High intensity circularly polarized waves in nonlinear dispersive media

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A class of exact solutions is found for the equations describing propagation of circularly polarized waves in a dispersive isotropic nonlinear medium. It is shown that the solutions include waves that are localized pulses, their duration being comparable to the rotation period of the field vector, and the form of each field projection changes continuously. Electromagnetic waves in a solid-state plasma are considered by way of example.

Exact solutions of the problem of wave propagation in nonlinear dispersive media can be obtained only in a few quite special cases. These include stationary waves propagating without change of waveform[1], certain self-similar solutions[2], and individual cases when the solution of the individual problem can be reduced to a solution of a linear boundary-value problem[6]. Usually, however, nonlinear waves are investigated by approximate methods based on the fact that the waveform is close to stationary (in particular, harmonic) and that its parameters vary slowly.

In the present paper we obtain one more class of exact solutions of the vector nonlinear partial differential equation by using for the dependent variables a transformation that reduces the initial autonomous solutions that do not depend explicitly on z and t) to other, likewise antonomous equations. For the latter it is possible to use, in particular, already known methods of finding the solutions (for example, stationary ones), which correspond in terms of the initial variables to certain new (essentially nonstationary) solutions. With the aid of such a transformation, we consider the propagation of circularly polarized waves in a solid-state plasma with a non-parabolic conduction band.

1. We consider a nonlinear isotropic dispersive medium described by a solution in the form

$$\sum_{m,n} f_{mn}(\{s_{ijkl}\})\frac{\partial^{m+n}\mathbf{E}_\perp}{\partial z^m\,\partial t^n}=0. \qquad (1)$$

Here f_{mn} are arbitrary functions of the aggregate of the scalar products

$$s_{ijkl}=\left(\frac{\partial^{i+j}\mathbf{E}_\perp}{\partial z^i\,\partial t^j}\frac{\partial^{k+l}\mathbf{E}_\perp}{\partial z^k\,\partial t^l}\right);$$

$\mathbf{E}_\perp$ is the sought vector, which lies in the plane (x, y); i, j, k, l, m, n = 0, 1, 2, 3 . . . In the investigation of circularly polarized solutions of equation (1) it is convenient to introduce the vectors $\mathbf{a}_+$ and $\mathbf{a}_-$, which rotate in the same direction

$$\mathbf{a}_+=\mathbf{e}e^{i\psi}+\mathbf{e}^*e^{-i\psi},\qquad \mathbf{a}_-=i(\mathbf{e}e^{i\psi}-\mathbf{e}^*e^{-i\psi})$$

and to make the substitution

$$\mathbf{E}_\perp=A\mathbf{a}_+, \qquad (2)$$

where the real functions A(z, t) and ψ(z, t) represent new depent variables that have the formal meaning of the amplitude and phase, and e is a constant complex vector, such that $(\mathbf{e}\cdot\mathbf{e})=(\mathbf{e}^*\cdot\mathbf{e}^*)=0$ and $(\mathbf{e}\cdot\mathbf{e}^*)=1/2$. e = (1/2; i/2; 0) for right-hand polarized waves and e = (1/2; −i/2; 0) for left-hand waves.

We substitute (2) in (1) and take into account the fact that the vectors $\mathbf{a}_+$ and $\mathbf{a}_-$ satisfy the relations

$$\frac{\partial\mathbf{a}_\pm}{\partial z}=\pm\mathbf{a}_\mp\frac{\partial\psi}{\partial z},\quad \frac{\partial\mathbf{a}_\pm}{\partial t}=\pm\mathbf{a}_\mp\frac{\partial\psi}{\partial t},\quad (\mathbf{a}_+\mathbf{a}_+)=(\mathbf{a}_-\mathbf{a}_-)=1,\quad (\mathbf{a}_+\mathbf{a}_-)=0,$$

and therefore none of the scalar products s_{ijkl} contain $e^{\pm i\psi}$. This substitution reduces (1) to the form

$$\Phi_1\mathbf{a}_+ + \Phi_2\mathbf{a}_-=0, \qquad (3)$$

where Φ_1 and Φ_2 are certain operators that do not contain explicitly the phase variable ψ, but only its derivatives. By virtue of the orthogonality of the vectors $\mathbf{a}_+$ and $\mathbf{a}_-$, expression (3) breaks up into two equations

$$\Phi_1=0,\qquad \Phi_2=0. \qquad (4)$$

Recognizing that Φ_1 and Φ_2 depend only on the derivatives of ψ, we change from the variable ψ to a new variable φ, connected with ψ by the relation

$$\psi(z,t)=\omega t+kz+\varphi(z,t), \qquad (5)$$

where ω and k are arbitrary constants. The transformation (5), which depends explicitly on the time and on the coordinate, leaves the system of equations (4) autonomous. We note that at A = const and φ = const we obtain from (4) a solution in the form of a harmonic stationary wave; such a solution is known, for example, for transverse waves in a relativistic plasma[4,5]. What is considered here is an essentially more general class of waves.

2. We examine now the propagation of nonlinear circularly-polarized electromagnetic waves in a solid-state plasma. It is well known that in a number of semiconductors the electron dispersion in the band is not parabolic. It can be shown that if damping is neglected the equation describing the propagation of transverse waves in a semiconductor with isotropic and non-parabolic conduction band can be written in the form of the vector equation

$$\frac{\partial^2\mathbf{p}}{\partial\xi^2}-\frac{\partial^2\mathbf{p}}{\partial\tau^2}=f(p^2)\mathbf{p}. \qquad (6)$$

Here p is the dimensionless quasimomentum, ξ and τ are the dimensionless coordinate and time, and $f(p^2)$ is determined by the band structure of the semiconductor.

Substituting the vector p in the form (2) in (6) with allowance for (5), and performing the transformations described above, we obtain new autonomous equations corresponding to Eqs. (4) and equivalent to Eq. (6) in the class of circularly polarized waves:

$$A_{\xi\xi}-A_{\tau\tau}=A[(\varkappa+\varphi_\xi)^2-(\Omega+\varphi_\tau)^2+f(A^2)], \qquad (7)$$

$$A(\varphi_{\xi\xi}-\varphi_{\tau\tau})=2[A_\tau(\Omega+\varphi_\tau)-A_\xi(\varkappa+\varphi_\xi)], \qquad (8)$$

where Ω and κ are the values of ω and k in terms of the dimensionless variables. Equations (7) and (8) are similar in form to the approximate equations obtained for the envelopes of quasiharmonic waves in a medium with small nonlinearity with the aid of the method of averaging and recently considered a number of times[6]. Since, however, no approximations whatever are used in the derivation of (7) and (8), the latter are valid for arbitrarily rapid variations of the "envelopes" of A and φ and for arbitrarily strong nonlinearity.

We consider for (7) and (8) stationary solutions that depend on one variable $\theta = \xi + v\tau$, where $v = \text{const}$. Equation (8) can be integrated:

$$(1-v^2)\varphi_\theta A^2 + (\Omega v - \varkappa)A^2 = J. \tag{9}$$

Here J is the integration constant. Equation (7) reduces to the equation of the nonlinear oscillator

$$A_{\theta\theta} - A\left[\frac{(1-\varkappa v/\Omega)^2}{(1-v^2)^2} - \frac{f(A^2)}{\Omega^2(1-v^2)} - \frac{J^2}{A^4(1-v^2)}\right] = 0, \tag{10}$$

which can be reduced to quadratures or investigated on the phase plane by standard methods.

To specify the concrete form of the function $f(p^2)$, we consider a semiconductor with a Kane-type dispersion[7]:

$$\varepsilon(p') = {}^1\!/_2\varepsilon_g[(1+2p'^2/m^*\varepsilon_g)^{1/2} - 1], \tag{11}$$

where $\epsilon(p')$ is the energy of a conduction-band electron having a quasimomentum p', m^* is the effective mass of the electron at the bottom of the conduction band, and ϵ_g is the width of the forbidden band. If we neglect damping, then in the drift approximation[8] the connection between the current density and the field is given by the expressions

$$\mathbf{j} = -\frac{en}{m^*}\frac{\mathbf{p}'}{(1+2\mathbf{p}'^2/m^*\varepsilon_g)^{1/2}}, \quad \frac{\partial \mathbf{p}'}{\partial t} = -e\mathbf{E}, \tag{12}$$

where n is the electron concentration. Using this connection, we can easily obtain from Maxwell's equations, for transverse electromagnetic waves, an equation of the type (6) with $f(p^2) = 1/\sqrt{1+p^2}$, where $\xi = \omega_0\sqrt{\epsilon_0}z/c$, $\tau = \omega_0 t$, $p = (2/m^*\epsilon_g)^{1/2}p'$ are the dimensionless coordinate, time, and momentum, ϵ_0 is the dielectric constant of the lattice, and $\omega_0 = (4\pi e^2 n/m^*\epsilon_0)^{1/2}$ is the plasma frequency[1)].

The possible character of the waves can be assessed in this case from the phase planes of Eq. (10), which are qualitatively analogous to those given in[6]. At $J \neq 0$ (in this case the "carrier" has only phase modulation) only periodic waves are possible. A case of interest is $J = 0$, when the phase is constant and $v\Omega/\kappa = 1$. The particular solution of (10) is then a single pulse with finite energy. This solution can be expressed in terms of elementary functions:

$$\theta = 2\arcsin\left[\frac{(1+A^2)^{1/2}+1}{(1+A_0^2)^{1/2}+1}\right]^{1/2} - \frac{2}{(1+A_0^2)^{1/2}+1}$$
$$\times\operatorname{Arch}\left[\frac{(1+A^2)^{1/2}+1}{(1+A_0^2)^{1/2}+1}\,\frac{(1+A_0^2)^{1/2}-1}{(1+A^2)^{1/2}-1}\right]^{1/2}, \tag{13}$$
$$A_0 = \frac{1}{4}\,\frac{1-\Omega^2(1-v^2)}{\Omega^2(1-v^2)},$$

where A_0 is the amplitude of the "envelope" of the packet. The duration and the spatial extent of such a packet expressed in periods (wavelengths) of the "carrier" at the level $0.5A_0$ are given by

$$T = v^{-1}\tau_u(A_0), \quad \Lambda = v\tau_u(A_0), \tag{14}$$

where the function $\tau_u(A_0)$ can be obtained from (13) and is shown in Fig. 1.

The obtained solution is characterized by two arbitrary constants, Ω and v. On the planes of these parameters (Fig. 2) there is separated region a, in which Eq. (10) admits of a solution in the form of solitary pulses; on the lines $v = 1$ and $\Omega = 0$, which limit this region, the amplitude of the packet is infinitely large; on the line b, corresponding to the linear dispersion relation, we have $A_0 = 0$. Near the line b, the solution goes over into wave packets with $A_0 \ll 1$, which contain many periods of the carrier. The latter have been well investigated by the averaging method.

It is of interest here to consider the behavior of the solutions at large values of the amplitudes. We note first that at appreciable packet amplitudes the frequency of the carrier can be much lower than unity ($\omega \ll \omega_0$), this being due to the "transparentization" of the plasma as a result of the increased effective mass of the electron. In addition, as seen from (14) and Fig. 1, when $A \gtrsim 1$ and $v \lesssim 1$ the duration (spatial extent) of the packet amounts to several periods (wavelengths)[2)], so that any projection of the pulse comprises a train consisting of several oscillations that change shape continuously as they propagate, since $v \neq \Omega/\kappa$ (Fig. 3). In the other limiting case $A \lesssim 1$ and $v \ll 1$, the spatial and temporal scales of the packet differ noticeably, namely, $T \gg 1$ and $A \ll 1$, and in the limit as $v \to 0$ the solution has the form of a stationary rotating "top" with finite amplitude $A_0 = 4(1/\Omega^2 - 1)$ and finite length (in the real spatial scale).

Of course, for real systems, such solutions are only an idealization, but one can expect waves of this type to occur also as a result of the evolution of perturbations of a broader class[3)], as is the case for ordinary solitons[2,3] (which are impossible for the considered type of function $f(p^2)$). We note in this connection that a monochromatic wave ($A = A_0$, $\varphi_0 = 0$), which is a particular solution of (9) and (10) and is characterized by the nonlinear dispersion relation $\Omega^2 = \kappa^2 + f(A_0^2)$, turns out to be unstable[4)]. Instability with respect to low-frequency perturbations sets in when

$$4f(A_0^2) > A_0^2 f'(A_0^2), \tag{15}$$

which is always satisfied when $f(p^2) = (1+p^2)^{-1/2}$. The increment increases in this case with frequency and reaches a maximum at frequencies on the order of the

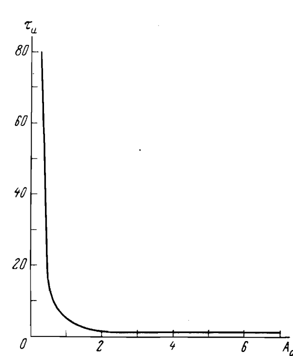

FIG. 1

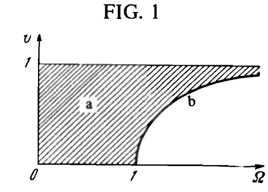

FIG. 2

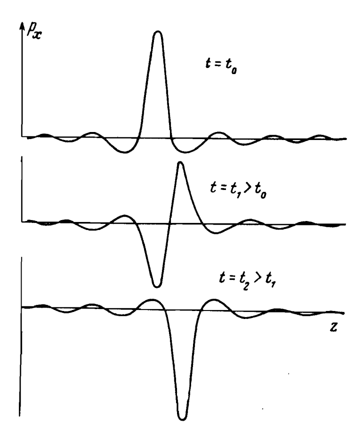

FIG. 3

reciprocal duration of the stationary pulse of corresponding amplitude (14), so that this instability can probably lead to solutions close to those discussed above.

In conclusion, let us discuss briefly the conditions for the existence of such pulses in real semiconductors. Equation (6) is valid if $\omega \gg \nu$, where ν is the electron collision frequency. The maximum intensity of the monochromatic wave, as a rule, is limited by the breakdown of the semiconductor[10]. If, however, the pulse is so short that $T \ll T_{br}$, where T_{br} is the breakdown development time, then there is no time for the breakdown to develop. In such a case, there can propagate in the semiconductor pulses of high intensity, and their damping is determined only by carrier scattering and not by breakdown.

In a pure n-InSb semiconductor at nitrogen temperatures, the collision frequency is $\nu \sim 10^{12}\ \mathrm{sec}^{-1}$ and the characteristic time of breakdown development is $T_{br} \sim 10^{-10}$ sec[11], and consequently there can propagate in the semiconductor pulses with duration from 10^{-10} sec ($A_0 \approx 0.4$) to durations on the order of the period of the carrier ($A_0 \approx 1$). Thus, the analysis considered here can be of definite interest for waves in the submillimeter and IR bands.

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1)In the derivation of (6) we have neglected the magnetic field of the wave, i.e., we have assumed that $v_* \ll c$, where $v_* = (\epsilon_g/2m^*)^{1/2}$ is the characteristic velocity that enter in the Kane dispersion law, for example $v_*/c \sim 3 \times 10^{-3}$ for InSb.

2)We note that definite durations and lengths of the nonstationary packet with respect to the characteristic scales of variation of the envelope A, when the latter is comparable with the period and wavelength of the carrier, is of arbitrary character.

3)This assumption is apparently confirmed by a numerical solution of Eq. (6), obtained by N. Zabusky (paper at European School on Plasma Physics, Tbilisi, 1970).

4)At low nonlinearities, such an instability is known as the self-modulation phenomenon. For the semiconductor InSb, this effect was investigated in [9].

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