The isotropization of homogeneous cosmological models

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A complete solution of the Einstein gravitational equations is constructed for anisotropic cosmological models with a hydrodynamic energy-momentum tensor. The isotropization process of the solution is followed through the expansion. It is shown that agreement with present-day astronomical observations is possible only if the expansion is isotropized at an extraordinarily early stage. In the type VII Bianchi models the isotropization of the expansion is accompanied by an isotropization of the curvature also. In models of the Bianchi type IX the solution becomes, in general, isotropic with respect to the deformation, but remains sharply anisotropic with respect to the curvature tensor of three-dimensional space. The problem of isotropy of the background radiation is considered.

1. INTRODUCTION

Numerous recent articles have been devoted to the analysis of the dynamics of the expansion of anisotropic homogeneous cosmological models near their singularity and to the consideration of the physical processes during that stage of their development (cf., e.g.,[1-11]). Anisotropic cosmological solutions have been considered as possible models for the beginning of the expansion of the universe. It is known that some of the anisotropic models are "isotropized" in the course of their expansion, i.e., they tend to Friedmann's isotropic solution. The most important problem is the construction of a solution for the anisotropic models valid throughout the whole time interval, i.e., to be able to describe the whole process of isotropization and to relate the initial data near the singularity to parameters at a later stage of the expansion. This will allow one to determine which of the models with parameters specified near the singularity agrees with the data from astronomical observations regarding the degree of isotropy of the cosmological expansion, and thus, will allow to decide the question of the generality of the class of initial conditions for anisotropic models can lead to the picture of the universe which is being observed now.

In resolving the problem of isotropization one must keep in mind that, according to astronomical observations, if the real universe really was originally anisotropic, the isotropization occurred sufficiently early: at the stage when the linear dimensions were at least a thousand times smaller than the present ones: $z_F \sim 10^3$ (from observations of the background radiation), and, very likely, earlier $z_F \sim 10^9$ (from data on the chemical composition of matter[12]; here z_F is the red shift). Thus, the isotropization must occur at a stage when the expansion takes place practically with parabolic speed and the Friedmann model is described by a solution corresponding to critical density. By isotropization we mean that the solution approaches an asymptotic behavior such that the expansion occurs uniformly in all directions (up to small corrections) and is described by Hubble's law.

We note that in the literature one can find other definitions of isotropization. Thus, S. P. Novikov[11] defines isotropization for closed models of the Bianchi type IX as closeness to each other of the principal values of the curvature tensor at the instant of maximal volume of the model. Such a definition of isotropization does not lead, in particular, to isotropy of the temperature of the background radiation in the model, and is therefore insufficient from an astronomical point of view. In other papers (cf., e.g., the paper of Collins and Hawking[13]) any approach to the Friedmann solution for $t \to \infty$ is considered as an isotropization. Such a definition is also insufficient from an astronomical point of view, since a late isotropization, when the model approaches the Friedmann solution at a stage, since a late isotropization, where the model approaches the Friedmann solution at a stage where the gravitation of matter no longer influences the dynamics (the Milne stage), is not compatible with the observations.

In the present paper we investigate the process of isotropization for models of the Bianchi types VII, VIII and IX. These models are the most general ones in the class of homogeneous anisotropic models[4,13] and are therefore of the greatest interest. For the simpler special cases of the models of types V and I the process of isotropization was considered before[14,15]. We note that the model of type VII may approach asymptotically the open or the flat Friedmann model; in the models of the other types there may exist lengthy stages of evolution when, with respect to many parameters, the solution corresponds to the Friedmann solution (e.g., the expansion is almost isotropic) and then it again deviates from it.

In the sequel we shall assume that the matter is at rest with respect to a homogeneous synchronous coordinate system (1) and is described by the hydrodynamic equation of state $p = \epsilon/3$ at the early stages and $p = 0$ at the late stages of expansion.

The occurrence of directed flows of matter and free particles in the homogeneous models[16,17] is considered separately. These phenomena which can appear at an early stage of cosmological expansion, lead to such a kinematics of expansion which leads to their damping out and to appearance of isotropization at a later stage, as described in this article (with small corrections).

2. ISOTROPIZATION OF MODELS OF TYPE VII

The metric of the homogeneous model can be written in the form (the Greek indices and the tetrad indices take on the values 1, 2, 3)

$$g_{00} = 1, \quad g_{0\alpha} = 0, \quad g_{\alpha\beta} = -\gamma_{ab} e_\alpha^a e_\beta^b; \tag{1}$$

the matrix $\gamma_{ab}(t)$ determines the time dependence of the metric. For $e_\alpha^a = e_\alpha^a(x^\beta)$ we have (x^α denote the spatial coordinates):

$$e_a^\alpha e_b^\beta (\partial_\beta e_\alpha^c - \partial_\alpha e_\beta^c) = C^c_{\cdot ab}. \tag{2}$$

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For the metric of the Bianchi type VII only the following structure constants are nonzero:

$$C^1_{\cdot 23} = C^2_{\cdot 31} = -1, \quad C^1_{\cdot 13} = C^2_{\cdot 23} = a \geqslant 0. \quad (3)$$

We assume that the matter is at rest in the synchronous reference system (1) with space being homogeneous (the four-velocity is $u^i = (1, 0, 0, 0)$). Then, without loss of generality, one may set $\gamma_{13} = \gamma_{23} = 0$. It is convenient to investigate the evolution of the model in a coordinate system where the matrix γ_{ab} is diagonalized. The transition to a new coordinate system is carried out by means of a rotation in the 1—2 tetrad plane by a time-dependent angle $\varphi(t)$:

$$\|\gamma_{ab}\| = S \begin{Vmatrix} \lambda_1(t) & 0 & 0 \\ 0 & \lambda_2(t) & 0 \\ 0 & 0 & \lambda_3(t) \end{Vmatrix} S^T, \quad S = \begin{Vmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix} \quad (4)$$

(here the superscript T denotes transposition).

Under the above assumptions the Einstein equations reduce to the form (with the Einstein constant and the speed of light set equal to one):

$$(\ln\sqrt{\lambda_1\lambda_2})^{\cdot\cdot} - 4a^2\lambda_1\lambda_2 = \gamma(\varepsilon - p),$$

$$(\ln\lambda_3)^{\cdot\cdot} - 4\lambda_1\lambda_2\,\mathrm{sh}^2\frac{\mu}{2} - 4a^2\lambda_1\lambda_2 = \gamma(\varepsilon - p),$$

$$\ddot\mu + 4\,\mathrm{sh}\,\mu\,(\lambda_1\lambda_2 - \dot\varphi^2) = 0, \quad \left(\dot\varphi\,\mathrm{sh}^2\frac{\mu}{2}\right)^{\cdot} = 2a\lambda_1\lambda_2\,\mathrm{sh}^2\frac{\mu}{2},$$

$$\frac{\dot\lambda_1}{\lambda_1}\frac{\dot\lambda_2}{\lambda_2} + \frac{\dot\lambda_2}{\lambda_2}\frac{\dot\lambda_3}{\lambda_3} + \frac{\dot\lambda_3}{\lambda_3}\frac{\dot\lambda_1}{\lambda_1} = 12a^2\lambda_1\lambda_2 + 4\,\mathrm{sh}^2\frac{\mu}{2}(\lambda_1\lambda_2 + \dot\varphi^2) + 4\gamma\varepsilon, \quad (5)$$

$$a\left(\ln\frac{\lambda_3}{\sqrt{\lambda_1\lambda_2}}\right)^{\cdot} = 2\dot\varphi\,\mathrm{sh}^2\frac{\mu}{2};$$

$$d\tau = \frac{dt}{\gamma^{1/2}}, \quad \gamma = \lambda_1\lambda_2\lambda_3, \quad \mu = \ln\frac{\lambda_1}{\lambda_2}$$

(the last two equations are first integrals of the equations (5)). The dots denote differentiation with respect to the time τ.

The energy density ϵ and the pressure p are related via the equation of state. As is well known, for the equation of state $p = \epsilon/3$, $\epsilon = L_1^2/\gamma^{2/3}$, and for $p = 0$, $\epsilon = L_2$ $\epsilon = L_2/\gamma^{1/2}$. The constants L_1 and L_2 have the dimension of length and are related to the instant up to which one may neglect the gravitational influence of matter on the evolution of the model. This early stage of expansion is called the "vacuum" stage and is characteristic for all homogeneous anisotropic models. We shall consider the constants L_1 and L_2 to be so large that the isotropization process should occur sufficiently early. As was remarked above, only under these conditions are the anisotropic models compatible with observations.

The constant a is the parameter of the model (3) and determines the beginning of that stage of the evolution when a principal role is played by the isotropic part of the curvature of the corresponding three-dimensional space: $-4a^2\lambda_1\lambda_2$ and the influence of the gravitation of matter can be neglected. From observations it is known that the term $-4a^2\lambda_1\lambda_2$ can become important only during the contemporary epoch. The requirement that there exist an isotropic stage of evolution when gravitation of matter plays an important role leads to the necessary condition (with the exception of the degenerate case $\lambda_1 \equiv \lambda_2$; cf. Appendix II):

$$a \ll 1 \quad (6)$$

Below we shall assume that this condition is satisfied.

Thus, the evolution of the model of type VII can be divided (by convention) into three large eras:

1. The vacuum stage, when matter gravitation and the term $4a^2\lambda_1\lambda_2$ do not have any influence.

2. The isotropization stage, when it is necessary to take into account matter gravitation, but one may still neglect the term $4a^2\lambda_1\lambda_2$.

3. The late stages of expansion, when the evolution is determined by the term $4a^2\lambda_1\lambda_2$ and matter does not affect the metric. At this stage it is necessary to take into account the rotation of the coordinate system which is oriented along the principal axes of the matrix γ_{ab}. We consider consecutively the character of the evolution of the type VII model.

In the vacuum stage the solution of the system (5) is well known. In the immediate vicinity of the singularity the Kasner solution[2,4,5] holds:

$$\lambda_a \sim t^{2p_a}, \quad \gamma^{1/2} = \Lambda_0 t, \quad p_3 < 0,$$

here p_a are the Kasner exponents:

$$\sum p_a = \sum p_a^2 = 1, \qquad \Lambda_0 = \mathrm{const.}$$

Then, under the influence of the anisotropic component of the curvature tensor, λ_1 and λ_2 begin to oscillate and λ_3 increases monotonically. This stage of evolution of the model under consideration is similar to the evolution of the "mixmaster universe" over one large period. The amplitude of oscillations of the functions λ_1 and λ_2 is in general large and decreases gradually with time. In the intervals between the maxima of the largest function the evolution corresponds again with a high degree of accuracy to the Kasner solution (the curvature terms λ_1^2 and λ_2^2 are not essential), the Kasner epochs taking up most of the evolution time.

In the most general case the amplitude of oscillations of the functions λ_1 and λ_2 is large at the time $t \sim t^*$ ($\tau \sim \tau^*$) when the influence of the term $\gamma(\epsilon - p)$ in (5) becomes essential. In this case the oscillations of λ_1 and λ_2 (with large amplitudes) stop under the influence of matter gravitation (cf. Appendices I and II). If the influence of matter begins to show up far from the maximum of the functions λ_1, λ_2 in the Kasner stage, there exists a long enough stage when $\gamma\epsilon \gg \lambda_1^2, \lambda_2^2$ and in this stage the equation of state $p = \epsilon/3$ leads to the following solution of the system (5)

$$\lambda_a = A_a \exp\left[2\Lambda\left(p_a - \frac{1}{3}\right)(\tau - \tau_0)\right]\mathrm{sh}^{-2}\frac{\Lambda}{3}(\tau - \tau_0)$$

$$\gamma\varepsilon = \frac{1}{3}\Lambda^2\,\mathrm{sh}^{-2}\frac{\Lambda}{3}(\tau - \tau_0). \quad (7)$$

For $\Lambda(\tau - \tau_0) \ll -1$ the solution (7) goes over into the Kasner solution with exponent p_a. The constants A_a, Λ, τ_0 are determined by smooth matching with the Kasner section of the vacuum solution which describes the oscillations of λ_1 and λ_2. The equations (7) describe an expansion which tends to isotropic expansion for $\Lambda(\tau - \tau_0) \to 0$. The tempo of the expansion is the same as in the Friedmann solution with critical density:

$$\lambda_a \sim t \sim (\tau - \tau_0)^{-2} \quad (\Omega = 12\gamma\varepsilon/(\ln\gamma)^{\cdot 2} \approx 1),$$

however, the quantities λ_1 and λ_2 are in general substantially different from one another. (The relative magnitude of λ_3 is not important, since Eqs. (5) admit the transformation $\tilde\lambda_3 = n\lambda_3$, $\tilde\epsilon = n^{-1}\epsilon$, n = const.)

Any solution of (5) under the condition $\lambda_1 \gg \lambda_2$ at $\tau \sim \tau^*$ tends to[1)]

$$\lambda_1 = \frac{1}{2}\sqrt{3}(\tau_1 - \tau)^{-1}, \qquad \lambda_2 = B_2(\tau_1 - \tau)^{-5/3},$$
$$\lambda_3 = B_3(\tau_1 - \tau)^{-5/3}, \qquad \gamma\varepsilon = \frac{21}{8}(\tau_1 - \tau)^{-2}. \tag{8}$$

The constants B_2, B_3, τ_1 are determined by matching the solution to (7). At this stage (8) the volume changes like in the Friedmann solution: $\gamma \sim t^3$. This may turn out to be important, e.g., for the synthesis of chemical elements, which depends on the speed with which the volume changes. The stage (8) starts at the time when $2\gamma\epsilon/7\lambda_1^2$ becomes of the order of unity (cf. Appendices I, II). Therefore the duration of the period (7) is the longer, the smaller the ratio λ_1/λ_2 for $\tau \sim \tau^*$ ($\frac{1}{3}\Lambda(\tau_0 - \tau^*) \sim 1$). If at the instant $\tau \sim \tau^*$ the quantity λ_1 is near its maximum, then without the intermediate asymptotic behavior (7) the solution goes over into the stage (8) ($p = \epsilon/3$)[2)]. The quantities λ_2 and λ_3 increase with time substantially faster than λ_1, and the stage (8) ends for $\lambda_1 \sim \lambda_2$ ($\mu \sim 1$). Damped oscillations of λ_1 and λ_2 occur again, but now with a smaller amplitude.

It is essential that the amplitude of the first oscillation $\mu_{max} \approx 0.36$ depends weakly on the initial parameters of the model. At the same time the initial parameters of the model are practically forgotten and in the most general case we have a single parameter depending on the initial conditions, $t_F(\tau_F)$ the instant where the isotropic stage begins, stage which is properly speaking the result of the isotropization. The evolution of the model at this stage, for an equation of state $p = \epsilon/3$ is described by the equations:

$$(\lambda_1\lambda_2)^{1/2} \approx -\frac{\tau_F}{\tau^2}N(\tau), \qquad \varepsilon\gamma \approx \frac{3}{\tau^2}\left[1 - \frac{1}{8}N(\tau)\right],$$
$$\langle\mu\rangle \approx \frac{\tau}{\tau_F}N^{-1/2}(\tau), \qquad N(\tau) = \frac{4}{\ln(\tau_F e^4/\tau)}, \qquad \mu \ll 1, \tag{9}$$

where $\langle\mu\rangle$ denotes averaging over the oscillation period, and the constant which determines the time shift in $\tau(\tau_1)$ has been set equal to zero.

The equation (9) is valid up to the change of the equation of state. Let us note several interesting peculiarities of the solution (9).

1. The principal peculiarity of the solution (9) is the friedmannian time dependence of the volume $\gamma \sim t^3$. Thus, throughout the second stage of evolution of the model, $t > t^*$, when matter gravitation plays an essential role, the density of matter changes according to the Friedmann solution:

$$\varepsilon \approx 3/4(t + t_0)^2$$

in the solution (7);

$$\varepsilon \approx 21/32t^2$$

in the solution (8) and

$$\varepsilon \approx 3\left[1 - 1/\ln\left(\frac{t}{t_F}e^8\right)\right] \Big/ 4t^2$$

in the solution (9).

2. In the solution (9) the anisotropy of the deformation tensor related to the oscillations of the quantity μ, and in the final count, to the anisotropy of curvature, decreases very slowly, proportionally to $(\ln t)^{-1}$. This means that in the models under consideration the anisotropy of the microwave background radiation related directly to the anisotropy of the deformation will be a much more sensitive "test," than, for instance, the chemical composition of prestellar matter.

During the late stages of expansion it is necessary to go over to the equation of state $p = 0$. In Subsection 1 above we gave the solution for a mixture of radiation ($p = \epsilon/3$) and nonrelativistic matter ($p = 0$). After the instant $t_c(\tau_c)$ determined by the equality of the densities of matter and radiation, all deviations from an exact isotropic expansion law are rapidly damped out according to a power law and the model approaches the Friedmann solution ($\Omega = 1$) for dust:

$$(\lambda_1\lambda_2)^{1/2} \approx -\frac{\tau_F}{\tau_c^{2/3}\tau^{4/3}}N_c, \qquad \langle\mu\rangle \approx \frac{\tau_c^{1/3}\tau^{2/3}}{\tau_F}N_c^{-1/2};$$
$$N_c = N(\tau_c) = \frac{8}{\ln(t_c e^8/t_F)} \tag{10}$$

The stage (10) continues up to the instant $t_M(\tau_M)$ when the term $4a^2\lambda_1\lambda_2$ becomes of order $\epsilon\gamma$, and is then followed by the concluding third stage of evolution of the model under consideration, determined by the isotropic curvature $4a^2\lambda_1\lambda_2$. It is important that at the instant t_M the oscillations of μ have already died out (cf. Subsection 2) and

$$\mu_{tM} = \mu_0 \approx a\left(\frac{t_c}{t_M}\right)^{1/6} N_c^{1/2} \ll a. \tag{11}$$

During the final stage any solution of the system (5) for $t > t_M$ approaches the solution (12) (for arbitrary $a > 0$):

$$\lambda_3^{1/2} = a(1 + \varkappa^2)t, \quad (\lambda_1\lambda_2)^{1/2} = \mathcal{P}t^{2/(1+\varkappa^2)},$$
$$\varphi = \frac{\ln t}{a(1 + \varkappa^2)} + \varphi_0 \tag{12}$$
$$\varkappa = \operatorname{sh}\frac{\mu}{2}\Big/ a = \text{const} < 1,$$

$\mathcal{P}$ and φ_0 are constants; the constant which determines the shift of time has been omitted. In our case

$$\varkappa^2 \approx \frac{\mu_0^2}{4a^2} \approx \left(\frac{t_c}{t_M}\right)^{1/3} N_c \ll 1, \qquad \Omega = \left(\frac{t_M}{t}\right)^{(1-\varkappa^2)/(1+\varkappa^2)}. \tag{13}$$

The parameter a of the model determines the instant (t_M) up to which the gravitational influence of matter is important, in particular, in order that there exist an isotropic stage $\Omega \approx 1$ (9), (10), it is necessary that the condition $a \ll 1$ be satisfied. However, after the transition to the concluding stage of the evolution (12) the value of the quantity a loses its meaning (physical meaning is possessed by the parameter κ) and by means of a scale transformation of the coordinates x^α for $a \neq 0$ one can always reduce a to a convenient form (e.g., to a $a = 1/(1 + \kappa^2)$). If $a = 0$, the stage (12) is absent and $\mu \to 0$ for $\gamma \to \infty$, cf. (10).

The quantity κ carries information on the duration of the isotropic stage (cf. (13)). If $t_M \sim t_F$ (the isotropic stage is practically absent; $a \sim 1$) then $\kappa \lesssim 1$. At the stage when the expansion of the Universe is determined by radiation ($p = \epsilon/3$), the quantity κ decreases logarithmically and after the change of the equation of state ($p = 0$) κ decreases according to a power law (13). Thus, the requirement of a sufficiently long isotropic stage (9), (10) leads to the condition $\kappa \ll 1$.

It is important to note that the solution (12) does not tend to the Friedmann solution but rather represents a "frozen" perturbation of the Friedmann solution in the Milne stage ($\kappa = 0$). Indeed, both the perturbation of the curvature, μ, and the relative difference of the Hubble "constants" κ^2 are constant in this case. It is true, that the ratio $\lambda_3/(\lambda_1\lambda_2)^{1/2}$ increases without bound for $t \to \infty$, but the equations (5) admit a scale transformation of λ_3 and by itself an increase of this ratio does not yet mean an increase of the deviations from the Friedmann solution. More precisely, at any instant of time $t > t_M$ the model is described by a Friedmann solution plus small corrections[3)]. In Sec. 4 it will be shown that the anisotropy of the microwave background radiation is also small[4)].

3. THE ISOTROPIZATION OF MODELS OF TYPES IX AND VIII

Let us consider the Bianchi models of type IX and VIII. In this case the following structure constants (2) are different from zero:

$$C^1_{\cdot 23} = C^2_{\cdot 31} = -1, \quad C^3_{\cdot 12} = -\delta \tag{14}$$

($\delta = 1$ for the type IX model and $\delta = -1$ for the type VIII model). We retain the same conventions regarding matter as in Section 2 ($u^\alpha = 0$). Then, without loss of generality, one may set $\gamma_{12} = \gamma_{23} = \gamma_{13} = 0$. The equations of gravity have the form

$$\begin{aligned}
&(\ln\lambda_1)^{\cdot\cdot} + \lambda_1^2 - (\lambda_2 - \delta\lambda_3)^2 = \gamma(\varepsilon - p),\\
&(\ln\lambda_2)^{\cdot\cdot} + \lambda_2^2 - (\lambda_1 - \delta\lambda_3)^2 = \gamma(\varepsilon - p),\\
&(\ln\lambda_3)^{\cdot\cdot} + \lambda_3^2 - (\lambda_1 - \lambda_2)^2 = \gamma(\varepsilon - p),
\end{aligned} \tag{15}$$

$$\frac{\dot\lambda_1}{\lambda_1}\frac{\dot\lambda_2}{\lambda_2} + \frac{\dot\lambda_2}{\lambda_2}\frac{\dot\lambda_3}{\lambda_3} + \frac{\dot\lambda_3}{\lambda_3}\frac{\dot\lambda_1}{\lambda_1} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2 - 2\delta\lambda_3(\lambda_1 + \lambda_2) + 4\gamma\varepsilon.$$

In the early "vacuum" state, when matter gravitation is unimportant, the expansion exhibits an oscillatory character, described in detail in refs.[2-5].

In the general case the term $\gamma(\epsilon - p)$ starts influencing the solution at the instant when all λ_a are different. We relabel them in order of their magnitude at that instant ($t \sim t^*$):

$$\lambda_1 \gg \lambda_2 \gg \lambda_3. \tag{16}$$

It can be seen from (15) that as long as λ_3 is sufficiently small:

$$\lambda_3 < (\lambda_1\lambda_2)^{1/2}\left\langle \mathrm{sh}^2\frac{\mu}{2}\right\rangle \Big/ \mathrm{ch}\frac{\mu}{2}, \tag{17}$$

the whole process of isotropization takes place exactly in the same way as in the model of type VII ($a = 0$) ((7)–(10). If (16) is valid the condition (17) can be violated only during the isotropic stage (9), (10) for $|\mu| \ll 1$. In this case, after the following condition is satisfied

$$\lambda_3/(\lambda_1\lambda_2)^{1/2} \sim \langle\mu^2\rangle \ll 1,$$

there appears a solution which is a perturbation of the axially symmetric solution of the system (15), $\lambda_1 = \lambda_2 \gg \lambda_3$.

For the equation of state $p = \epsilon/3$ under the condition $\lambda_1 = \lambda_2 = \lambda \gg \lambda_3$, the solution of Eq. (15) has the form

$$\varepsilon\gamma = \frac{\lambda\lambda_3}{3\omega^2} = 3c^2\omega(\alpha + \omega - \delta\omega^3), \qquad \left(\frac{\lambda}{\lambda_3}\right)^{1/2} = Q\frac{\alpha + \omega - \delta\omega^3}{\omega}, \tag{18}$$

where $c > 0$, α, $Q \gg 1$ are integration constants and the function $\omega = \omega(\tau) > 0$ is determined from the equation

$$(\ln\omega)^{\cdot} = c(\alpha + \omega - \delta\omega^3). \tag{19}$$

A characteristic peculiarity of this solution is the presence of a long stage of evolution (for $|\alpha| \ll 1$) where the expansion takes place according to Friedmann laws, but the ratio $\lambda_3/\lambda_1 \ll 1$ and is approximately constant[5)], i.e., the most general conditions at the instant when matter gravitation becomes important, $t \sim t^*$ (cf. (16)), do not lead to a special equalization of all three principal values of the curvature tensor in the models of types IX and VIII.

In the general case there appears an isotropic stage with $\lambda_3 \ll \lambda_1 \approx \lambda_2$, the duration of which is determined by only one constant—the ratio $(\lambda_1\lambda_2)^{1/2}/\lambda_3 = Q_F^3$ at the instant when the isotropic stage starts $t \sim t_F$ (t_M is the instant when the isotropic stage ends ($\omega \sim 1$); $p = \epsilon/3$):

$$\left(\frac{\gamma_M}{\gamma_\Phi}\right)^{1/6} = \frac{\tau_F}{\tau_M} \approx \frac{Q_F^{3/2}}{\ln^{1/2} Q_F} \gg 1, \qquad Q \approx \frac{Q_F}{\ln Q_F} \gg 1, \quad |\alpha| \lesssim (\ln Q_F)^{-3/2} \ll 1. \tag{20}$$

In the model of type IX the instant t_M coincides with the time of maximum volume[6)]. As can be seen from the solutions (18) and (19), the maximum of the volume is followed immediately by the vacuum stage and the solution goes over into the Kasner solution with exponents

$$p_1 = p_2 = -2/3, \quad p_3 = 1/3.$$

This solution is applicable up to the maximum of λ_3, as long as the curvature related to λ_3 does not affect the solution (cf. Fig. 1). At the instant of maximal λ_3 we have:

$$\frac{\lambda_3}{(\lambda_1\lambda_2)^{1/2}} \approx \left(\frac{Q_F}{\ln Q_F}\right)^{3/2} \gg 1. \tag{21}$$

After that the function λ_3 decreases, becomes smaller than λ_1, λ_2 and a stage of small-amplitude oscillations of λ_1, λ_2 starts, as described in[2]. The number of oscillations of λ_1, λ_2 during the first large cycle, while λ_3 decreases monotonically, is

$$u_0 \sim (Q_F^9/\ln^7 Q_F)^{1/4} \gg 1.$$

In the sequel, as it approaches the singularity, the solution becomes oscillatory with large amplitudes, like in the vicinity of the first singularity. For the equation of state $p = 0$ the solution of Eqs. (14) for the condition $\lambda_1 = \lambda_2 \gg \lambda_3$ has a more complicated form, however the qualitative peculiarities of the solution are similar to those described above (cf. for more details[19,20]). It is easy to estimate the path followed by light along the principal directions of the deformation tensor during the isotropic stage ($\alpha = 0$):

$$L_\lambda = 1/2\int(\lambda\lambda_3)^{1/2}\,d\tau = 3/2\int_0^1 (1 - \delta\omega^2)^{-1/2}\,d\omega = \begin{cases} 3/4\,\pi & \text{for IX type} \\ 3/2\sinh^{-1} 1 & \text{for VIII type}\end{cases}$$

$$L_{\lambda_3} = 1/2\int\lambda\,d\tau \approx Q^{3/2}L_\lambda. \tag{22}$$

(We recall that in the closed Friedmann model ($\lambda = \lambda_3$; $\delta = 1$) up to the maximum of γ the light traverses a path $L_\Phi = \pi/2$.)

In the model of type VIII the gravity of matter becomes inessential after the time t_M, (18),[7)] and the model evolves asymptotically to the solution (23) ($p = \epsilon/3$):

$$(\lambda_1\lambda_2)^{1/2} \approx t^2, \quad \lambda_3 \sim \ln t. \tag{23}$$

For the equation of state $p = 0$ the matter gravitation ($t > t_M$) has a "logarithmic" influence on the metric:

$$(\lambda_1\lambda_2)^{1/2} \approx t^2, \quad \lambda_3 \sim \ln^2 t, \quad \varepsilon \sim (t^2\ln t)^{-1}. \tag{24}$$

The most important conclusions are the following.

First, for arbitrary initial conditions the gravitation of matter does not automatically lead to an isotropization of the solution where all λ_a are practically equal to each other for a lengthy time. For such an isotropization a special selection of initial conditions is necessary.

Secondly, for arbitrary initial conditions (within a

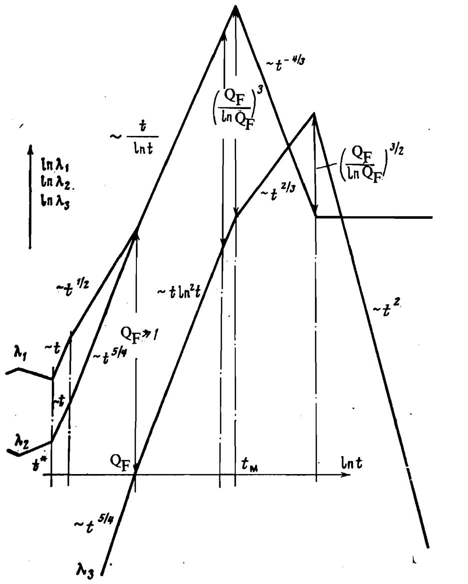

FIG. 1. The evolution of a model of type IX ($p = \epsilon/3$).

wide range) the solution exhibits a lengthy period where the expansion is practically isotropic (with small corrections) and is well described by the Friedmann solution for critical density. However, the values of the curvature tensor remain different and $\lambda_1 \approx \lambda_2 \gg \lambda_3$.

4. THE ANISOTROPY OF THE BACKGROUND RADIATION

We now turn to the anisotropy of the background radiation during the late stages of expansion. We discuss this problem in detail in[21], here we list only the main conclusions. Starting from some instant of time t_1 ($t_1 > t_c$) matter becomes transparent for the microwave background radiation and the photons propagate freely. The different velocities of expansion in different directions leads, after matter has become transparent, to an anisotropy of the temperature of the background radiation for an observer at rest with respect to the matter.

As was shown above, the anisotropy of deformation of the model during the late stages practically does not depend on the initial conditions. Therefore the amplitude of anisotropy of the background radiation depends only on the instant t when the anisotropic stage (9) begins, on the instant t_c when the equation of state changes and on t_0. The amplitude of anisotropy of the temperature of the background radiation in the isotropic stage for $\Omega \approx 1$ is related to the anisotropy of the deformation tensor and is determined by the formula

$$\Delta T/T \approx (t_c/t_i)^{2/3} N_c. \tag{25}$$

A peculiarity of this formula is its extremely weak dependence on the time t_F of isotropization of the model. It follows from (25) that if one adopts for the instant of matter becoming transparent the same value as the time of recombination and assumes that $\Delta T/T \sim 10^{-3}$, then the instant of isotropization will be close to the quantum theory limit of validity of the general theory of relativity $t_g \sim 10^{-44}$ sec.

The angular distribution of the intensity of the background radiation on the celestial sphere depends on the anisotropy of deformation along null-geodesics[22] and varies strongly from model to model. In the models considered by us the simplest structure of the anisotropy of the background is found in the model of type IX (VIII). In this model light propagates along the principal directions of the deformation tensor, and therefore the background anisotropy has a quadrupole character.

An interesting peculiarity of the model discussed above (Sec. 3) is the condition $\lambda_3 \ll \lambda_1 \approx \lambda_2$ near the maximum of expansion. This allows the light during the late stages of expansion to manage to go a round the Universe many times along the direction λ_3. The number of times the light goes around is determined by the equation

$$N_0 \approx (\lambda_1/\lambda_3)^{1/2} \approx Q^{3/2} \gg 1.$$

The astronomical consequences of this will be discussed separately.

In models of type VII the distribution of anisotropy of the background is much more complicated. If the observations are carried out at a density which is close to the critical value, when the equations (10) are still applicable, the angular dependence of the background anisotropy is close to quadrupolar. In the sequel, if $a \neq 0$, the quadrupole character of the background anisotropy is destroyed, and similar to the case of the model of type V[15] a spot begins to form.

In distinction from the models of the type V, in the case under consideration the anisotropy possesses an axial symmetry with respect to λ_3. At the stage (12) the temperature anisotropy of the background radiation related to the differences in expansion velocities along the axes $\lambda_{1,2}$ and λ_3 is given by the equation

$$\frac{\Delta T}{T} \approx \varkappa^2 \left\{ \frac{\sin^2 \bar{\varphi}}{1 + \cos \bar{\varphi} \cdot \operatorname{th} \ln (t/t_0)} \right\} \operatorname{th} \ln \left(\frac{t}{t_0} \right), \tag{26}$$

where $\bar{\varphi}$ is the angle between the ray of vision and the direction of the minimum of T. In Eq. (26) it is assumed that at the stage (12), at the time $t_0 > t_M$ the ratio $\Delta T/T = 0$. For $t \gg t_0$, as can be seen from (26), the size of the spot is $\Delta\bar{\varphi} \sim t_0/t \ll 1$. At this stage the maximal anisotropy (owing to different speeds of expansion along the axes λ_3 and $\lambda_{1,2}$) is of the order κ^2.

In the general case two effects contribute to the anisotropy of the background radiation: $\Delta T/T$ in the stage $\Omega \approx 1$ (25) and $\Delta T/T \sim \kappa^2$ in the stage (12) since $t_1 \ll t_M$ the maximal anisotropy is determined by Eq. (25) with the angular distribution from (26), $t_0 \sim t_M$ (up to the time t_M the angular distribution has a quadrupolar character):

$$\frac{\Delta T}{T} \approx \varkappa^2 \left(\frac{t_M}{t_i} \right)^{2/3} \begin{cases} \sin^2 \bar{\varphi} & t < t_M \\ \dfrac{\sin^2 \bar{\varphi}}{1 + \cos \bar{\varphi} \operatorname{th} \ln (t/t_M)}, & t > t_M \end{cases} \tag{27}$$

We stress that $\Delta T/T \to \text{const} \ll 1$ for $t \to \infty$; this conclusion differs from the conclusion of Collins and Hawking[13], who have considered that the anisotropy of the background radiation tends to infinity.

APPENDIX I

We consider the Eq. (5) with the condition $a = 0$. In this case $\dot{\varphi} = 0$, and by means of a rotation of the tetrad (2) in the 1—2 plane one can always select $\varphi = 0$.

In the most general case at the instant $\tau \sim \tau^*$, when the gravitation of matter begins to affect the metric,

$$\lambda_1 \gg \lambda_2. \tag{I.1}$$

The system (5) with the condition (I.1) reduces to the form ($p = \epsilon/3$):

$$(\ln z)^{\cdot\cdot} = z + y, \qquad (\ln y)^{\cdot\cdot} = 2z - 6y; \tag{I.2}$$

where

$$z = \gamma(\varepsilon - p) = {}^2/_3\varepsilon\gamma, \quad y = \lambda_1^2/3.$$

For $z \gg y$, i.e., in the case when the gravitation of matter becomes substantial far from the maximum of the function λ_1, the solution of (I.2) (cf. (7)) is

$$z = \frac{2\Lambda^2}{9\,\mathrm{sh}^2\,{}^1/_3\Lambda(\tau - \tau_0)}, \qquad y = \frac{A_1^2}{3}\frac{\exp[4\Lambda(p_1 - {}^1/_3)(\tau - \tau_0)]}{\mathrm{sh}^4\,{}^1/_3\Lambda(\tau - \tau_0)}. \tag{I.3}$$

After the conclusion of the vacuum stage (${}^1/_3\Lambda(\tau^* - \tau_0) \sim -1$) we have the isotropic Friedmann stage ($\Omega = 1$):

$$z \approx \frac{2}{(\tau - \tau_0)^2}, \quad y \approx \frac{A^2}{(\tau - \tau_0)^4}, \quad A = \frac{3\sqrt{3}}{\Lambda^2}A_1. \tag{I.4}$$

We shall show that every solution of (I.2) approaches the asymptotic behavior (cf. (9):

$$z = \frac{7}{4(\tau - \tau_1)^2}, \quad y = \frac{1}{4(\tau - \tau_1)^2}, \quad \tau_1 = \text{const}. \tag{I.5}$$

We introduce the notations

$$\zeta = \ln(7y/z), \quad d\xi = z^{1/2}d\tau. \tag{I.6}$$

Then (a dash denotes the derivative with respect to ξ)

$$\begin{gathered}\zeta'' + {}^1/_2(\ln z')\zeta' + (e^\zeta - 1) = 0,\\ E(\zeta, \zeta') = {}^1/_2\zeta'^2 + (e^\zeta - \zeta) = \text{const} - {}^1/_2\int \zeta'^2(\ln z)'\,d\xi.\end{gathered} \tag{I.7}$$

Further, the condition $(\ln z)^{\cdot\cdot} > z$, (I.2) yields

$$(\ln z)' = \frac{\dot z}{z^{3/2}} > \frac{\dot z}{z(z + 2(\Lambda_0/3)^2)^{1/2}} > 2^{1/2}. \tag{I.8}$$

Thus, the equation (I.7) describes damped oscillations with amplitude tending to zero. Indeed, $E(\zeta, \zeta') > 0$ decreases monotonically to zero with the increase of ξ. Hence $\zeta \to 0$, $\zeta' \to 0$ for $\xi \to \infty$.

We now show that any solution of (I.2) manages to get close to (I.5). For this we estimate the amplitude of the first oscillation ζ_{max} after the conclusion of the stage of anisotropic expansion (I.4). In the phase plane $(\zeta; \zeta')$ the solutions of the equation (I.7) are represented by converging spirals (Fig. 2). The zero-derivative line $d(\zeta')/d\zeta = 0$ is:

$$\zeta_0'(\zeta, \xi) = \frac{2(1 - e^\zeta)}{(\ln z)'} < \frac{2}{(\ln z)'} < \sqrt{2}. \tag{I.9}$$

At the instant $\tau \sim \tau^*$ we have from (I.3)

$$\zeta \ll -1; \; |\zeta'| \sim \sqrt{2}|-1 + 6p_1| < 3\sqrt{2}. \tag{I.10}$$

A point in the phase plane with coordinates in the region (I.10) will obviously move in the sequel along the line $\zeta_0'(\zeta, \xi)$ (the spirals a, b); this is the Friedmann portion $\Omega = 1$ (from (I.4) we have $\zeta' \approx 2^{1/2}$). Since the zero-derivative line is situated below the level $\zeta' = 2^{1/2}$ we have $\zeta' < 2^{1/2}$ at the point $\zeta = 0$. Hence we obtain from (I.7)

$$\begin{gathered}E(\zeta_{max}, 0) = \exp(\zeta_{max}) - \zeta_{max} < E(0, \zeta')\\ = {}^1/_2\zeta'^2 + 1 < 2, \quad \zeta_{max} < 1{,}15.\end{gathered} \tag{I.11}$$

Thus, in the most general case, the oscillations around the solution (I.4) will start with the amplitude $\zeta_{max} \lesssim 1$ and will be damped out rapidly according to the law

$$\begin{gathered}\zeta = \zeta_0(\tau_1 - \tau)^{1/2}\sin\left(\sqrt{\frac{3}{2}}\ln\frac{A_0}{\tau_1 - \tau}\right), \quad |\zeta| \ll 1,\\ z = \frac{7}{4(\tau - \tau_1)^2}\left(1 - \frac{\zeta}{15}\right),\end{gathered} \tag{I.12}$$

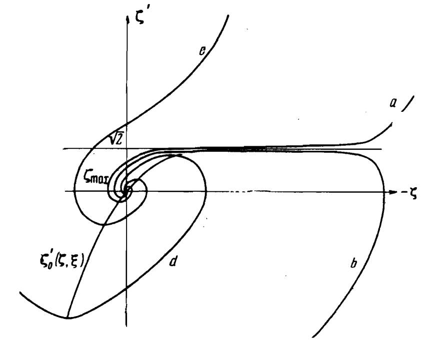

FIG. 2. The phase plane (ζ, ζ') of the equation (I.7). The spirals (a, b, c, d) correspond to different solutions of the (I.7). The point $\zeta' = 0$ corresponds to the solutions (I.5). The spiral b corresponds to the solution represented in Fig. 1 (up to the instant t_F).

where ζ_0, A_0 are constants. The value of τ_1 is determined by matching the solution to (I.4) at $\zeta \sim -1$. Obviously, the closer τ^* is to the point of maximum of the function λ_1, the shorter will be the duration of the isotropic expansion (I.4). If at the time $\tau \sim \tau^*$, $z \sim y$ (the gravitation of matter is switched on in the region of the maximum of λ_1), the amplitude of the first oscillation is $\zeta_{max} \lesssim 2$–2.5 (the spirals c, d) and the amplitude of the second oscillation is already much smaller than 1. In this case the vacuum stage is followed immediately by the solution (I.5), omitting the stage (I.4). If for $\tau \sim \tau^*$, $\lambda_1 \approx \lambda_2$ and $z \gg y$, the period of isotropic expansion (I.4) bypasses (I.5) and is directly replaced by the solution (I.14). And finally, if $\mu < 1$ in the vacuum stage, the stage (I.14) follows immediately after the instant $\tau \sim \tau^*$.

Such a character of the evolution of (I.5) is typical for all homogeneous models (with the exception of those of types I and V). The solutions of the type (I.5), (8) where the gravitation of matter is essential, can be continued up to the singularity, however, they are explicitly unstable for $t \to 0$ ($\tau \to -\infty$)—small perturbations grow near the singularity and lead to the general vacuum (Kasner or oscillating) asymptotic behavior.

The solution (I.5) is applicable up to the time when $\mu \sim 1$, after which follows the solution (9) with small amplitudes $|\mu| \ll 1$. In order to estimate the amplitude of the first oscillation, μ_{max}, it suffices to note that

$$\lambda_1\lambda_2\mu^2/3z \sim y/z \sim {}^1/_7 < 1$$

for $\mu \approx 1$. Therefore one may assume that over the first few oscillations of $(\mu)z$, and consequently also $\lambda_1\lambda_2$, do not have time to change much (cf. (9)). Then we have for $\mu < 1$

$$\ddot\mu + \frac{2\sqrt{3}B_2}{(\tau_1 - \tau)^{7/2}}\mu = 0, \quad \mu = \sqrt{\tau_1 - \tau}\,Z_{-2/3}(2^{5/2}3^{-3/4}B_2^{1/2}(\tau_1 - \tau)^{-3/4}), \tag{I.13}$$

where $Z_{-2/3}(\tau)$ is a cylindrical (Bessel) function. Matching (I.5) and (I.13) at $\mu \approx 1$ we obtain for the amplitude of the first oscillation[8] $\mu_{max} \approx 0.36$. This estimate is approximate but agrees well with a computer calculation.

The stage of small oscillations $|\mu| \ll 1$ is, properly speaking, the result of the isotropization. We shall call this stage "isotropic" (not to be confused with the stage (I.4)). It is important that if the condition (I.1) holds we have a single parameter depending on the initial conditions—the instant of beginning of the isotropic stage (9) $\tau_\Phi(\mu \sim 1)$ (in the sequel the constant τ_1 will be set equal to zero, since it has no physical meaning in the isotropic stage).

There remains to investigate the behavior of the solution of the system (5) in the asymptotic region $|\mu| \ll 1$. This case is discussed in detail in refs.[19,20]. Here we list only the result. For the equation of state $p = \epsilon/3$ the solution has the form

$$z = \frac{2}{\tau^2}[1-\beta(\tau)], \quad 0<\beta(\tau)\ll 1, \quad (\lambda_1\lambda_2)^{1/2} = -4h\frac{\tau_F}{\tau^2}\theta(\tau),$$
$$\mu = \frac{h_0}{2}\frac{\tau}{\tau_F}\theta^{-1/2}\sin 8h\tau_\Phi \int \frac{\theta(\tau)}{\tau^2}d\tau + O\left(\frac{\tau^2}{\tau_F^2}\theta^{-3/2}\right), \quad h_0 = \frac{\sqrt{3}}{2\sqrt{2}\,h}. \tag{I.14}$$

The functions $\beta(\tau)$ and $\theta(\tau)$ are represented as series in the powers of $1/\nu$ ($\nu = \ln(D/\tau) \gg 1$):

$$\beta = \frac{1}{2\nu}\left[1 - \frac{D_0 + 1/2}{\nu} - \frac{\ln\nu}{2\nu} + O\left(\frac{1}{\nu^2}\right)\right],$$
$$\theta = \frac{1}{\nu}\left[1 - \frac{D_0}{\nu} - \frac{\ln\nu}{2\nu} + O\left(\frac{1}{\nu^2}\right)\right]; \tag{I.15}$$

D, D_0, and h are constants. (The oscillatory part has an exponentially small factor $e^{-2\nu}$ for $\beta(\tau)$ and $e^{-4\nu}$ for $\theta(\tau)$.) Matching this solution with (7) yields the orders of magnitude

$$\nu_F \sim 4, \quad D \sim \tau_F e^4, \quad h \sim 1.$$

For a mixture of radiation ($p = \epsilon/3$) and nonrelativistic matter ($p = 0$) the solution has the form

$$\gamma^{1/6} = \frac{c_0^2}{\mathrm{sh}^2\psi}\left[1 - \frac{1}{2}\beta(\psi)\right], \quad 0<\beta(\psi)\ll 1, \quad (\lambda_1\lambda_2)^{1/2} = -4h\frac{\tau_F\theta(\psi)}{\tau_c^2\,\mathrm{sh}^4\psi}, \tag{I.16}$$

$$\mu = \frac{h_0}{2}\frac{\tau_c}{\tau_F}\theta^{-1/2}\mathrm{sh}^2\psi\sin 16h\frac{\tau_\Phi}{\tau_c}\int\frac{\theta(\psi)}{\mathrm{sh}^2\psi}d\psi + O\left(\frac{\tau_c^2}{\tau_F^2}\theta^{-3/2}\mathrm{sh}^4\psi\right), \tag{I.17}$$

where

$$1/2\,\mathrm{sh}\,2\psi - \psi = -\tau/\tau_c$$

is the solution of the Friedmann model $\Omega = 1$ ($\beta = 0$, $\theta = \mathrm{const}$); $\tau_c = -3^{1/2}/c_0^2L_1$ determines the instant when the density of radiation and the density of matter are equal ($|\tau_c| \ll |\tau_F|$), $c_0^2 = L_1^2/L_2$ (cf. (5)).

Up to first order in ν^{-1} ($\nu = \ln[(D/\tau_c)\sinh^{-2}\psi] \gg 1$):

$$\beta = 1/2\,\theta(3\,\mathrm{sh}^{-2}\psi + 1 - 3\psi\,\mathrm{sh}\,\psi\,\mathrm{sh}^{-3}\psi),$$
$$\theta^{-1} = -3/2\,\psi\,\mathrm{ch}\,\psi\,\mathrm{sh}^{-3}\psi + \ln(4De^{2\psi}/\tau_c) - e^{2\psi} + 3/2\,\mathrm{sh}^{-2}\psi. \tag{I.18}$$

In the two limiting cases we have from (I.18):

$$\psi \ll -1, \quad \beta \approx \frac{1}{2\ln(D/\tau)}, \quad \theta^{-1} \approx \ln\frac{D}{\tau}, \text{(see. I.15)} \tag{I.19}$$
$$|\psi| \ll 1, \quad \beta \approx \frac{1}{5\ln(D/\tau_c)}\left(\frac{3\tau}{2\tau_c}\right)^{2/3}, \quad \theta^{-1} \approx \ln\frac{D}{\tau_c} - \frac{9}{5}\left(\frac{3\tau}{2\tau_c}\right)^{2/3}.$$

After the changeover of the equation of state all deviations from the exact Friedmann solution die out quickly:

$$(\lambda_1\lambda_2)^{1/2} \approx -\frac{4h_1}{\ln(\tau_F e^4/\tau_c)}\frac{\tau_F}{\tau_c^{2/3}\tau^{4/3}} \tag{I.20}$$
$$\mu \approx h_2\frac{\tau_c^{1/6}\tau^{2/3}}{\tau_F}\left(\frac{1}{4}\ln\left(\frac{\tau_F}{\tau_c}e^4\right)\right)^{1/2}\sin\left\{\frac{\tau_F}{\tau_c^{2/3}\tau^{1/3}} + \vartheta_0\right\},$$
$$\vartheta_0 = \mathrm{const}, \quad h_1 = (2/3)^{4/3}h \sim 1, \quad h_2 = (3/2)^{2/3}h_0 \sim 1.$$

APPENDIX II

We consider (5) with the condition $a > 0$. In this case the isotropic stage (I.16), (I.18) continues up to the instant τ_M, after which follows the third era of evolution (cf. supra), when the gravitation of matter is inessential. The duration of the isotropic stage is

$$4a^2\lambda_1\lambda_2 \approx \frac{2}{\tau_M^2}, \quad \left(\frac{\tau_M}{\tau_F}\right)^{1/3} \approx \left(\frac{\tau_F}{\tau_c}\right)^{2/3} aN_c \ll 1 \tag{II.1}$$

($a \ll 1$; for $a \sim 1$, $\tau_M/\tau_F \sim 1$). It is easy to show that the first and second stages in the open model of type VII ($a \neq 0$) occur in the same manner as in the case $a = 0$, i.e. effects related to rotations of the tetrad oriented along the principal axes of the matrix γ_{ab} must be taken into account only during the third stage.

In[9] it was shown that the rotation of the principal axes of γ_{ab} leads to an effect of reflection of λ_1 and λ_2, i.e., μ can have only one sign ($\mu > 0$). In the most general case in the "vacuum" stage we have $\mu_{max} \gg \mu_{min}$ and the condition (I.1) has the form

$$\mu_{\tau\sim\tau^*} \gg \max\left\{1, \dot\mu_{min} = 2\,\mathrm{arsh}\frac{2a|3p_3-1|}{|p_1-p_2|}\right\}. \tag{II.2}$$

If (II.2) holds, all conclusions regarding the stage (I.5) are valid. Further, we obtain from (5)

$$2\dot\varphi\,\mathrm{sh}^2\frac{\mu}{2} = a\left(\ln\frac{\lambda_3}{(\lambda_1\lambda_2)^{1/2}}\right)^{\cdot} = \frac{3a}{4(\tau_1-\tau)}$$

for

$$\mu \sim 1, \quad \dot\varphi \approx \frac{3a}{4(\tau_1-\tau)} \approx \frac{\sqrt{3}}{2}a(\lambda_1\lambda_2)^{1/2} \ll (\lambda_1\lambda_2)^{1/2}. \tag{II.3}$$

Consequently, all conclusions relative to the transition to a stage of small oscillations of μ remain in force, and the solution (5) goes over into the isotropic stage (I.18) with a minimal gap μ_{min} much smaller than the oscillation amplitude μ_{max}. In this case the solutions for $\mu(\tau)$ in the region of μ_{min} coincide with those determined in[9]: in the vicinity of the point $\mu_{min} \ll 1$ there occurs a mirror reflection of the function $\mu(\tau)$ relative to the level $\mu \equiv 0$.

Thus, the maximal amplitude of oscillations coincides with (I.17) and the solution (I.18) is valid:

$$\mu_{max} \approx (\tau_F^2\lambda_1\lambda_2)^{-1/4}. \tag{II.4}$$

The size of the gap μ_{min} varies according to the law ($p = \epsilon/3$):

$$\mu_{min} \approx 6a\Big/\left(\ln\frac{D}{\tau}\right)^{1/2} \tag{II.5}$$

(II.4) is valid as long as $\mu_{max} \gg \mu_{min}$. In the case $\mu_{max} \approx \mu_{min}$ we have from (5)

$$\dot\varphi \approx (\lambda_1\lambda_2)^{1/2}, \quad \mu \approx (\tau_F^2\lambda_1\lambda_2)^{-1/4}\exp\left\{a\int_{\tau_F}(\lambda_1\lambda_2)^{1/2}d\tau\right\}, \quad \mu \ll 1. \tag{II.6}$$

We estimate the integral (II.6) at the end of the isotropic stage[9]:

$$a\int^{\tau_M}(\lambda_1\lambda_2)^{1/2}d\tau \approx \begin{cases} 2^{-1/2}, & \text{if } |\tau_c| \lesssim |\tau_M| \\ 3\cdot 2^{-1/2}, & \text{if } |\tau_c| \gg |\tau_M| \end{cases}. \tag{II.7}$$

Consequently, the solution (I.18) is valid up to a time of the order of τ_M. We show that towards the end of the isotropic stage we always have (II.6)

$$\dot\varphi \sim \frac{2a}{\mu^2}\int_{\tau_F}\lambda_1\lambda_2\mu^2 d\tau \sim (\lambda_1\lambda_2)^{1/2}\left\{a\int(\lambda_1\lambda_2)^{1/2}d\tau\right\} \sim (\lambda_1\lambda_2)^{1/2} \tag{II.8}$$

(cf. (II.7)).

Thus, towards the beginning of the third period[10]

$$\mu = \mu_0 \approx \left(\frac{a\tau_M}{\tau_\Phi}\right)^{1/2} \approx a\left(\frac{\tau_M}{\tau_c}\right)^{1/4}N_c^{1/2}. \tag{II.9}$$

From (5) it follows that during the third stage

$$(\lambda_1\lambda_2)^{1/2} = -1/2a(\tau-\tau_2), \quad \tau_2 = \mathrm{const} = 0{,}14\tau_M. \tag{II.10}$$

It is important that (II.10) is the asymptotic behavior for all the solutions of the system (5). (The corrections to (II.10) related to the presence of matter decrease according to a power law.) It is easy to show that any solution of the equations (5) tends asymptotically (for $\gamma \to \infty$) to (II, 11) (for any $a > 0$):

$$\mu = \text{const}, \quad \dot\varphi = (\lambda_1\lambda_2)^{1/2}. \qquad \text{(II.11)}$$

Let us denote

$$\dot\varphi = (\lambda_1\lambda_2)^{1/2}\left(1 - f/4\,\text{sh}\frac{\mu}{2}\right), \quad d\eta = (\lambda_1\lambda_2)^{1/2}\,d\tau. \qquad \text{(II.12)}$$

Then it follows from Eq. (5) (the accent means derivative with respect to η

$$(f^2+\mu'^2)^{1/2} = -\int (f^2+\mu'^2)^{1/2}(\ln\sqrt{\lambda_1\lambda_2})'\,d\eta + 4\int\frac{f}{(f^2+\mu'^2)^{1/2}}\text{sh}\frac{\mu}{2}\{(\ln\sqrt{\lambda_1\lambda_2})' - 2a\}\,d\eta + \text{const}. \qquad \text{(II.13)}$$

Further, it follows from (5) that (cf. (II.10)):

$$2a < (\ln\sqrt{\lambda_1\lambda_2})' \to 2a, \quad \eta\to\infty. \qquad \text{(II.14)}$$

During the third era, when the evolution of the metric is determined by the isotropic curvature $4a^2\lambda_1\lambda_2$, the quantity $\{(\ln(\lambda_1\lambda_2)^{1/2})' - 2a\}$ tends to zero exponentially with respect to η, (5) (II.10). Consequently, for $\eta\to\infty$ the corresponding integral in (II.13) converges exponentially (with respect to η). Thus, we obtain from (II.13)

$$\eta\to\infty, \quad (f^2+\mu'^2)^{1/2}\to 0, \quad f\to 0, \quad \mu'\to 0, \quad \mu\to\mu_0 = \text{const}.$$

(If a = 0, then f = 4 sinh($\mu/2$) (cf. (II.12)) and it follows from (II.13) that $\mu\to 0$.) Since towards the beginning of the third period the oscillations of μ are already damped out (II.6)[11], we have immediately after the time τ_M (II.11) with $\mu = \mu_0$ from (II.9).

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1)Such an evolution is characteristic for all anisotropic models except those of types I and V.

2)Solutions of the type (8) have been investigated in detail by Ruban [18] for the axially symmetric case ($\lambda_2 = \lambda_3$).

3)The conclusion that $\mu\to$ const for $t\to\infty$ was derived by Collins and Hawking [13] from their analysis of the first-order deviations from the Friedmann solution. In the present paper this is an exact statement, following from the general solution (12) (without assuming the smallness of $(\lambda_1 - \lambda_2)/\lambda_1$).

4)We note that, in principle, an experiment is possible to detect the effect of unbounded growth of the ratio $\lambda_3/(\lambda_1\lambda_2)^{1/2} \sim t^{2x2}$ for $t\to\infty$. Indeed, a sphere of particles at rest or moving slowly in the synchronous coordinate system (1) will, in the course of time, become an ellipsoid with prolateness increasing without bound. However, such an experiment is not practically realizable.

5)A more detailed investigation of the solutions (18), (19) can be found in [19,20].

6)We recall that in the model of type IX the volume of the model increases monotonically from zero at the singularity up to some maximal value and then decreases monotonically to the second singularity. In all other models under consideration the quantity γ increases monotonically from zero to infinity.

7)It should be noted that two completely different types of evolution are possible for models of type VIII, depending on the relation between the functions λ_1, λ_2, and λ_3 at the moment $t\sim t^*$ when the gravitation of matter is "switched on". The most interesting case is $\lambda_1, \lambda_2 \gg \lambda_3$, which we consider. If at the time $t\sim t^*$ we have $\lambda_1, \lambda_3 \gg \lambda_2$, the model evolves in agreement with the ideas developed above, up to the time $t\sim t_\Phi$, $|\ln(\lambda_1/\lambda_3)| \sim 1$; after that starts a stage where $\lambda 1 \approx \lambda_3 \approx -\sqrt{3}/2\tau$, $\lambda_2 \sim \tau^{-4}$, $\gamma\epsilon \approx 3/2\tau^2$ for
for $p = 0$.

It is important that the volume here varies according to the Friedmann law, however the expansion along the axes is anisotropic. This stage goes on until $\lambda_2 \approx \lambda_{1,3}$. After this the λ_3 axis slows down its expansion, and the solution approaches one of the asymptotic forms (23), (24) depending on the equation of state.

8)The amplitude μ_{max} of the first oscillation depends weakly on the degree of decrease of $\lambda_1\lambda_2$.

9)a^{-1} can be interpreted as the number of oscillations in the isotropic stage (cf. (II.7)).

10)If at the instant $\tau\sim\tau^*$, $\mu\sim\mu_{min}\gg 1$, the solution of (5) misses (1.5) and goes over into the stage $\dot\phi \approx (\lambda_1\lambda_2)^{1/2}$, sinh $\mu \approx$ const $(\lambda_1\lambda_2)^{-1/4}$ exp $\{a\int(\lambda_1\lambda_2)^{1/2}d\tau\}$. The estimates for the integral agree with (II.7).

11)It is important to note that this conclusion does not depend on the presence of matter. One can show that in an open model of type VII ($a > 0$) always after the stage of "vacuum" oscillations, towards the time $t_M(\tau_M)$ the oscillations of μ are damped out, therefore directly after the time τ_M any solution of the system (5) goes over into the stage (II.11). for details, cf. [19].

1. I. M. Khalatnikov and E. M. Lifshitz, Phys. Rev. Lett. **24**, 76 (1970).
2. V. A. Belinskiĭ and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **56**, 1700 (1969) [Sov. Phys.-JETP **29**, 0000 (1970)].
3. E. M. Lifshitz, I. M. Lifshitz and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **59**, 322 (1970) [Sov. Phys.-JETP **32**, 173 (1970)].
4. V. A. Belinskiĭ, E. M. Lifshitz and I. M. Khalatnikov, Usp. Fiz. Nauk **102**, 463 (1970) [Sov. Phys.-Uspekhi 100, 000 (0000)].
5. A. G. Doroshkevich and I. D. Novikov, Astron. Zh. **47**, 948 (1970) [Sov. Astron.-AJ **14**, 673 (1971)].
6. Ya. B. Zel'dovich, ZhETF Pis. Red. **11**, 300 (1970) [JETP Lett. **11**, 197 (1970)].
7. C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969); Phys. Rev. **186**, 1319, 1328 (1970).
8. V. A. Belinskiĭ, E. M. Lifshitz and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **60**, 1969 (1971) [Sov. Phys.-JETP **33**, 1061 (1969)].
9. L. P. Grishchuk, A. G. Doroshkevich and V. N. Lukash, Zh. Eksp. Teor. Fiz. **61**, 3 (1971) [Sov. Phys.-JETP **34**, 1 (1972)].
10. G. Ellis and M. MacCallum, Commun. Math. Phys. **12**, 108 (1969).
11. S. P. Novikov, Zh. Eksp. Teor. Fiz. **62**, 1977 (1972) [Sov. Phys.-JETP **35**, 1031 (1972)].
12. A. G. Doroshkevich, I. D. Novikov, R. A. Sunyaev and Ya. B. Zel'dovich, Highlights of Astronomy, **2**, 318 (1971).
13. C. B. Collins and S. W. Hawking, Preprint: "Why is the Universe Isotropic?", Inst. of Theor. Astron., Cambridge, 1972.
14. E. Schucking, in "Gravitation: An Introduction to Current Research", ed. by L. Witten, Wiley, N.Y., 1962.
15. L. P. Grishchuk, A. G. Doroshkevic and I. D. Novikov, Zh. Eksp. Teor. Fiz. **55**, 2281 (1968) [Sov. Phys.-JETP **28**, 1210 (1969)].
16. I. D. Novikov, Astron. Zh. **47**, 1203 (1970) [Sov. Astron.-AJ **14**, 960 (1971)].
17. A. G. Doroshkevich, Ya. B. Zel'dovich and I. D. Novikov, Zh. Eksp. Teor. Fiz. **53**, 644 (1967) [Sov. Phys.-JETP **26**, 408 (1968)].
18. V. A. Ruban, Preprint 355, Leningrad Physico-Techn. Inst. 1971.
19. A. G. Doroshkevich, V. N. Lukash and I. D. Novikov, Preprint, Inst. Appl. Math. 1973.
20. V. N. Lukash, Astron Zh. **50**, 1000 (1973) [sic!]
21. A. G. Doroshkevich, V. N. Lukash and I. D. Novikov, Preprint, Inst. Appl. Math., 1973.
22. I. D. Novikov, Astron. Zh. **45**, 538 (1968) [Sov. Astron. AJ **12**, 427 (1968)].

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