits of the symmetry of order m. However, inclusion of the terms of next order of smallness should apparently lead to attraction of the bubbles to one another. After coalescence, provided N was not equal to unity from the very outset, the single bubble should be located on the axis of the filament, so that the plasma cross section becomes annular.

It follows from these considerations that the minimum of the potential energy is reached for a tubular plasma configuration with outside radius a_* and inside radius $\rho_0 = (a_*^2 - a^2)^{1/2}$, where a_* is determined from the minimum-energy condition. For the sake of greater generality, however, we carry out the analysis for N bubbles and put N = 1 where necessary in the final formulas.

Let the initial azimuthal magnetic field on the filament boundary be $B_a = \xi\alpha a B_0$. If $\xi = 1$, then the field goes over continuously into the field inside the plasma, and consequently we have on the surface $q = m/l$, i.e., the small perturbations are flute-like also with respect to external perturbations. It is known that in this case the filament has neutral stability, i.e., the perturbation growth increment vanishes. This is easily seen from the equations obtained by us: since the magnetic field $B_* = aB_a/r - \alpha r B_0$ vanishes on the boundary of the unperturbed plasma at $\xi = 1$, the work of the external forces in (25) vanishes in the approximation quadratic in the perturbation if the perturbations are small. At $\xi > 1$ the value of q outside the plasma decreases, $q = m/l\xi < m/l$, and the plasma becomes unstable in the linear approximation (if ξ is not very large). This follows quite naturally also from our expressions, since the external "pressure" $B_*^2/8\pi$ decreases away from the plasma boundary. To the contrary, at $\xi < 1$, i.e., $q = m/l\xi > m/l$, the external pressure increases away from the plasma boundary, and the plasma is stable in the linear approximation. We are interested primarily, however, not in linear perturbations but in strongly nonlinear perturbations that lead to formation of bubbles in a plasma.

At constant I the magnetic field B_* outside a filament with cylindrical external surface retains its initial value, so that W is obviously determined by the relation

$$8\pi W = -\int_{a_*}^{b} B_*^2 2\pi r\,dr = -\frac{\pi\alpha^2 B_0^2}{2N}(a_*^2 - a^2)^2, \tag{39}$$

where

$$B_* = \alpha B_0 a(\xi a/r - r/a). \tag{40}$$

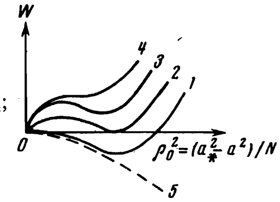
We introduce the quantity $P' = -8\partial W/\partial a_*^2$, which characterizes, as it were, the effective "pressure" of the bubbles: they tend to expand at $P' > 0$ and to contract at $P' < 0$. From (39) we get

$$P' = \alpha^2 a^2 B_0^2\left\{\frac{1}{N}\left(\frac{a_*^2}{a^2} - 1\right) - \left(\frac{\xi a}{a_*} - \frac{a_*}{a}\right)^2\right\}. \tag{41}$$

At $a_* = a$ we have $P' = -\alpha^2 a^2 B_0^2(\xi - 1)^2 \leq 0$, i.e., at $\xi \neq 1$ there is an energy barrier to bubble formation, and consequently bubbles can develop only from some other perturbations, for example from linear perturbations in the region of linear instability.

The case of one bubble (N = 1) is somewhat special, since the terms quadratic in a_* cancel in (41), which consequently becomes proportional to $(2\xi = -1 - \xi^2 a^2/a_*^2)$. We see therefore that if $\xi > 1/2$, the radius a_* increases without limit after passage through the barrier, and there is no second stable equilibrium state.

At $N \neq 1$, the force P' becomes negative at sufficiently large a_*, i.e., large bubbles should contract. The dependence of W on a_*^2 for $N \neq 1$ is shown qualitatively in the figure. The potential energy has a minimum that lies certainly below the initial energy W_0 at $\xi = 1$, when there is no potential barrier. When $\xi \neq 1$, a barrier appears, and at the same time the value of W at the minimum can change. The equilibrium value of



Potential energy W of plasma filament as a function of the size of the bubbles: 1) at $\xi = 1$; 2) $\xi = \xi_c$; 3) $\xi_c < \xi < \xi_e$; 4) $\xi = \xi_e$; 5) energy in the case of a single bubble; ρ_0 is the bubble redius and N is the number of bubbles.

a_* corresponding to the minimum of W is determined from the conditions $P' = 0$ and $\partial P'/\partial a_* < 0$, and is equal to

$$a_*^2/a^2 = 1 + \{2\xi - 1 + [1 + 4N\xi(\xi - 1)]^{1/2}\}/2(N-1). \tag{42}$$

Expression (42) becomes meaningless at $1 + 4N\xi(\xi - 1) < 0$. The minimum on the plot of the potential energy against a_* then vanishes, as seen in the figure. The limiting value of ξ for which a minimum still exists will be designated ξ_e:

$$\xi_e = (1 + \sqrt{1 - 1/N})/2. \tag{43}$$

Even if the minimum does exist, it is of interest to determine under what conditions it is absolute, i.e., it lies below the energy in the initial state. The critical value of ξ at which the minimum coincides with W_0 will be designated ξ_c. It can be easily obtained with the aid of (39), but in the general case the expression for ξ_c is very cumbersome, and we present only the asymptotic expression:

$$\xi_c = 1 + (3 \pm 2\sqrt{3})/2N, \quad a_*/a = 1 + 3(2 \pm \sqrt{3})/4N, \quad N \gg 1. \tag{44}$$

We note also that in the linear approximation the filament considered by us is unstable according to[1], when $(a/b)^m \ll 1$, in the interval

$$1 < \xi < (m+1)/(m-1). \tag{45}$$

Comparing expressions (44) and (45) at large N = m, we easily see that the region of linear stability is narrower than the region in which the presence of m bubbles is energywise favored. This means that in the region of linear stability, but at $\xi_{c_1} < \xi < \xi_{c_2}$, the so-called "hard" onset of bubbles can take place, which is connected with the passage of the bubbles through the energy barrier. The region $\xi > \xi_e$, where W has a minimum and the bubbles are in the metastable state, is still much broader.

We note that according to (42) and (44), the equilibrium value of the bubble radius $\rho_0 = [(a_*^2 - a^2)/N]^{1/2}$ turns out to be of the order of a/m at $N = m$ for values of ξ where the potential energy is minimal, i.e., of the order of the transverse wavelength of the perturbation of the mode m. Accordingly, perturbations with sufficiently large amplitudes can "grow" in the case of nonlinear instability into bubbles of optimal radius ρ_0, so that the region of the potential barrier in the figure, corresponding to very small bubbles, can be easily jumped through in the case of real perturbations with transverse wavelength $\sim a/m$.

We determine now the change of the azimuthal magnetic flux Φ_θ per unit filament length following formation of a bubble. According to (20), we have $\delta\Phi_\theta = \delta\psi_b$. But the value of the stream function on the jacket is altered by the fact that when the bubble is formed the function ψ at the point a_* decreases to zero, i.e., $\delta\psi_b = -\psi(a_*)$. Consequently, taking (42) into account, we easily obtain

$$\delta\Phi_\theta = -\alpha a^2 B_0\left\{\Delta\xi - \frac{1 + (1 + 4m\Delta\xi)^{1/2}}{4m}\right\}\frac{1 + (1 + 4m\Delta\xi)^{1/2}}{4m} \tag{46}$$

where $\Delta\xi = \xi - 1$. We see that $\delta\Phi_\theta$ reverses sign at $\Delta\xi = 3/4m$, i.e., approximately at the midpoint of the linear-instability region $0 < \Delta\xi < 2/m$. At $\Delta\xi > 3/4m$ the formation of bubbles leads to a decrease of $\delta\Phi_\theta$, i.e., to a crowding out of a certain fraction of the magnetic flux outside the jacket, which should correspond to negative spikes on the loop voltage. At $\Delta\xi < 3/4m$, the bubble formation should be accompanied by capture of a fraction of the poloidal flux into the bubbles.

We have considered so far the case of constant I. It turns out that at constant Φ_θ the picture remains approximately the same, but the plasma stability increases somewhat. In fact, the bubble energy (39), as can be easily verified, does not depend on whether expression (28) or (30) is used for the energy. The energy (28) outside the filament differs from (30). But if the derivative of W with respect to a_* is calculated, recognizing that the expression for B_* outside the filament can be expressed in the form (40), where ξ should be regarded as a function of a_* determined from the condition ψ_b = const, we obtain exactly the expression (41), where

$$\xi(a_*) = \left\{ \xi_0 \ln\frac{b}{a} - \frac{1}{2}\left(\frac{a_*^2}{a^2} - 1\right)\right\}\left(\ln\frac{b}{a_*}\right)^{-1}, \qquad \xi_0 = \xi(a).$$

The dependence of ξ on a_* leads to an increase of the energy $W(a_*)$, so that the bubble dimensions decrease somewhat. In addition, at constant Φ_θ there appears a minimum in the potential energy of the filament with one bubble, i.e., N = 1. Accordingly, we can obtain the equilibrium stable value of a_* of a hollow filament. If the jacket is close to the filament, $b - a \ll a$, then the stable value of a_* is given by the relation

$$b - a_* = (b - a)^{3/2} / \sqrt{2a}. \qquad (47)$$

Thus, at constant Φ_θ the mode m = 1 is not catastrophic, leads only to a certain expansion of the filament, and brings its boundary closer to the jacket without the two touching.

We now forgo the limitation $B_* = 0$ inside the plasma. Bubble formation should then be accompanied by a perturbation of the magnetic field inside the plasma, and accordingly by an additional growth of the energy. Near the limit of the helical instability, however, when B_* is close to zero on the plasma boundary, bubble formation under the surface of the plasma continues to be energywise favored. In fact, the change of the energy of N bubbles following variation of the radius ρ_0 can be represented in the form $8\pi\delta W = (B_{*e}^2 + AB_{*i}^2 - B_{*\rho}^2)2\pi a_* da_*$, where B_{*e} is the field outside the filament on its boundary, B_{*i} is the field inside the plasma in the region where the bubbles are located, A is a numerical factor of the order of unity and depends on the shape of the bubble, and $B_{*\rho}$ is a certain effective field of the bubble, given by $B_{*\rho}^2 = \alpha^2 a^2 B_0^2 (a_*^2/a^2 - 1)N^{-1}$. The value of $B_{*\rho}^2$ is proportional to the bubble area, so that sufficiently large bubbles are energywise favored. Addition of the quantity $B_{*i}^2 \sim B_{*e}^2$ does not change the qualitative picture of bubble formation, and only leads to a certain increase of the barrier for their production, and prevents the bubbles from moving freely through the plasma.

5. FORMATION AND EVOLUTION OF THE BUBBLES

In our idealized scheme with a homogeneous current and a sharp plasma-filament boundary, the range of ξ in which the bubbles are energywise favored is wider than the region of linear instability. Outside the stability region, however, bubble formation calls for overcoming a certain potential barrier, so that bubble formation in the stability region is not very probable if the plasma is quiescent. In the instability region, i.e., for $m \gg 1$ at $0 < \Delta\xi < 2/m$, bubbles can arise from linear perturbations that increase in time, if no potential barrier is again encountered in their path, i.e., if no equilibrium with a wavy helical surface can be produced. Near the limits of the instability interval, the equilibrium with helical symmetry can be sought by expansion in the small perturbation amplitude. Such an investigation was carried out in[6]. It has shown that near the right-hand stability boundary, $\Delta\xi = 2/m$, when the jacket is far away, $(a/b)^m \ll 1$, the higher-order terms correspond to a stabilizing but very weak effect, so that equilibrium exists only very close to the instability boundary. On the other hand, if the jacket is located nearby, then hard excitation takes place, and the terms of higher order of smallness lead to a faster growth of the perturbations. Near the left-hand boundary, $\Delta\xi = 0$, under the condition that $B_* = 0$ inside the plasma, there is no equilibrium with wavy boundary, which likewise corresponds to "hard" excitation of helical perturbations.

Thus, within the framework of the model with homogeneous current, bubble formation can be expected in the entire region of linear stability, and can be accompanied in this case by either capture or expulsion of the azimuthal flux. However, if the approach to the instability region is connected with contraction of the plasma filament and a decrease of q on the filament boundary, then the bubbles should be produced near the boundary $\xi = 1$, where their formation is connected with capture of the azimuthal flux. Moreover, were we to assume a more realistic model with low current density on the plasma boundary, then the instability region would be narrowed down as a result of the approach of its right-hand ξ boundary to the value $\xi = 1$, i.e., the entire linear-instability region would correspond to positive spikes of the loop voltage when bubbles are formed.

In experiment, however, only negative spikes are observed. It must therefore be concluded that the negative spikes are not connected directly with bubble formation. However, the fact that rapid expansion of the plasma filament does accompany the spike in experiment, or more accurately, takes place ahead of the spike itself, can be very naturally attributed to the formation of large-amplitude helical perturbations such as bubbles. As to the spike itself, it can be due to the toroidal character of the filament, i.e., to the displacement of the filament along the major radius due to the change of the equilibrium conditions when the minor radius is increased. More natural, however, is another mechanism, connected with the filament touching the walls (or a diaphragm).

In fact, we note that when a bubble is formed, say when $\xi = 1$ on the original filament boundary, the quantity $\partial\psi/\partial n$ becomes different from zero on the plasma boundary, namely positive in the bubble and negative on the outer boundary of the filament. If $B_* = 0$ under the plasma surface, this means the appearance of a longitudinal surface current, which is positive on the bubble boundary and negative on the outer boundary of the plasma. If a plasma filament with negative surface current touches a diaphragm, then this negative surface current should "drop" because of the abrupt cooling of the edge of the filament and the rapid decrease of its conductivity. But the vanishing of a negative current is equivalent to transfer of a positive current from the plasma to the edge of the filament, i.e., to a decrease of the inductance and to ejection of a fraction of the magnetic azimuthal flux to the region outside the jacket. From this point of view, a negative spike should occur at the instant when the filament touches the diaphragm, in agreement with the experimental data. As to the process of formation of helical perturbations, in a real experiment it apparently proceeds more smoothly than in our model with an abrupt filament boundary. This may be due to the effect of finite conductivity, which produces an admixture of the Thirring mode, and also owing to the presence of a field $B_* \neq 0$ inside the plasma, which prevents the bubble from collapsing completely and from becoming separated from the vacuum region. It is possible that the experimentally observed growing helical perturbations ahead of the spike are indeed bubbles that penetrate from under the plasma surface.

The slow evolution of the already produced bubbles should be determined by current-redistribution processes resulting from the finite conductivity. When the finite conductivity is taken into account, the equation for ψ takes the form [7]

$$\frac{4\pi\sigma}{c^2}\left(\frac{\partial\psi}{\partial t}+\mathbf{v}\nabla\psi\right)=\Delta\psi+2\alpha B_0-\frac{4\pi\sigma E_0}{c}, \qquad (48)$$

where E_0 is the longitudinal electric field and σ is the conductivity.

When very slow evolution is considered, the velocity $\mathbf{v}$ in (48) should be determined from the equilibrium condition, which according to (14) can be easily shown to reduce to the equation

$$\Delta\psi=F(\psi), \qquad (49)$$

where F is an arbitray function of ψ.

Let us consider again the very simplest case when we have in the initial state $B_* = 0$ inside the plasma and σ is constant, i.e., $E_0 = \alpha B_0 c/2\sigma\pi$. Assume that development of helical instability in the filament has caused a bubble to be produced under the filament surface. Neglecting the slow diffusion of the plasma, to the interior of the bubble, the bubble evolution at a skin-effect time can be described by Eqs. (48) and (49). The former can be interpreted as the equation of thermal conductivity for a certain "temperature" ψ. At $\sigma \neq \infty$, as seen from (48), it is necessary to take into account the finite "specific heat" $4\pi\sigma/c^2$. In vacuum, the equation retains its form (27) and, as noted above, the quantity $\partial\psi/\partial n$ has opposite signs on the plasma boundary in the bubble and on the external boundary. Accordingly, the bubble will "heat" the plasma, and the outer boundary will "cool" the plasma, i.e., the fluxes from these regions have opposite signs. Less formally, we can state that surface currents having opposite signs on the bubble and on the outer boundary will diffuse to the interior of the plasma. But currents flowing in opposite directions repel each other, so that the bubble should start to move away from the plasma boundary into the interior of the filament. At $B_* = 0$, the penetration of the bubble to the interior of the plasma should be sufficiently rapid, and at $B_* \neq 0$ it proceeds first at the skin-effect rate, and then, when the surface current becomes completely smeared out, the bubble motion can slow down.

All this can of course be obtained also purely formally from (48) and (49), from which it follows that $d\psi/dt$ is a function of ψ only. Consequently, the contour ψ = const will move with the plasma in such a way that ψ along the contour remains constant but dependent on the time. On the bubble boundary, ψ is likewise only a function of the time, first increasing with time and then assuming a certain limiting equilibrium value.

Thus, the bubble exhibits no tendency to be pushed out of the plasma. The repeating process of helical-instability development and penetration of bubbles into the plasma can lead to an increased plasma diffusion from the peripheral sections of the filament.

6. CONCLUSION

We have thus shown that if we consider the helical instability of a current-carrying plasma filament in a strong longitudinal magnetic field, without confining ourselves to a linear approximation, then the penetration of helical cavities to the interior of the plasma becomes energywise favored. These cavities have bubble-like cross sections. The number of bubbles in the cross section coincide with the number m of the helical-stability mode. Most favored energywise are configurations with a single cylindrical cavity inside the filament, where all the bubbles coalesce into one (or if the mode m = 1 develops from the very outset). At a given current, there is no energy minimum in such a filament at all, and the filament expands all the way to the wall. If a jacket without a transverse joint is used, such a minimum does appear, and the filament can stop growing without reaching the jacket even when the Kruskal-Shafranov limit is reached. At q > 1, when higher modes develop, bubbles of radius $\rho_0 \sim a/m$ should develop (a is the filament radius and m is the number of the mode). If the helical perturbation had initially a pitch equal to the pitch of the force lines on the plasma boundary, then as the bubble dimensions increase the bubble gathers in its interior the force lines that are farther from the boundary, with smaller pitch, and thus this leads to a drawing in of a certain fraction of the azimuthal flux from the vacuum region into the bubbles. A surface negative current then appears on the outer boundary of the plasma. When the filament touches the diaphragm, this surface current vanishes, owing to the decrease of the conductivity, and a negative spike should appear on the plot of the loop voltage. It seems to us that it is precisely these processes which occur in the case of the so-called disruptive instability of the plasma, although under real conditions they become somewhat more complicated because of the toroidal character of the filament and its finite conductivity.

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*[rot BB] ≡ curl B × B.

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