Nonlinear theory of a quasimonochromatic packet of electrostatic waves in an inhomogeneous plasma

Ya. N. Istomin, V. I. Karpman, and D. R. Shklyar
Institute of Terrestrial Magnetism, Ionosphere, and Radio Wave Propagation, USSR Academy of Sciences;
Submitted December 13, 1972
Zh. Eksp. Teor. Fiz. **64**, 2072-2083 (June, 1973)

A nonlinear theory is developed of the interaction of a quasimonochromatic packet of electrostatic waves with resonant particles in a weakly inhomogeneous plasma. Expressions are obtained for the distribution function in the resonance range of phase space and also for the nonlinear decrement (increment) of the wave. In particular, it is shown that the nonlinear growth rate at large distances from the front boundary is determined by the difference between the mean distribution functions of the trapped and untrapped resonant particles. A study has been made of the qualitative peculiarities of the nonlinear evolution of the packet in a weakly inhomogeneous plasma.

1. INTRODUCTION

As is well known[1,2] an infinite wave in an homogeneous plasma ceases to change after the establishment of the so-called "ergodic state" (mixing phase) in the resonant region of phase space. In an inhomogeneous plasma, owing to the change in the wave phase velocity, the composition of the resonant region varies continuously, so that the damping (amplification) of the wave does not stop, generally speaking, even at times exceeding the nonlinear time of the phase mixing. This question was considered earlier in[3] within the framework of the initial-condition problem for infinite (quasiplane) monochromatic waves. On the other hand, in a real situation it is frequently necessary to deal with wave packets. The latter case becomes of particular interest in connection with the experiments of Helliwell et al.[4] on magnetospheric propagation of packets of "whistlers" radiated by a transmitter on earth. The corresponding nonlinear theory of the evolution of such packets in a homogeneous plasma[5] has a far reaching analogy with electrostatic waves. This analogy, which is naturally conserved also in an inhomogeneous plasma, makes it possible to extend the theory developed for packets of electrostatic waves to the case of whistlers propagating in a real magnetosphere, where the effects of the inhomogeneity of the medium are frequently very significant. This analogy remains in force also for the so-called "ionic whistlers"[6,7], which are registered in the form of quasimonochromatic pulsations of the geomagnetic field. Thus, besides being of independent interest, the question discussed in the present paper is important also for an analysis of a number of active and passive experiments in a magnetospheric plasma.

From the results obtained in this paper it will be seen that the physical effects that take place for packets whose length is comparable with the characteristic length determined by the inhomogeneity have many peculiarities that distinguish them from infinite waves[3] and packets in a homogeneous medium[5]. On the other hand, in the corresponding limiting cases we obtain the results of[3,5].

2. KINETIC EQUATIONS FOR RESONANT ELECTRONS

We write down the basic equations in the form

$$\frac{\partial F}{\partial t}+\dot{x}\frac{\partial F}{\partial x}-\frac{e\mathscr{E}(x,t)}{m}\frac{\partial F}{\partial \dot{x}}=0, \tag{2.1}$$

$$\mathscr{E}(x,t)=\mathscr{E}_0(x,t)\sin\left[\int_0^x k(x')\,dx'-\omega t+\varphi\right], \tag{2.2}$$

where ω (assumed constant) and $k(x)$ are connected by the dispersion relation $\omega^2 = \omega_p^2(1 + 3k^2r_d^2)$; here $\omega_p(x)$ is the plasma frequency of the electrons, which is a slowly varying function of x.

We are interested in the distribution function of the resonant particles, for which

$$\dot{x}-\omega/k\lesssim 1/k\tau \ll \omega/k, \tag{2.3}$$

where τ is the characteristic period of the oscillations of the velocity of the resonant particles in the wave field, and is defined by the expression

$$\tau=(m/e\mathscr{E}_0 k)^{1/2}. \tag{2.4}$$

It is now convenient to change over in Eqs. (2.1) and (2.2) to new independent variables x, ξ, and $\dot{\xi}$:

$$2\xi=\int_0^x k(x')\,dx'-\omega t+\varphi,\quad 2\dot{\xi}=k(x)\left[\dot{x}-\omega/k(x)\right] \tag{2.5}$$

(the second relation is formally the total derivative of the first with respect to time). Then, taking condition (2.3) into account, the kinetic equation for the resonant particles can be written in the form

$$\frac{\omega}{k(x)}\frac{\partial F}{\partial x}+\dot{\xi}\frac{\partial F}{\partial \xi}-\left(\frac{\sin 2\xi}{2\tau^2}+\alpha\right)\frac{\partial F}{\partial \dot{\xi}}=0, \tag{2.6}$$

$$\alpha=-(\omega^2/2k^2)\,dk/dx. \tag{2.7}$$

The quantity α (in the corresponding units) is equal to the force of the inertia in a reference frame where the phase of the wave does not depend on the time: $d[\omega/k(x)]/dt \equiv a = 2\alpha/k$. We shall assume that the inhomogeneity of the plasma is small enough, so that $\alpha\tau^2 \ll 1$. This condition means that the effects due to the inhomogeneity become manifest quite slowly in comparison with the phase mixing time τ. In the opposite limiting case $\alpha\tau^2 \gg 1$, the particles have time to go out of resonance before the nonlinear stage sets in. The linear theory is therefore applicable in the latter case.

Neglecting, for simplicity, the group velocity of the packet $v_g = 3(\omega/k)(kr_d)^2$ in comparison with ω/k, we can assume that in the laboratory frame, in which (2.6) has been written out, the amplitude of the packet, and consequently τ, depend little on the time. Assuming that the amplitude changes little during the time when the resonant particle passes through the entire length of the packet[1)], we can, by solving the kinetic equation, assume that τ is independent of the time. According to the Liouville theorem,

$$F(x, \xi, \dot{\xi}) = F(0, \xi_0, \dot{\xi}_0), \tag{2.8}$$

where $x = 0$ is the start of the packet and ξ_0 and $\dot{\xi}_0$ are the initial values of the variables ξ and $\dot{\xi}$, with the function $\dot{\xi}(\xi_0, \dot{\xi}_0, x)$ etc. determined from the equations for the characteristics corresponding to (2.6):

$$\frac{d\xi}{d\tilde{t}} = \dot{\xi}, \quad \frac{d\dot{\xi}}{d\tilde{t}} = -\left(\frac{\sin 2\xi}{2\tau^2(x)} + \alpha(x)\right), \quad \tilde{t} = \int_0^x \frac{k(x')\,dx'}{\omega}. \tag{2.9}$$

The right-hand side of (2.8) is equal to the unperturbed distribution function ahead of the packet $f(\dot{x}_0)$, where $\dot{x}_0$ is expressed in terms of $\dot{\xi}_0$ and k_0

$$\dot{x}_0 = \omega / k_0 + 2\dot{\xi}_0 / k_0$$

Here k_0 is the value of the wave number at the start of the packet. Since we are interested only in the resonant region, for which $\dot{\xi}_0/k_0 \ll v_T$, we can expand f in powers of $\dot{\xi}_0$ and confine ourselves to the first two terms, so that (2.8) takes the form

$$F(x, \xi, \dot{\xi}) = f_0 + \frac{2\dot{\xi}_0(x, \xi, \dot{\xi})}{k_0} f_0'; \tag{2.10}$$

$$f_0 = f\left(\frac{\omega}{k_0}\right), \quad f_0' = \frac{\partial}{\partial \dot{x}} f\left(\dot{x} = \frac{\omega}{k_0}\right). \tag{2.11}$$

The problem is thus reduced to a solution of the system (2.9). At τ = const and $\alpha = 0$ it coincides with the system of equations for particles moving in the field of an electrostatic wave of constant amplitude $1/2\tau^2$, and its energy integral takes the form κ = const, where

$$\kappa^{-2} = \dot{\xi}^2\tau^2 + \sin^2\xi. \tag{2.12}$$

The particles having $|\kappa| > 1$ are trapped by the wave, while those with $|\kappa| < 1$ are untrapped. The sign of κ is chosen such as to coincide with the sign of $\dot{\xi}$. The solutions of the system (2.9)

$$\xi(\tilde{t}, \xi_0, \kappa), \ \dot{\xi}(\tilde{t}, \xi_0, \kappa) \ \text{at}\ \tau = \text{const}, \ \alpha = 0$$

are expressed in terms of elliptic functions (see, e.g.,[2]) and are periodic functions of $\tilde{t}$ with period $T \sim \tau$. The corresponding averages with respect to time are

$$\overline{\dot{\xi}} = \frac{\pi}{2\tau\kappa K(\kappa)}, \quad \overline{\dot{\xi}^2} = \frac{1}{\kappa^2\tau^2}\frac{E(\kappa)}{K(\kappa)}, \quad |\kappa| < 1;$$
$$\overline{\dot{\xi}} = 0, \quad \overline{\dot{\xi}^2} = \frac{E(1/\kappa) - (1-\kappa^{-2})K(1/\kappa)}{\tau^2 K(1/\kappa)}, \quad |\kappa| > 1. \tag{2.13}$$

Here $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind.

We proceed now to the case when $d\tau/dx \neq 0$ and $\alpha \neq 0$. Now κ is a slowly varying function of x. Differentiating (2.12) with respect to x and using the equations of motion (2.9), we obtain

$$\frac{\omega}{k(x)}\frac{d}{dx}\left(\frac{1}{\kappa^2}\right) = \dot{\xi}^2\frac{\omega}{k(x)}\frac{d\tau^2}{dx} - 2\alpha\tau^2\dot{\xi}.$$

Averaging over the oscillations with the period τ, we can replace $\dot{\xi}$ and $\dot{\xi}^2$ by their mean values of $d\tau/dx = 0$ and $\alpha = 0$. As a result we obtain for the untrapped particles ($|\kappa| < 1$)

$$\frac{d}{dx}\left[\frac{E(\kappa)}{\tau\kappa}\right] = -\frac{\pi k(x)}{2\omega}\alpha(x) \tag{2.14}$$

and for the trapped particles ($|\kappa| > 1$)

$$\frac{d}{dx}\left[\frac{E(1/\kappa) - (1-\kappa^{-2})K(1/\kappa)}{\tau}\right] = 0. \tag{2.15}$$

These equations generalize the corresponding laws of conservation of the adiabatic invariants in a homogeneous plasma, where $\alpha = 0$. Relation (2.15) for an inhomogeneous plasma was obtained also in[8].

We introduce further in place of κ the new variables μ and ν:

$$\mu = E(\kappa)/\kappa, \quad |\kappa| < 1;$$
$$\nu = E(1/\kappa) - (1-\kappa^{-2})K(1/\kappa), \quad |\kappa| > 1; \tag{2.16}$$

$\mu(\kappa)$ increases from 1 to ∞ when κ decreases from 1 to 0, with $\mu(\kappa) = -\mu(-\kappa)$, sign μ = sign κ. As to $\nu(\kappa)$, it is an even function with $\nu(1) = 1$ and $\nu(\infty) = 0$. From (2.14) and (2.15) we obtain

$$\frac{\mu(x)}{\tau(x)} - \frac{\mu_0}{\tau_0} = -\sigma(x), \quad \frac{\nu(x)}{\tau(x)} = \text{const}; \tag{2.17}$$

where $\mu_0 = \mu(0)$, $\tau_0 = \tau(0)$,

$$\sigma(x) = \frac{\pi}{2\omega}\int_0^x \alpha(x')k(x')\,dx' = -\frac{\pi}{4}\omega \ln\frac{k}{k_0}. \tag{2.18}$$

We now trace the motion of the resonant particles moving from the region ahead of the packet ($x < 0$) towards its maximum, the coordinate of which is x_m (see Fig. 1). At $x < x_m$ the function $\tau(x)$ decreases with increasing x, i.e., $d\tau/dx < 0$. We assume also, for concreteness, that $\alpha(x) > 0$ over the entire length of the packet. At the start of the packet practically all the particles are untrapped, i.e., $|\kappa_0| < 1$ for these particles. To be able to visualize better all the possible cases, it is useful to introduce the potential energy in the form $U(\xi, x) = \sin^2\xi/2\tau^2(x) + \alpha(x)\xi$ (see Fig. 2). With increasing x (i.e., the "time" $\tilde{t}$), the amplitude of the wave increases (at $x < x_m$). If $|\mu|$ becomes very close to unity as the particle moves, then the validity of the adiabatic approximation, on the basis of which (2.17) was derived, is violated. The particle then is either reflected from the potential wall[2)] and then μ reverses sign jumpwise, or crosses one of the separatrices $\mu = \pm 1$ and becomes trapped.

We now analyze in greater detail the possible cases. For particles that remain trapped during the entire time of their motion from the start of the packet $x = 0$ to the point x, the connection between the initial and final states is determined by relation (2.17). For particles that reach the separatrices at the point x' and have become reflected, we have at $\mu_0 > 0$

$$\frac{\mu_0}{\tau_0} = \frac{1}{\tau(x')} + \sigma(x'), \tag{2.19}$$

$$\frac{\mu}{\tau(x)} + \sigma(x) = \sigma(x') - \frac{1}{\tau(x')}. \tag{2.20}$$

Equation (2.20) determines the function $x'(\mu, x)$ which makes it possible, together with (2.19) to determine the function $\mu_0(\mu, x)$.

It should be noted that particle reflection is possible only under definite conditions. Indeed, noting that $\mu < -1$ for the reflected particles, we obtain from (2.20)

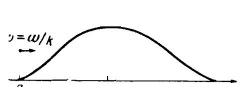

FIG. 1.

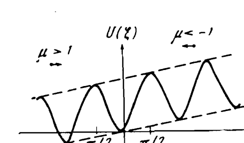

FIG. 2.

FIG. 1. Motion of resonant particles relative to the packet.

FIG. 2. Form of the effective potential energy $U(\xi, x) = \sin^2\xi/2\tau^2(x) + \alpha(x)\xi$

$$\sigma(x)-\sigma(x')>\tau^{-1}(x)-\tau^{-1}(x').$$

Letting now $x \to x'$, we obtain

$$\frac{d\sigma(x')}{dx'}>\frac{d}{dx'}\left[\frac{1}{\tau(x')}\right], \qquad (2.21)$$

which is the necessary condition for the reflection. If this condition is not satisfied, then the particles that have reached the separatrices are always captured by the wave (this holds true, in particular, in a homogeneous plasma, where $d\sigma/dx = 0$). On the other hand, even in the case when (2.21) is satisfied, some of the particles are captured by the wave (this will be discussed later on). We note also that when (2.21) is satisfied the particles with $\mu_0 < 0$ cannot reach the separatrices at all (this can be easily verified with the aid of (2.17)).

Assume now that a condition opposite to (2.21) is satisfied, viz.,

$$\frac{d\sigma(x')}{dx'}<\frac{d}{dx'}\left[\frac{1}{\tau(x')}\right]. \qquad (2.22)$$

In this case all the particles that have reached the separatrices at the point x' are trapped by the wave, and μ_0 can be of either sign. It is easy to verify that for particles trapped by the wave the relations (2.19) and (2.20) are replaced by

$$\frac{\mu_0}{\tau_0}=\frac{\operatorname{sign}\mu_0}{\tau(x')}+\sigma(x'), \qquad \frac{\nu}{\tau(x)}=\frac{1}{\tau(x')}. \qquad (2.23)$$

Differentiating further (2.19) and (2.20) at constant τ_0 and x, and then eliminating dx', we obtain

$$\frac{d\mu}{\tau}=\frac{\sigma'(x')-[1/\tau(x')]'}{\sigma'(x')+[1/\tau(x')]'}\frac{d\mu_0}{\tau_0} \qquad (2.24)$$

It follows similarly from (2.23) that

$$\frac{d\nu}{\tau}=\frac{[1/\tau(x')]'}{[1/\tau(x')]'+\operatorname{sign}\mu_0\sigma'(x')}\frac{d\mu_0}{\tau_0} \qquad (2.25)$$

Relations (2.24) and (2.25) determine the connection between the phase-space elements of the particles at the points x = 0 and x. Indeed, from the general expression for phase space in the $(\xi, \varkappa)$ plane

$$d\xi\, d\dot{\xi}=\frac{d\xi\, d\varkappa}{\tau\varkappa^2(1-\varkappa^2\sin^2\xi)^{1/2}}$$

we find that the value of the phase space between $\varkappa$ and $\varkappa + d\varkappa$ in the region of the trapped particles ($|\varkappa| < 1$) is given by

$$\frac{d\varkappa}{\tau\varkappa^2}\int_{-\pi/2}^{\pi/2}\frac{d\xi}{(1-\varkappa^2\sin^2\xi)^{1/2}}=\frac{2K(\varkappa)}{\tau\varkappa^2}=2\left|\frac{d\mu}{\tau}\right|. \qquad (2.26)$$

Analogously for the trapped particles ($|\varkappa| > 1$) we have

$$\frac{d\varkappa}{\tau\varkappa^2}\int_{-\arcsin(1/|\varkappa|)}^{\arcsin(1/|\varkappa|)}\frac{d\xi}{(1-\varkappa^2\sin^2\xi)^{1/2}}=2\left|\frac{d\nu}{\tau}\right|. \qquad (2.27)$$

In the case (2.21), when part of the particles reaching the separatrices are reflected, and some are trapped, it follows from (2.24) and (2.25) (where $\mu_0 > 0$)

$$2\left|\frac{d\nu}{\tau}\right|+\left|\frac{d\mu}{\tau}\right|=\left|\frac{d\mu_0}{\tau_0}\right|, \qquad (2.28)$$

i.e., the sum of the phase spaces of the trapped and reflected particles (which were initially in the element $d\mu_0/\tau_0$) is equal to the initial phase space[3)].

In the case (2.22) (when all the particles that have reached the separatrices are trapped, and the particles arrive at the given element of volume $2d\nu/\tau$ from different regions of μ_0, viz., $\mu_0 = \mu_0^+ > 0$ and $\mu_0 = \mu_0^- < 0$), it follows from (2.25) that

$$\left|\frac{d\mu_0^+}{\tau_0}\right|+\left|\frac{d\mu_0^-}{\tau_0}\right|=2\left|\frac{d\nu}{\tau}\right|, \qquad (2.29)$$

i.e., the phase space of the trapped particles is equal to the sum of the initial phase spaces. Relations (2.28) and (2.29) generalize the law of phase-space conservation to include the case when the phase trajectories of the particle become branched and move apart, or, to the contrary, come together.

The foregoing results make it possible to determine uniquely the ergodic distribution functions in various regions. For particles that stayed trapped during their entire motion, the ergodic distribution function in terms of the variables $(x, \varkappa)$ is obtained from (2.10) by replacing $\dot{\xi}_0$ by the ergodic mean value $\bar{\dot{\xi}}_0$, where $\bar{\dot{\xi}}_0(\varkappa_0)$ is determined by the same formula as $\dot{\xi}$ in (2.13), but with $\varkappa$ replaced by $\varkappa_0$[2]. Here $\varkappa_0$ is a function of $\varkappa$ (according to formulas (2.17) and (2.16)). If we use in place of $\varkappa$ the variable μ, then, for the untrapped particles the ergodic distribution function takes the form

$$F(x,\mu)=f_0+\frac{\pi f_0'}{k_0\tau_0}R(\mu_0). \qquad (2.30)$$

The function R in (2.30) is defined in parametric form:

$$R(w)=1/\varkappa K(\varkappa), \quad w=E(\varkappa)/\varkappa. \qquad (2.31)$$

The main properties of R(w) are described in[3]. We note here that this is a monotonically increasing odd function, with

$$R(w)=-\frac{2}{\ln(w-1)}, \quad w-1\ll 1; \quad R(w)=\frac{4}{\pi^2}w-\frac{\pi^2}{128w^3}, \quad w\gg 1. \qquad (2.32)$$

In addition to the data given in[3], it is expedient to define R(w) additionally in the following manner: R(w) = 0 at $|w| < 1$ (relations (2.31) define R(w) only for $|w| > 1$). For particles that do not reach the separatrices in the course of their motion, the function $\mu_0(\mu, x)$ is determined from (2.17), and the region of values of μ is defined by the following inequalities:

$$\mu>1, \quad \mu<\min(-1, -\tau(x)/\tau_0-\tau(x)\sigma(x)).$$

For particles that experience reflection (we recall that this is possible only under condition (2.21)), and that in (2.30) it is necessary to substitute the function $\mu_0(\mu, x)$ defined by formulas (2.19) and (2.20). Then

$$-\tau(x)/\tau_0-\tau(x)\sigma(x)<\mu<-1.$$

To determine the distribution function of the trapped particles in the case of (2.21), we write down the particle conservation law with allowance for the branching out of the phase trajectories near the separatrix:

$$2\left|\frac{d\nu}{\tau}\right|F(x,\nu)+\left|\frac{d\mu}{\tau}\right|F(x,\mu)=\left|\frac{d\mu_0}{\tau_0}\right|F(0,\mu_0).$$

Substituting here the expression for the distribution function $F(x, \mu)$ of the reflected particles from (2.30), we obtain

$$F(x,\nu)=F(0,\mu_0)=f_0+\frac{\pi f_0'}{k_0\tau_0}R(\mu_0), \qquad (2.33)$$

where the function $\mu_0(\nu, x)$ is determined from (2.23) (we recall that $\mu_0 > 0$ under the condition (2.21)).

In the case (2.22), the particle-number conservation law is

$$2\left|\frac{d\nu}{\tau}\right|F(x,\nu)=\left|\frac{d\mu_0^+}{\tau_0}\right|F(0,\mu_0^+)+\left|\frac{d\mu_0^-}{\tau_0}\right|F(0,\mu_0^-),$$

where $\mu_0^+ > 0$ and $\mu_0^- < 0$. Substituting here (2.25), we get

$$F(x,\nu)=F(0,\mu_0^+)S^+ + F(0,\mu_0^-)S^-, \qquad (2.34)$$

where the function $\mu_0^\pm(\nu, x)$ is determined from (2.23) and

$$S^{\pm}=\frac{1}{2}\left\{\left[\frac{1}{\tau(x')}\right]'\pm\sigma'(x')\right\}\Big/\left[\frac{1}{\tau(x')}\right]'.$$

In particular, in a homogeneous plasma, where $\sigma'(x') = 0$, we obtain[5]

$$F(x,\nu)=\frac{1}{2}[F(0,\mu_0^+)+F(0,\mu_0^-)]=f_0.$$

We consider now the region lying behind the maximum of the packet, i.e., at $x > x_m$. In this case

$$\frac{d}{dx}\frac{1}{\tau(x)}<0$$

and the untrapped particles, when they reach the separatrices, can only be reflected. This takes place only under the condition

$$\frac{d\sigma}{dx}>-\frac{d}{dx}\frac{1}{\tau(x)}. \tag{2.35}$$

On the other hand, part of the trapped particles leaves the potential well and becomes untrapped (with $\mu < -1$).

Under a condition inverse to (2.35), the untrapped particles cannot reach the separatrices, and the trapped particles, leaving the potential wells, can have μ of either sign. The corresponding general expressions for the distribution functions behind the maxima will not be presented here, for the sake of brevity; they can be obtained by applying a reasoning analogous to that used above for the region ahead of the maximum. We shall need only the distribution function of the trapped particles in the case when the leading front of the packet is steep enough, i.e., where the condition (2.22) is satisfied. Then the expression for $F(x, \nu)$ coincides formally with (2.34). The point x' that figures in (2.23) then lies ahead of the maximum ($x' < x_m$).

3. EQUATION FOR THE FIELD OF THE WAVE

We start from the equation[4)]

$$\frac{\partial U}{\partial t}=-\overline{j\mathscr{E}} \tag{3.1}$$

(the bar denotes averaging over the period of the wave), where U is the energy density of the wave, $U = \mathscr{E}_0^2/8\pi$, and j is the current density of the particles moving in the field of the wave. In the calculation of the right-hand side of the wave in (3.1) we make use of the fact that

$$\overline{j\mathscr{E}}=-\frac{2e\omega\mathscr{E}_0}{\pi k^2}\int\limits_{-\pi/2}^{\pi/2}d\xi\int\limits_{-\infty}^{\infty}d\dot{\xi}(F-f)\sin 2\xi, \tag{3.2}$$

where f is the plasma distribution function in the absence of the wave field; in the resonant region we have for this function, the equation

$$\frac{\omega}{k(x)}\frac{\partial f}{\partial x}+\dot{\xi}\frac{\partial f}{\partial \xi}-\alpha\frac{\partial f}{\partial \dot{\xi}}=0. \tag{3.3}$$

(This equation is obtained from (2.6) by putting in it $\tau = \infty$:) The solution of (3.3) is (under the same boundary conditions as for the function F)

$$f=f_0+\frac{2f_0'}{k_0}\left[\dot{\xi}+\frac{2}{\pi}\sigma(x)\right]. \tag{3.4}$$

It is easy to verify that the asymptotic form of the function F, obtained in the preceding section, takes the form (3.4) at large $\dot{\xi}$, and that only the resonant region contributes to (3.2), so that we can confine ourselves to the expressions obtained above for F and f.

Subtracting further Eq. (3.3) from (2.6) and integrating with respect to ξ and $\dot{\xi}$, we obtain after simple transformations

$$\int\limits_{-\pi/2}^{\pi/2}d\xi\int\limits_{-\infty}^{\infty}d\dot{\xi}\,\delta F=0,\qquad \delta F=F-f, \tag{3.5}$$

where it is recognized that $\delta F = 0$ at $x = 0$. Multiplying further both halves of Eqs. (2.6) and (3.3) by x^2 and subtracting one from the other, we obtain after a number of transformations, with allowance for (3.5),

$$\frac{\omega}{k(x)}\frac{\partial}{\partial x}\int\delta F\dot{\xi}\,d\xi\,d\dot{\xi}=-\frac{1}{2\tau^2}\int\sin 2\xi\delta F\,d\xi\,d\dot{\xi}.$$

Comparing this expression with (3.2), we obtain $\overline{j\mathscr{E}} = (4\omega^2 m/\pi k^4)\chi$, where

$$\chi=\frac{\partial}{\partial x}\int\limits_{-\pi/2}^{\pi/2}d\xi\int\limits_{-\infty}^{\infty}d\dot{\xi}\,\dot{\xi}(F-f). \tag{3.6}$$

To obtain now an equation for the amplitude in closed form, it remains to calculate the quantity χ in (3.6). Substituting the expressions obtained in the preceding section for the distribution function in different regions of χ, we obtain, after cumbersome but straightforward calculations, the following expression, which holds true in the region $x < x_m$ (x_m is the coordinate of the maximum of the amplitude):

$$\chi=\frac{8f_0'}{k_0}\frac{\partial}{\partial x}\left\{\frac{1}{\tau}\int\limits_0^x R\left[\frac{\tau}{\tau(x')}+\tau|\sigma(x)-\sigma(x')|\right]\right.$$
$$\left.\times\left[\frac{\partial}{\partial x'}\left(\frac{1}{\tau(x')}\right)-\left|\frac{\partial\sigma(x')}{\partial x'}\right|\right]\frac{dx'}{\tau(x')}-\frac{2}{9\tau^3}\right\}. \tag{3.7}$$

If the wave front is steep enough in the section $(0, x)$, namely, the condition (2.22) is satisfied, then, as can be easily verified,

$$\tau^{-1}(x)-\tau^{-1}(x')>\sigma(x)-\sigma(x').$$

The argument of the R-function is then less than unity (and more than zero), so that the term containing the integral with respect to x vanishes, and we obtain the same expression as in a homogeneous plasma[5]:

$$\chi=\frac{16f_0'}{9k_0}\frac{\partial\tau^{-3}(x)}{\partial x}.$$

Thus, under the condition (2.22) the wave front evolves with time in the same way as in a homogeneous plasma (for more details see[5,10]).

Assume now that (2.21) holds. We introduce the characteristic distance L from the start of the packet, for which $|\dot{\xi}(L)-\dot{\xi}_0|$ is of the order of the width of the resonance region, i.e., $|\alpha|L \sim \omega/k\tau$:

$$L=2\omega/\pi k\tau|\alpha|. \tag{3.8}$$

The quantity L can be called the characteristic length of the renewal of the resonant region owing to the inhomogeneity of the plasma. By virtue of the condition $\alpha\tau^2 \ll 1$, the length L greatly exceeds the nonlinear "ergodization" length $\tau\omega/k$ (i.e., the length over which an ergodic distribution function is established). At $x - x_0 \ll L$ (x_0 is the coordinate of the point where $d[1/\tau(x)]/dx = d\sigma(x)/dx$) the argument of the R-function in the expression (3.7) for χ is close to unity. Using the corresponding asymptotic expression for the R function (see (2.32)) we obtain

$$\chi=\frac{8\pi k}{k_0\omega}f_0'\frac{|\alpha|}{\tau^2}\frac{1}{\ln[(x-x_0)/L]}. \tag{3.9}$$

At large distances from the start of the packet ($x \gg L$) the asymptotic expression for χ takes the form (at $d\sigma/dx \gg d(1/\tau)/dx$)

$$\chi(x)=-\frac{8k}{k_0\omega^2}f_0'\alpha(x)\int\limits_{x_0}^{x}\frac{dx'}{\tau(x')}\alpha(x')k(x'). \tag{3.10}$$

Comparing (3.10) and (3.9) we can verify that the effects of the inhomogeneity become manifest most strongly at $x > L$.

Expression (3.10) can be written also in another form that is useful for a general analysis. To this end we introduce the average distribution function of the trapped particles at the point x:

$$\overline{F_T}=\int_0^1 F(x,\nu)\frac{d\nu}{\tau}\Big/\int_0^1\frac{d\nu}{\tau}=\int_0^1 F(x,\nu)\,d\nu \qquad (3.11)$$

(where we used the expression for the phase volume of the trapped particles (2.27)). The total density of the trapped particles is expressed in terms of F_T by the relation $n_T = (8/\pi k\tau)\overline{F_T}$.

In the region before the maximum, under the condition (2.21), the distribution function of the trapped particles is determined by formula (2.33). Substituting here $\mu_0(x, \nu)$ from (2.23) and calculating the asymptotic form of $R(\mu_0)$ at $\tau_0/\tau \gg 1$, we obtain (at $d\sigma/dx \gg d(1/\tau)/dx$)

$$\overline{F_T}=f_0+\frac{4\sigma}{\pi k_0}f_0'-\frac{4\tau}{\pi k_0}f_0'\int_{x_0}^{x}\frac{d\sigma(x')}{dx'}\frac{dx'}{\tau(x')}. \qquad (3.12)$$

Taking into account the definition of the quantities f_0 and f_0' in (2.11), we can write

$$f_0+\frac{4\sigma}{\pi k_0}f_0'=f\left(\frac{\omega}{k_0}+\frac{4\sigma}{\pi k_0}\right)=f\left(\frac{\omega}{k}\right).$$

Thus, (3.12) takes the form

$$\overline{F_T}=f\left(\frac{\omega}{k}\right)-\frac{4\tau}{\pi k_0}f_0'\int_{x_0}^{x}\frac{d\sigma(x')}{dx'}\frac{dx'}{\tau(x')}. \qquad (3.13)$$

Taking (3.13) into account, we can rewrite the expression for χ in the form

$$\chi(x)=\frac{4k\alpha}{\omega\tau}\left[\overline{F_T}(x)-f\left(\frac{\omega}{k}\right)\right]. \qquad (3.14)$$

The same expression can be obtained by considering the region lying behind the minimum. The quantity $f(\omega/k)$ in this expression is, as can be easily verified, the average distribution function of the resonant untrapped particles at $x \gg L$ (in the same sense as (3.11) is for the trapped ones): $\overline{F_{UT}}(x) = f(\omega/k)$.

Thus, energy exchange between the wave and the plasma at large distances from the start of the packet is determined by the difference between the average distribution functions of the trapped and resonant untrapped particles. This result can be useful in the study of the evolution of waves of large length with slowly varying amplitude (cf. the note added in proof in[3]).

4. EVOLUTION OF WAVE PACKET IN AN INHOMOGENEOUS PLASMA

By way of a simple example we consider the evolution of a packet whose amplitude varies quite steeply in the front and rear regions and is almost constant between them (Fig. 3a), so that the condition (2.22) is satisfied in regions AB and CD, and condition (2.21) in region BC. In region AB, where the amplitude varies steeply over very small distances, the effects of the inhomogeneity are inessential, since this section will vary in the same manner as in a homogeneous medium, i.e., it will become even steeper in a stable plasma, and will spread out, to the contrary, in an unstable one (for more details see[5,10]).

At $x > x_0$ (x_0 is the coordinate of the point B), effects of inhomogeneity become important. At $x - x_0 \ll L$, using (3.9), we obtain the following equations for the wave amplitude:

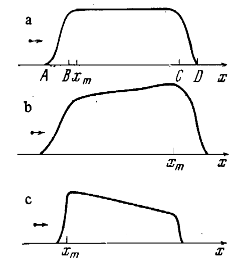

FIG. 3. Evolution of the shape of the packet: a) t = 0, b) t > 0 ($\gamma_L > 0$), c) t > 0 ($\gamma_L < 0$).

$$\frac{\partial\varepsilon}{\partial t}=-\beta\ln^{-1}\frac{x-x_0}{L}, \qquad (4.1)$$

$$\beta=64\pi^{-1}\gamma_L\tau_m^2|\alpha|,\quad \varepsilon=\mathscr{E}_0/\mathscr{E}_{0m},\quad \tau_m=(m/e\mathscr{E}_{0m}k)^{1/2} \qquad (4.2)$$

(γ_L is a linear increment). At large distances from the start of the packet $x - x_0 \gg L$, using (3.10), we obtain[5)]

$$\frac{\partial\varepsilon^2}{\partial t}=\beta_1\int_{x_0}^{x}\varepsilon^{1/2}(x')\,dx',\qquad \beta_1=\frac{128}{\pi^2\omega}\gamma_L\tau_m^3\alpha^2k. \qquad (4.3)$$

We consider now the evolution of the packet. Assume that $\epsilon = 1$ ($x > x_0$) at $t = 0$. It follows then from (4.1) that at $x - x_0 \ll L$ we have

$$\varepsilon(x,t)=1-\beta t\ln^{-1}\frac{x-x_0}{L}. \qquad (4.4)$$

As to Eq. (4.3), it can be solved by successive approximation. In the first approximation we put in the right-hand side $\epsilon(x, t) = \epsilon(x, 0) = 1$. Then

$$\varepsilon^2(x,t)=1+\beta_1(x-x_0)t. \qquad (4.5)$$

Since the quantity ϵ^2 enters in the right hand side of (4.3) raised to the power $1/4$, expression (4.5) has a high enough accuracy. The quantities β and β_1 have the same sign as γ_L. Therefore at $\gamma_L > 0$ (i.e., in an unstable plasma) the rear part of the packet is amplified more rapidly than the front part (see Fig. 3b). As to the packet in a stable plasma, it follows from the foregoing equations that the packet should have the form shown in Fig. 3c. It should be noted here, however, that since the maximum of the packet is located at the very start of the packet, it is necessary to take into account also particles that emerge from the trapping region. To take this circumstance into account, it is necessary to substitute in the general expression (3.14) the distribution function of the trapped particle behind the maximum, which was discussed at the end of Sec. 2. In this case the equation for the field amplitude becomes

$$\partial\varepsilon^2/\partial t=\beta_1\varepsilon^{1/2}(x-x_0). \qquad (4.6)$$

Solving (4.6), we obtain

$$\varepsilon^{3/2}=1+{}^3/_4\beta_1(x-x_0)t. \qquad (4.7)$$

Thus, Eqs. (4.3) and (4.5) pertain the an unstable plasma ($\nu_L > 0$), and (4.6) and (4.7) pertain to a stable plasma ($\gamma_L < 0$).

The expressions obtained above were derived under the assumption that the field varies little over the time of flight of the resonant particle through the entire region of the packet. It follows then from (4.3) and (4.6) that $\beta_1 x \ll \omega/kx$, i.e.,

$$x^2 \ll L^2/\gamma_L\tau. \qquad (4.8)$$

This condition is compatible with the condition $x - x_0$ at $\gamma_L\tau \ll 1$, as assumed throughout in this paper.

[1]It is shown at the end of Sec. 4 that this assumption is valid at $\gamma_L \tau \ll 1$, where γ_L is the linear decrement.

[2]In the laboratory frame, reflection corresponds to the wave overtaking the particle.

[3]The factor 2 in front of the first term in (2.28) corresponds to allowance for particles moving in opposite directions (we recall that for the trapped particles the value of ν is the same for both signs of the velocity).

[4]We neglect here the spreading of the packet as a result of dispersion. As shown in [9], where dispersion effects are discussed for nonlinear evolution of a packet in a homogeneous plasma, the latter are quite small if the wave amplitude is not very large (see condition (18) of [9]). It is important that we do not consider here the nonlinear frequency shift. This problem will be dealt with separately.

[5]In the derivation of (4.2) and (4.3) we have assumed for simplicity that α is constant, and also neglected the difference between k_0 and k; this imposes a limitation on x, namely $x \ll L\,(\omega\tau)$.

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Translated by J. G. Adashko
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