Fluctuation theory of the two-dimensional mixed state in type 1 superconductors

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The structure of the surface ”mixed” state in type 1 superconductors is elucidated under conditions when, due to the presence of an electric field, the superconductivity in the layer is strongly suppressed and exists in the form of superconducting fluctuations. It is shown that an arbitrarily weak electric field leads to practically complete destruction of the mixed-state layer for currents close to the second critical current. The electromagnetic impedance of the surface in the presence of the mixed-state layer is calculated.

The picture of the destruction of superconductivity by an electric current in a multiply-connected sample possesses an important feature in comparison with the case of a singly-connected superconductor. Namely, in addition to the purely superconducting, normal, and intermediate states, the so-called two-dimensional mixed state should also be realized in the present case.

The idea of the existence of such a ”mixed” state was expressed by L. D. Landau as long ago as 1937 (private communication to D. Shoenberg, see[1,2] in connection with the question of the destruction of the superconductivity of a hollow cylinder by a current. The point here is that the intermediate state must vanish with an increase of the current because the radius of the intermediate-state region becomes smaller than the radius of the cylinder’s inner surface. However, in this connection the purely normal state is also unstable with respect to the onset of superconductivity close to the inner surface, where for an arbitrary value of the current the magnetic field is as small as desired. This superconductivity cannot exist in the form of a layer of the usual superconducting phase since an electric field exists in the normal phase surrounding it, and by virtue of the continuity conditions this electric field must also exist in the superconducting region. Thus, the inside surface of the cylinder must be covered by a ”mixed” state layer in which superconductivity and an electric field co-exist. This two-dimensional mixed state was experimentally observed and investigated by I. Landau and Sharvin.[3,4]

In the article by ‘Tzekel’ and one of the authors,[5,6] the properties of the two-dimensional mixed state were theoretically investigated under the assumption that a sufficiently weak electric field does not turn out to have any substantial influence on the properties of the layer, which in this connection turn out to be approximately the same as the properties of an ordinary superconducting layer of finite thickness. Such a picture is confirmed by the experiments of I. Landau and Sharvin for weak currents in sufficiently pure metals. It is clear, however, that the existence of an electric field is the most important feature of the mixed state under investigation, and a strong electric field must lead to a significant suppression of the superconductivity.

In the present article the structure of the two-dimensional mixed state will be investigated under conditions when, due to the presence of an electric field, the superconductivity in the layer is almost completely suppressed and exists in the form of small ”superconducting” fluctuations. The results obtained in this connection make it possible to clarify under what conditions the electric field plays an essential role. It will be shown, in particular, that the presence of the electric field is extremely important even in an arbitrarily pure metal, when the current is not small. It is precisely the effect of the electric field which determines the maximum value of the current, the value at which the two-dimensional mixed state vanishes and the sample goes over to an essentially purely normal state. Owing to their small conductivity, the picture of the two-dimensional mixed state in type 1 superconducting alloys practically always turns out to be strongly dependent on the electric field.

1. GENERAL RELATIONSHIPS

Let us consider a hollow cylindrical superconductor with outer radius $r_2$ and inner radius $r_1$ through which a current $J$ is flowing. In this connection, if the sample exists in a purely normal state, then the magnetic field distribution as a function of the radial distance is determined by the formula

$$H(r) = \frac{2J}{cr} \frac{r_2^3 - r_1^3}{r_2^3 - r_1^3}.$$ 

For our purposes only the region near the inner surface ($r \approx r_1$) will be important; in this region the magnetic field is small and may be written in the form $H(x) = (4J/c)(r_2^3 - r_1^3)^{1/2}$, where $x = r - r_1$ denotes the distance from the inner surface. The corresponding vector potential of the magnetic field can be represented in the form $A_x = A_y = 0$, $A_z = - (2J/c)(r_2^3 - r_1^3)$, where the $z$ axis is directed along the axis of the sample. In addition to the magnetic field there is also an electric field $E$, resulting from the finite conductivity of the normal metal and directed along the $z$ axis. With allowance for this field, the total vector potential is given by

$$A_z = - \frac{2J}{c} \frac{r^3 - r_1^3}{r_2^3 - r_1^3} \epsilon_{0}$$ \tag{1}$$

We shall assume that the superconductivity which arises near the inner surface is strongly suppressed by the presence of the electric field and only exists in the form of superconducting fluctuations.[6,7] The conditions under which this will actually occur will be clarified below.

In investigating the superconducting fluctuations we shall use the method which is most convenient for our purposes, based on the introduction of an external force into the temporal equation for the superconducting order parameter. This method was first used by Schmid,[8] and was subsequently extended by Kulik[9] to the temperature region $T < T_c$, in which the smallness of the
fluctuations may be due to the presence of the electric field. The latter result was previously derived by a microscopic method in an article by Gor'kov.\cite{11}  

With the external force taken into account, the time-dependent equation for the superconducting order parameter $\psi$ has the following form:

$$\frac{\partial \psi}{\partial t} - \gamma \mathbf{F} \psi = f(t, \mathbf{r}),$$  

where $\xi = \xi(T)$ is the coherence length, $\nu = \theta(T_c - T)/\hbar$, and $f$ is the external force satisfying the condition

$$\langle f(t, \mathbf{r}) f(\mathbf{r}', t') \rangle = 4mV2\delta^2(\mathbf{r} - \mathbf{r}')\delta(t - t'),$$  

where $m$ denotes the electron mass, and the angular brackets denote average values with respect to an ensemble.

Substituting (1) into (2) and changing to the dimensionless quantities

$$\hat{\psi} = \frac{1}{2} \left( \frac{\hbar}{mF} \right)^{1/2} \psi, \quad \epsilon = \frac{2eE}{\hbar}, \quad \hat{\gamma} = \frac{4eJ^2}{\hbar^2} \frac{1}{c^2} r_1^2 - r_1^2,$$

we obtain

$$\frac{\partial \hat{\psi}}{\partial \tau} - \hat{\gamma} \mathbf{F} \hat{\psi} = \hat{f}(\tau, \mathbf{r}),$$  

$$\mathbf{D} = - \frac{\partial^{2}}{\partial \mathbf{r}^{2}} - \frac{\partial^{2}}{\partial \mathbf{y}^{2}} - \left( \frac{\partial}{\partial \xi} + i\epsilon \mathbf{e} \right)^{2},$$

where

$$\langle \hat{f}(\tau, \mathbf{r}) \hat{f}(\mathbf{r}', \tau') \rangle = \delta(\mathbf{r} - \mathbf{r}')\delta(\tau - \tau').$$  

We shall use the dimensionless quantities below, and wherever it does not lead to ambiguities we shall omit the tilde above the corresponding letters.

We shall seek a solution of Eq. (5) in the form

$$\psi = \sum_{\lambda, k} a_{\lambda, k}(t) \phi_{\lambda, k}(x, t),$$  

where

$$\phi_{\lambda, k}(x, t) = L^{-1} \exp \left( i(k_y y - k_z z) \right) \psi_{\lambda, k}^{(0)}(x)$$

are the eigenfunctions of the operator $\mathbf{D}$ with eigenvalues $(\lambda + k^2)/2m$

$$\mathbf{D} \phi_{\lambda, k}^{(0)}(x) = (\lambda + k^2) \phi_{\lambda, k}^{(0)}(x),$$

satisfying the normalization condition

$$\int \phi_{\lambda, k}^{(0)}(x) \phi_{\lambda, k}^{(0)*}(x) dx = \delta_{\lambda, \lambda'},$$

where $k = k_x + i\epsilon$, and $L$ is a normalization length. Equation (6) is equivalent to the condition, which is imposed on the function $\phi$:  

$$d\phi_{\lambda, k}^{(0)}/dx^2 + \nu_\lambda - (k + \beta x)^2) \phi_{\lambda, k}^{(0)} = 0,$$

from which it is seen that the functions $\phi_{k_x k}^{(0)} = \phi_{k_x k}^{(0)}(\lambda)$ depend on $k_x$.

Let us consider the most interesting case when the electric field is not too strong. Namely, let us assume that the electrical energy $\epsilon \mathbf{E}$ is small in comparison with the relaxation frequency $\nu$, that is

$$\epsilon \ll 1.$$  

Substituting the expansion (7) into Eq. (5), after some simple transformations we obtain

$$\frac{\partial a_{\lambda, k}(t)}{\partial t} - \frac{\partial a_{\lambda, k}(t)}{\partial t} + \left[ \lambda_{\lambda}(k) \right] \phi_{\lambda, k}^{(0)} - \hat{f}_{\lambda, k}(t),$$  

where

$$f_{\lambda, k}(t) = \int \hat{d}x \exp \left( -i(k_y y - k_z z) \right) \psi_{\lambda, k}^{(0)}(x) f(t, \mathbf{r}).$$

are the new components of the external force, satisfying the condition

$$\langle f_{\lambda, k}(t) f_{\lambda', k'}(t') \rangle = \delta_{\lambda, \lambda'} \delta_{\lambda', \lambda} \delta(t - t').$$

In deriving formula (11) we neglected expressions of the form $a_{\lambda, k}^{(0)} \partial \phi_{\lambda, k}^{(0)}/\partial t$ in comparison with $\phi_{k_x k}^{(0)} a_{\lambda, k}^{(0)}/\partial t$, which is legitimate since the time-dependence of the eigenfunctions $\phi_{\lambda}^{(0)}$ is due to the dependence $k(t)$ associated with a given $k_x$. In the case when $\epsilon \ll 1$ this dependence is extremely weak, and it can be neglected in comparison with the explicit time dependence of the coefficients $a_{\lambda, k}^{(0)}$.

The solution of Eq. (11) has the form

$$a_{\lambda, k}(t) = \exp \left( - \rho_{\lambda, k}(t) \right) \int \exp \left( a_{\lambda, k}(t') \right) f_{\lambda, k}(t') dt',$$

where

$$\rho_{\lambda, k}(t) = \int \left[ 1 - \lambda_{\lambda}(k') - k_x^{(0)} \right] dt', \quad k' = k_x + \epsilon t'.$$

The density of the fluctuating superconducting current is determined by the well-known Ginzburg-Landau formula:

$$j = \frac{2\epsilon}{m} \int \left( \psi \left( \frac{\partial}{\partial \gamma} + i\epsilon \mathbf{e} \right) \right)^2.$$

which in our dimensionless form can be rewritten in the form

$$\langle j, \gamma \rangle = -\frac{2\epsilon}{1 + \epsilon} \int \left( \psi \left( \frac{\partial}{\partial \gamma} + i\epsilon \mathbf{e} \right) \right)^2.$$

Substituting here the expansion (7), the solution (13), and averaging with the aid of formula (12), we find

$$\langle j, \gamma \rangle = \frac{2\epsilon}{1 + \epsilon} \int \frac{dk_x}{(2\pi)^{2}} \left( k^2 + \beta^2 \right)^{1/2} \psi_{\lambda, k}^{(0)}(x) \int dt \phi_{\lambda, k}^{(0)}(t').$$

One can transform the difference $2p_{\lambda, k}^{(0)}(t') - 2p_{\lambda, k}^{(0)}(t)$ in the following way:

$$2p_{\lambda, k}^{(0)}(t') - 2p_{\lambda, k}^{(0)}(t) - 2 \int \left[ 1 - \lambda_{\lambda}(k') - k_x^{(0)} \right] dt'$$

$$= \int \left[ (t - t') \left( 1 - k_x^{(0)} \right) - \frac{1}{\epsilon} \int \lambda_{\lambda}(k') dk' \right].$$

After substituting the last expression into (15), carrying out the simple integration with respect to $k_x$ and introducing the new variable $\tau = \epsilon(t - t')$ we obtain

$$\langle j, \gamma \rangle = \left( \frac{2}{\pi \epsilon} \right)^{1/2} \frac{2\epsilon}{1 + \epsilon} \int \frac{dk_x}{(2\pi)^{2}} \int \frac{dk}{(2\pi)^{2}} \int \frac{dt}{(2\pi)^{2}} \psi_{\lambda, k}^{(0)}(x) \int dt \phi_{\lambda, k}^{(0)}(t').$$

where

$$\int \frac{dk}{(2\pi)^{2}} \int \frac{dt}{(2\pi)^{2}} \frac{1}{\epsilon} \int \lambda_{\lambda}(k') dk' = \int \frac{dk}{(2\pi)^{2}} \int \frac{dt}{(2\pi)^{2}} \frac{1}{\epsilon} \int \lambda_{\lambda}(k') dk' = 1.$$
equal to the derivative of the corresponding eigenvalue 
\((1/2)\alpha \Lambda_{n}/B\). Therefore the surface current density is 
given by

\[ I_{s} = (2\pi \epsilon / me^{2}) \sum_{n=0}^{\infty} \int d\mathbf{k} \frac{\partial \Lambda_{n}}{\partial \mathbf{k}} \int \frac{d\mathbf{k}}{4\pi} \exp \left\{ \frac{\chi}{\pi} \int_{k_{0}}^{k_{0}+\pi} \lambda_{n}(k') dk' \right\}. \]

(17)

The total superconducting current flowing through the sample is obtained by multiplying \(I_{s}\) by \(2\pi r_{o}\).

Below we shall consider two important limiting cases when the fluctuating superconducting current determined by formulas (16) and (17) can be completely calculated.

2. THE DOMAIN OF WEAK CURRENTS

Let the current passing through the sample be such that the dimensionless parameter \(\beta\) is small in comparison with unity. In order of magnitude the value of \(\beta\) is equal to the ratio \(I/J_c^2\), where \(J_c^2\) denotes the second critical current which was introduced in article [13].

For small values of \(\beta\) one can solve Eq. (9) for the functions \(\varphi_{n}(k)\), and the functions \(\lambda_{n}(k)\) can be determined for all important values of \(k\). In fact, if \(k\) is positive and satisfies the condition \(k > \beta^{1/3}\) then Eq. (9) coincides with the Schrödinger equation for an oscillator with mass 1/2, eigenenergy \(\lambda_{n} = k^2\), and small anharmonic term of fourth order. Using the well-known formula (see [13], problem 3 in Sec. 38) for the energy levels of the anharmonic oscillator, we find

\[ \lambda_{n}(k) = k^2 + (32 \beta k^2)^{1/3} (n + 1/4) + 3 \beta k^2 (n^2 + 1/2 n + 1/4), \]

(18)

where it has been taken into consideration that in our case the functions \(\varphi_{n}(k)\) must satisfy the Ginzburg-Landau boundary condition \(\varphi_{n}(0) = 0\), and therefore only the even parity (even with respect to the transformation \(x \rightarrow -x\)) solutions have any meaning. Accordingly formula (18) is obtained from the formula for the linear oscillator by the substitution \(n \rightarrow -n - 2n\).

For negative values of \(k\) such that \(|k| \gg \beta^{1/3}\), the potential energy \((k + \beta x^2)^2\) has a sharp minimum when \(x = x_{0} = (|k|/\beta)^{1/3}\). Introducing the new variable \(x' = x - x_{0}\), one can easily rewrite Eq. (9) in the form of the Schrödinger equation for an oscillator with mass 1/2, energy \(\lambda_{n} = k^2\), and small anharmonic terms of third- and fourth-order. Again using the well-known formulas of quantum mechanics, we obtain

\[ \lambda_{n}(k) = (4|k|\beta)^{1/3} (n + 1/2) - \beta/2k^{2} (3n^2 + 3n + 1). \]

(19)

Since the large factor \(2/\epsilon \) occurs in the argument of the exponentials appearing in formulas (16) and (17), a small neighborhood of the maximum of the function

\[ F_{c}(k, \tau) = \tau - 1/\epsilon \lambda_{n}(k') dk'. \]

gives the major contribution to the integrals over \(k\) and \(\tau\). The position of the maximum \((k_{0}, \tau_{0})\) is determined by the conditions \(\partial F_{c}(k_{0}, \tau_{0})/\partial k = 0\) and \(\partial F_{c}(k_{0}, \tau_{0})/\partial \tau = 0\), where the derivative \(\partial F_{c}(k, \tau)/\partial k\) must be positive for \(k = k_{0}\) and negative for \(k = k_{0} - \tau_{0}\). Using formulas (18) and (19) we find

\[ k_{0} = \frac{(8\beta)\chi}{(n + 1/4)}, \quad \lambda_{n} = \frac{1}{4\beta(2n + 1)^{3/2}} - \beta(3n^2 + 3n + 1). \]

(20)

At the maximum the value of the function \(F_{c}\) is given by

\[ F_{c}(k_{0}, \tau_{0}) = -\frac{1}{12\beta(2n + 1)^{3/2}} + \frac{2}{3} \frac{4\beta}{3} \frac{(3n^2 + 3n + 1)}{6n(6n - 1)} \frac{4\beta}{3} \frac{(3n^2 + 3n + 1)}{6n(6n - 1)}. \]

(21)

Now if we expand the function \(F_{c}\) near its maximum in powers of \(k - k_{0}\) and \(\tau - \tau_{0}\), substitute into (17) and perform the simple integrations, then we obtain the following result upon taking into consideration that, as is clear from (21), the first term with \(n = 0\) gives the major contribution to the summation over \(n\):

\[ I_{s} = \frac{4e\epsilon}{3} \left( \frac{8\beta}{2n} \right)^{1/3} \exp \left\{ \frac{1}{6n\beta} - \frac{4}{3e} \frac{8\beta}{2n} \frac{3n^2 + 3n + 1}{6n(6n - 1)} \right\}. \]

(22)

According to Eq. (16) the coordinate dependence of the density of the fluctuating current is determined by the function \(\varphi_{n}(k)\) with \(n = 0\) and \(k \approx \tau_{0} - \tau\), that is, by the wave function of the oscillator ground state, normalized over the interval \((0, \tau_{0})\):

\[ \langle \psi(x) \rangle = \langle 2(2\beta)^{1/3} \rangle \exp \left\{ -\frac{1}{2x^2} / d \right\}, \]

(23)

where \(d = (2\beta)^{1/3}\) obviously plays the role of the thickness of the fluctuational "mixed" state layer. In the usual units we have

\[ d = \left( \frac{e\epsilon}{\hbar} - \frac{n}{\epsilon} \right)^{1/3}. \]

(24)

Thus, the thickness of the layer decreases rather slowly with increasing current.

Now let us discuss the question of the range of validity of the obtained results. The major limitation is associated with the fact that we treated the fluctuations as a small perturbation and, in particular, we did not take into account the influence of the fluctuation current on the form of the vector potential which, according to Eq. (1), is determined by only the normal current. In order for this to be correct, the current \(I_{s}\) must be small in comparison with the normal current passing through a region of thickness \(d\) near the inner surface:

\[ I_{s} < \frac{j_{0}d}{1/\epsilon} \left( \frac{\tau_{0}^{3} - \tau_{0}^{3}}{1/\epsilon} \right) - \frac{\tau_{0}^{3} - \tau_{0}^{3}}{1/\epsilon} \frac{4\beta}{3} \frac{(3n^2 + 3n + 1)}{6n(6n - 1)} e\hbar \]

(25)

where \(\Delta_{c} \sim e\epsilon^{1/3}\) is the critical magnetic field, and \(\kappa\) is the parameter of the Ginzburg-Landau theory.

In order of magnitude the factor in front of the exponential in formula (22) can be written down in the following form:

\[ e\epsilon / \xi \sim c\hbar / (\xi / \beta_{0}) \approx (a_{o} / \lambda_{o})^{1}, \]

where \(\Delta_{o}\) and \(\lambda_{o}\) denote the energy gap and the London penetration depth in a superconductor at zero temperature, \(\xi / \beta_{0} = \hbar v_{F} / 2a_{o}\), and \(a_{o} \sim h / v_{F}\) is the interatomic distance.

Thus, the following quantity should be small:

\[ (a_{o} / \lambda_{o})^{1} \epsilon / \hbar v_{F} \ll 1, \]

(26)

where we have omitted some factors that are not essential for an estimate. The parameter \(\epsilon\) can be expressed in terms of the conductivity \(\sigma\) of the metal's normal phase:

\[ \epsilon = \beta e^{1/16} a_{o}^{2} (T_{c} - T), \]

from which it is seen that the argument of the exponential in (26) is equal to \((\beta k_{F}^{2} x_{c}^{2})^{-1}\) in order of magnitude, where \(l\) denotes the free path length of the electrons.
Although the factor in front of the exponential in (26) is extremely small, nevertheless, condition (26) cannot be satisfied in pure metals due to the enormous magnitude of the exponential. This means that the electric field is too weak, and therefore in the present case the results of article [13] are actually valid.

However, owing to the smallness of the ratio $a_0/\lambda_0$ in type 1 superconducting alloys with $\kappa \sim 1$ and $l \sim \xi_0$ there exists a wide range of values of current through the sample for which condition (26) is satisfied and the properties of the two-dimensional mixed state should be described by the formulas derived above.

3. THE REGION NEAR THE SECOND CRITICAL CURRENT

Now let the current be close to the second critical current. The latter is defined as the maximum current which can grow with time can still exist. In our dimensionless units this means that for $J < J_{C2}$ values of $n$ and $k$ exist such that $\lambda_n(k) < 1$. For $J > J_{C2}$ the opposite inequality, $\lambda_n(k) > 1$, holds for arbitrary $n$ and $k$. If the current is equal to the second critical current, then the minimum value of $\lambda$, which is reached for $n = 0$ and some value $k = k_{m}$, is exactly equal to unity. Therefore it is clear that the following formula (n = 0) holds for $J = J_{C2}$ and $k \approx k_{m}$:

$$\lambda(k) = 1 + a - \frac{J - J_{C2}}{J_{C2}} \cdot \frac{b}{k} (k - k_{m}),$$

where $a$ and $b$ are certain coefficients, whose order of magnitude is unity.

Substituting (27) into (17) and assuming that the term with $n = 0$ gives the major contribution, after a simple integration we obtain

$$I_s = \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \int dx \exp \left[ \frac{2a}{e} \left( \frac{J_{C2} - J}{J_{C2}} \right) \right].$$

The written expression significantly depends on the relative magnitude of the parameters $(J - J_{C2})/J_{C2}$ and $e^{1/2}$, and also on the sign of the difference $J - J_{C2}$.

For $J < J_{C2}$ and $(J_{C2} - J)/J_{C2} \ll e^{1/2}$ the integral in (28) can easily be evaluated by the method of steepest descents, and we obtain

$$I_s = \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \int dx \exp \left[ \frac{2a}{e} \left( \frac{J_{C2} - J}{J_{C2}} \right) \right].$$

Similarly, for $J > J_{C2}$ and $(J - J_{C2})/J_{C2} \gg e^{1/2}$ the integral in (28) can easily be evaluated by the method of steepest descents, and we obtain

$$I_s = \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \int dx \exp \left[ \frac{2a}{e} \left( \frac{J_{C2} - J}{J_{C2}} \right) \right].$$

If $J_{C2} - J < J_{C2}$, then one can neglect the first term in the argument of the exponential in (28), and then we easily find

$$I_s = \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}}.$$ (30)

Finally, for $J > J_{C2}$ and $(J - J_{C2})/J_{C2} \gg e^{1/2}$ one can neglect, on the other hand, the second term in the exponential in (28). In this connection the superconducting current is given by

$$I_s = \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}} \left( \frac{b}{2} \right)^{1/2} e^{(1/2)^{1/2}}.$$ (30)

In the present case the electric field is unimportant, and therefore the last formula is identical to the result of article [13].

Since an exact calculation of the coefficients $a$ and $b$ is difficult, we shall use a variational principle for their determination, assuming a trial function of the form

$$\psi(x) = \left( \frac{8\pi}{\alpha} \right)^{1/2} e^{-x^2/\alpha}.$$

Multiplying Eq. (9) by $\psi(x)$ and then integrating the obtained result with respect to $x$, we obtain the following result after substituting the explicit form of the functions $\psi$:

$$\lambda = \alpha + b + \frac{2b}{3} \beta + \frac{2b}{3} \alpha.$$ (31)

Minimizing this expression with respect to $\alpha$ and $b$, we obtain

$$k_{m} = -\left( \frac{2}{3} \right)^{1/2} \beta$$ and $\lambda = \left( \frac{2}{3} \right)^{1/2} \beta$.

From the condition $\lambda(k_{m}) = 1$ we obtain the value of the parameter $\beta$ corresponding to the second critical current, $\beta_{c2} = 2(2/3)^{1/2}$, and in this manner we find

$$I_{c2} = \left( \frac{1}{2} \right)^{1/2} e^{(1/2)^{1/2}}.$$ (31)

Expanding (31) in powers of $k - k_{m}$ and of the difference $J_{C2} - J$, we find

$$I_{c2} = \left( \frac{1}{2} \right)^{1/2} e^{(1/2)^{1/2}}.$$ (31)

which gives zero upon integrating with respect to $x$.

This indicates that the direction of the fluctuation current varies over the depth of the layer, and it also indicates that oppositely directed currents practically cancel each other. The total current $I_{c2}$ is a quantity of a higher order of smallness in $(J_{C2} - J)$ than the current density.

The formulas derived in the present section enable us to elucidate the distinctive features of the picture of the destruction of the mixed-state layer upon increasing the current in pure metals. In fact, we saw above that owing to the smallness of the electric field, the superconducting current in such metals is large for $\beta \approx 1$, and in order of magnitude is given by $I_{c2} \sim \frac{1}{2} \chi_{H}$. However, formula (30) indicates that for currents close to $J_{C2}$ the fluctuation current is given by

$$I_{c2} = \left( \frac{1}{2} \right)^{1/2} e^{(1/2)^{1/2}}.$$ (31)

i.e., it is so insignificant that for all practical purposes one can regard the sample as existing in the purely normal phase. Thus, in pure metals and for $J < J_{C2}$, an electric field weakly influences the properties of the mixed-state layer. However, in a sufficiently small neighborhood of $J_{c2}$ any arbitrarily weak electric field leads to practically complete destruction of the layer and to the transition of the metal into the normal state. The width of this neighborhood decreases with increasing purity of the metal. Formula (29) enables us to trace the picture of the destruction in more detail. Let us write the total current through the sample as the sum of the normal current $I_{n} \approx J$ and the fluctuation current $2\pi r_{l} I_{s}$. According to Eq. (29) the derivative $\partial I_{s}/\partial J_{n}$ can be written in the form

$$\frac{\partial I_{s}}{\partial J_{n}} = \left( \frac{12}{b} \right)^{1/2} \left( \frac{2a}{3} \right)^{1/2} \left( \frac{3J_{C2} - J}{J_{C2}} \right) I_{s}.$$ (31)
and therefore the derivative of the total current through the sample with respect to the electric field is given by

\[
\frac{\partial I}{\partial E} = \frac{\partial I_0}{\partial E} \left\{ - \left( \frac{12}{5} \right) \left( \frac{2n_0}{3} \right) \sqrt{\frac{6 \pi n_0}{kT}} \right\} \frac{I_0 - I}{I_0}.
\]

The order of magnitude of the second term inside the curly brackets of the last equation is given by

\[
e^{-1/2}' \left( \frac{1}{w_{\text{bulk}}} \right) \frac{I_0 - I}{I_0},
\]

where \( R \) is the characteristic size of the sample. At the boundary of the range of validity of formula (29) from the weak current side, the superconducting current is of the order \( R \) is an important experimentally observable characteristic of the mixed state under consideration. In this connection the second term in Eq. (32) is of the order of \( R \) is a small correction proportional to the vector potential \( A(r, t) \) of the variable field. Since the expression \( \xi \varphi \) can be replaced by \( \delta \varphi \), we find \( \frac{\partial I}{\partial E} \) in the linear approximation with respect to the amplitude of the variable field, and using relations (12):

\[
\delta I_f = \frac{16\pi T}{c_\pi} \int \left\{ \frac{\partial I_0}{\partial E} \right\} \left\{ k_1 \cdot (\partial \varphi(x)) \right\} \left\{ \frac{\partial x(x)}{\partial \varphi(x)} \right\} \left\{ \frac{\partial \varphi(x)}{\partial x(x)} \right\} \delta \varphi.
\]

One can evaluate the integrals over \( k_1, k_2 \), and \( t' \) appearing in Eq. (33) by using the method of steepest descents:

\[
\delta I_f = \frac{16\pi T}{c_\pi} \int \left\{ \frac{\partial I_0}{\partial E} \right\} \left\{ k_1 \cdot (\partial \varphi(x)) \right\} \left\{ \frac{\partial x(x)}{\partial \varphi(x)} \right\} \left\{ \frac{\partial \varphi(x)}{\partial x(x)} \right\} \delta \varphi.
\]

Here \( k_1 = k_2 + c(t_1 - t) \), and \( k_0, \tau_0 \) are the parameters introduced above, determining the position of the saddle point and satisfying the conditions \( \lambda(k_0) = \lambda(k_0 - \tau_0) = 1 \). The restriction on the range of integration with respect to \( t_1 \) appears on the right-hand side of the last two equations, arises from the condition that the saddle point was located inside the region of integration.

The vector potential of the variable field depends on \( k_1, k_2 \), and \( t' \) appearing in Eq. (33) by using the method of steepest descents:

\[
\delta I_f = \frac{16\pi T}{c_\pi} \int \left\{ \frac{\partial I_0}{\partial E} \right\} \left\{ k_1 \cdot (\partial \varphi(x)) \right\} \left\{ \frac{\partial x(x)}{\partial \varphi(x)} \right\} \left\{ \frac{\partial \varphi(x)}{\partial x(x)} \right\} \delta \varphi.
\]

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Substituting here the solution (13) for the coefficients \( a_{k_1, k_2} \) and solving the inhomogeneous equation for \( \delta a_{k_1, k_2} \) just as above, we find

\[
\delta a_{k_1, k_2}(t) = -\frac{4e\pi}{c} \exp \left\{ \frac{1}{\tau} \int dt' \exp \left\{ \frac{1}{\tau} \int dt' \right\} \right\} \times \left\{ k_1 \cdot (A_1(t)) \right\} \left\{ k_1 \cdot (A_1(t)) \right\} \delta a_{k_1, k_2}(t),
\]

where \( k_1 = k_2 + c(t_1 - t) \).
the first formula of (34) is equal to zero since in this case
\[ A_z(x) = \text{const}, \quad \langle (k + \beta \mathbf{r}) \rangle = \frac{1}{2} \kappa_0 \frac{\partial}{\partial k}, \]
and the values of the function \( \lambda(k) \) are identical (equal to unity) at both limits of integration. Large (in absolute value), negative values of \( k_1 \) play a major role in calculating the following approximations. The function \( \psi_k(x) \) associated with these values of \( k_1 \) are localized in small neighborhoods of the points \( x = x_0 \equiv (|k|/|\beta|)^{1/2} \), and therefore \( A_{z}(x) \) may be taken outside of the average value sign if we set \( x = x_0(k) \) in it. The integral over \( k_1 \) arising after this has been done can be easily evaluated if it is taken into consideration that the equality \( \lambda(k) = 2(\beta|k|)^{1/2} \) holds for negative values of \( k_1 \) and thus we obtain
\[ \delta_{j_1} = \frac{2e}{c} I_0 A_z(0) |\psi_x(x)|^2 \left\{ \omega \frac{1}{2\pi} \beta, \frac{1}{4} \right\} - \frac{1}{\beta} \left( \frac{e}{\varepsilon_0} \right) \).

We determine the other component of the current in similar fashion:
\[ \delta_{j_1} = \frac{2e}{c} I_0 A_z(0) |\psi_x(x)|^2 \left( \frac{1}{\beta} \right) - \frac{1}{\beta} \left( \frac{e}{\varepsilon_0} \right). \]

The surface impedance \( Z \) is determined in the usual way as the ratio of the electric field \( E = (\omega/\varepsilon) A \) at \( x = 0 \) to the total variable current, integrated with respect to \( \xi \). Since this total current is equal to the sum of the normal and fluctuation currents, by introducing the impedance \( Z_0 \) of the normal metal we may write down the following expression:
\[ Z = 1 + \frac{cZ_0}{\omega A_0} \int_{-\xi}^{\xi} A_z(x) \xi dx. \]
If the frequency satisfies the condition $\Omega \ll 1 (\omega \ll \epsilon)$, then $I = 4/9$ and we have

$$\delta I = 2e \frac{2}{e} \left[ \frac{\partial}{\partial k} \langle \varphi (x) \rangle \right] - \frac{A_z (x) + \partial f}{2} \left( \frac{2b}{a} \Omega \right)^{\frac{3}{4}} A_t (0).$$

In similar fashion we find

$$\delta I = 2e \frac{2}{e} \left[ \frac{\partial}{\partial k} \langle \varphi (x) \rangle \right] - \frac{A_z (x) + \partial f}{2} \left( \frac{2b}{a} \Omega \right)^{\frac{3}{4}} A_t (0).$$

Integrating the obtained expressions with respect to $\xi dx$ and using Eq. (35), we obtain

$$\frac{Z_n}{Z_s} = 1 + 4e e_{i o} b_0 Z_n \left( \frac{2b}{a} \Omega \right)^{\frac{3}{4}} H \left( \frac{4}{e} + 1 + \frac{4}{e} \Delta \right).$$

We obtain the impendence for the other polarization of the field in a quite similar manner:

$$\frac{Z_n}{Z_s} = 1 + 4e e_{i o} b_0 Z_n \left( \frac{2b}{a} \Omega \right)^{\frac{3}{4}} H \left( \frac{4}{e} - 4 \theta e^{-\Omega} e^{\Omega} \Omega^{-\frac{1}{4}} \right).$$

Thus, at high frequencies of the variable field the surface impedance of the sample is an oscillating function of the parameter $2b \omega (\epsilon^2 - \epsilon^3)$ (in ordinary units) of the parameter $2b \omega (\epsilon^2 - \epsilon^3)$ (in ordinary units). The oscillations can therefore be observed in the presence of a variation of the current $J$ and a fixed frequency of the field.

6. THE SURFACE IMPEDANCE OF PURE METALS

Using the general formulas of section 4 and the results of section 3, we may calculate the surface impedance of pure metals for values of the current close to the second critical current. In this connection we shall treat the most interesting region $J < J_c 2$, $(J_c 2 - J)/J_c 2 \gg \epsilon^3$, where the fluctuating superconducting current is not small.

Near the second critical current we have

$$k_c - k_e + \left( \frac{2c}{b} \frac{I_c - I}{I_c} \right)^{\frac{1}{2}}, \quad \Omega = \frac{8b}{b} \frac{I_c - I}{I_c} \left( \frac{2b}{a} \Omega \right)^{\frac{3}{4}} A_t (0).$$

In addition, the integration in formula (34) takes place over an extremely narrow region near $k_m$. We can therefore take the average values $1 / \theta$ outside of the integral sign, having set $k = k_m$ in them. After doing this the remaining integrals can easily be evaluated. Integrating at once with respect to the coordinate $x$, we obtain the following formulas for the impedance:

$$Z_n = 1 + 4e e_{i o} b_0 \left( \frac{\partial}{\partial k} \langle \varphi (x) \rangle \right) - \frac{A_z (x)}{A_t (0)} \left( \frac{\partial}{\partial k} \langle \varphi (x) \rangle \right) - \frac{2b}{a} \left( \frac{I_c - I}{I_c} \right)^{\frac{1}{2}}.$$
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