Some features of viscous flow of vortices in superconducting alloys near the critical temperature

L. P. Gor'kov and N. B. Kopnin
(L. D. Landau Institute of Theoretical Physics, USSR Academy of Sciences
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Time equations for slow ($\omega \ll \Delta$) motion of the vortex structure in alloys with $l \ll \xi(T)$ near $T_c$ are derived on the basis of the microscopic superconductivity theory. The major role is played by the term responsible for the slow diffusion mechanism of relaxation.

The motion of vortex lines is considered for the case of a magnetic field $H \ll H_{c2}$. This relaxation mechanism leads to a large value of the viscosity coefficient.

1. INTRODUCTION

It is known that the finite resistance of a type-II superconductor in a magnetic field exceeding the lower critical field $H_{c1}$ is due to energy dissipation that occurs when the vortex structure moves in the superconductor. This question has been under experimental study for quite some time, whereas the theoretical study of the motion of vortex filaments was based until recently only on phenomenological models. These turned out (see [1-3]) not to cover all the significant kinetic processes that occur in superconductors. A consistent approach to this problem obviously calls for the use of microscopic time-dependent equations of the theory of superconductivity [4].

The motion of a vortex structure in strong magnetic fields (near $H_{c2}$), where the superconductivity is greatly suppressed, was investigated on the basis of the microscopic theory, for example, by Caroli and Maki [5], and by Baba and Maki [6]. Great interest, however, attaches to the wide range of fields $H \ll H_{c2}$. In an earlier paper [4], using alloys with paramagnetic impurities as an example, we have proposed a scheme by which to describe the vortex motion within the framework of microscopic differential time-dependent equations of the Ginzburg-Landau type.

Our result [4] is valid for the case of large paramagnetic-impurity concentrations, strictly speaking, only for an uncharged Fermi gas, since we took no account of the term due to the normal energy loss in the center of the vortex. This circumstance was pointed out by a number of authors [10, 11].

It follows from [6] that at finite temperatures, when the anomalous terms in the equations for the superconductors become important, the viscosity experienced by the moving vortex increases strongly, i.e., the vortex speed is low. This is caused by a circumstance already noted by Eliashberg and one of the authors [4], that the anomalous term becomes larger in order of magnitude than the remaining terms with time derivatives when the damping of the excitations is determined by slow diffusion processes. The reason for this increase in viscosity is that to ensure stationary motion of the vortex it is essential that the perturbations produced around the moving vortex have time to attenuate within a time on the order of $\xi(T)/u$, where $u$ is the vortex speed and $\xi(T)$ is the scale of variation of the ordering parameter. In superconductors, the times of homogeneous energy relaxation are quite long ($\Theta^2/T^3$ and $E_F/T^3$ for electron-phonon and electron-electron interactions, respectively). The damping of the excitations is therefore determined by a diffusion mechanism which, nonetheless, is still slow enough at temperatures $T_c - T \ll T_c$.

This article is devoted to a study of these features of relaxation for vortex filaments moving at $T_c - T \ll T_c$ under the more favorable experimental conditions of superconducting alloys with ordinary impurities. It will be shown that the relaxation mechanism mentioned above plays the principal role in this case, so that an analytic solution of the resultant equations can be obtained.

2. DERIVATION OF EQUATIONS FOR THE ANOMALOUS GREEN’S FUNCTIONS IN ALLOYS AT A TEMPERATURE CLOSE TO $T_c$

The general scheme for deriving the time-dependent equations for superconductors was developed by Eliashberg and one of the authors [3]. It was shown that when the expressions for the ordering parameter $\Delta$ and for the current are analytically continued to the real frequency axis, the result consists of terms that are regular in the upper or lower half of the complex $\omega$ plane, as well as the so-called “anomalous terms,” which do not possess this property. We write down the expressions derived in [3], and choose immediately for $\Delta$ a gauge whereby we obtain throughout only the gauge-invariant combinations

$$|\Delta|, \quad Q = A - \frac{e}{2\pi} \nabla \mu = \eta + \frac{1}{2\pi} \chi$$

where $\chi$ is the phase shift of the ordering parameter (we shall henceforth omit the absolute-magnitude symbols of $\Delta$):

$$\frac{\Delta(r)}{|\Delta|} = \int \frac{d\mathbf{r}'}{4\pi} \left[ \frac{\theta}{2\pi} \alpha \phi \phi_{\alpha}(r,r') - \frac{\theta}{2\pi} \phi \phi_{\alpha}(r,r') \right]$$

$$+ \int \frac{d\mathbf{r}'}{4\pi} \left( \mathbf{p} \cdot \mathbf{p}' \right) \left[ \frac{\theta}{2\pi} \alpha \phi \phi_{\alpha}(r,r') - \frac{\theta}{2\pi} \phi \phi_{\alpha}(r,r') \right] - \frac{\alpha}{\pi} \mu Q(r). \quad (1)$$

Here $GR(A)$ and $FR(A)$ are retarded and advanced Green’s functions, $\mathbf{p} = -i\nabla$, and $\mathbf{p}' = -i\nabla'$. It is convenient to write down the definitions of the anomalous functions directly in matrix form. To this end, we introduce the matrices

$$G^{\alpha\beta}(r, r') = \begin{pmatrix} G^{\alpha\beta}_{R}(r, r') & F^{\beta\gamma}_{R}(r, r') \\ F^{\gamma\alpha}_{R}(r, r') & G^{\gamma\alpha}_{R}(r, r') \end{pmatrix},$$

$$G^{\alpha\beta}_{i}(r, r') = \begin{pmatrix} G^{\alpha\beta}_{i}(r, r') & F^{\beta\gamma}_{i}(r, r') \\ F^{\gamma\alpha}_{i}(r, r') & G^{\gamma\alpha}_{i}(r, r') \end{pmatrix},$$

$$\tilde{H}_{\alpha}(r) = \begin{pmatrix} -eQ(r)\mathbf{p} + e\nu_{\alpha} - \Delta_{\alpha} \\ e\nu_{\alpha} + e\mathbf{p} \cdot \mathbf{p}' - \Delta_{\alpha} \end{pmatrix}.$$
The equation for the anomalous functions $g^{(a)}_{\xi}(\nu)$ contains retarded and advanced functions $g^{R}_{\xi}(A)$ that depend on two frequencies (see [1]). In view of the slowness of the considered processes ($\omega \ll \Delta$), the regular functions can be replaced by the equilibrium functions $g^{R}_{\xi}(A)$$2\pi\delta(\varepsilon - \epsilon_{1})$, and we can omit from $\bar{H}_{\nu}$ the $\mu$-containing terms, which are of the order of $\omega$. Ultimately we get

$$
S^{\nu}_{\xi}(r, r') = -\frac{\hbar}{2T} \frac{1}{\varepsilon - \epsilon_{1}} \int \frac{d\varepsilon_{1}}{2\pi} S^{\nu}_{\xi}(r, r') \bar{S}^{\nu}_{\xi}(r, r') d\varepsilon_{1},
$$

(2)

This formula corresponds to the diagram equation shown in the figure. The dashed lines denoting average over the impurities corresponds, as usual, to the factor $(2\pi)^{-1}$, where $\tau$ is the free path time.

We introduce also the notation

$$
\tilde{\gamma}(r) = \left( \begin{array}{cc} \nu_{1} & \nu_{2} \\ -\nu_{1} & \nu_{2} \end{array} \right)\frac{2\pi}{m_{p}} S^{\nu}_{\xi}(r, r),
$$

$$
\tilde{\gamma}(r) = \left( \begin{array}{cc} \nu_{1} & \nu_{2} \\ -\nu_{1} & \nu_{2} \end{array} \right)\frac{2\pi}{m_{p}} \bar{S}^{\nu}_{\xi}(r, r').
$$

(3)

$p_{0}$ is the Fermi momentum.

Expansions of the regular parts of (1) in powers of $\Delta/T$ are well known [18]. We can use them to write down expressions for $A$ and for the current:

$$
\frac{\pi}{8T} \left[ -\frac{\partial}{\partial t} + D_{t} \left( \nu_{1} - 4\pi c^{2} \right) \right] + \frac{T_{r}}{T_{c}} - \frac{\gamma \nu_{1}}{2} = 0,
$$

(4a)

$$
\frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi} + D_{1} \frac{2\mu_{0} e^{2}}{2c},
$$

(4b)

$$
\bar{J}_{k} = \frac{\partial}{\partial \xi} - \frac{m_{p}}{2c} v_{1} D_{1} Q_{k},
$$

(4c)

$$
\bar{J}_{k} = \frac{\partial}{\partial \xi} - \frac{m_{p}}{2c} v_{1} D_{1} Q_{k},
$$

(4d)

Here

$$
D_{t} = -\frac{\nu_{1}}{3} \gamma(\nu_{1}), \quad y(\nu) = \frac{\nu_{1}}{3} \sum_{n=1}^{N} \left( \begin{array}{c} \mathbb{E}^{2} \\ \mathbb{E}^{2} \end{array} \right) \frac{1}{2} (2n + 1)(2n + 1) 2n^{2} + 1.
$$

At $T \ll 1$, we have $y(\nu_{1}) = 1$ and at $T \gg 1$ we get $y(\nu_{1}) = 7\xi(3)/2\nu_{1}^{2}$.

We mention that conditions that $\Delta$ be real (Eq. (4b)) follows from the expression for the current with allowance for the electroneutrality of the superconductor, div $J = 0$. The problem thus reduces to a solution of (2) for the anomalous functions.

We confine ourselves to alloys in which $l \ll \xi(\nu)$, where $l = \nu_{0}T$ is the mean free path. In this region we can draw the following conclusions concerning the spatial behavior of the functions $g^{R}_{\xi}(A)(r, r')$. We represent them in the form

$$
g^{R}_{\xi}(A)(r, r') = \frac{m e^{|r - r'|/2\nu_{0}}}{2\pi e^{2}} q_{\nu_{0}}(r, r') \exp[ip_{\nu_{0}}(r - r')],
$$

(5)

Applying to the right and to the left of (5) the operator

$$
g^{(a)}_{\xi}(A)(r, r') = \frac{1}{2\pi e^{2}} \frac{\nu_{0}}{m_{p}} q_{\nu_{0}}(r, r')
$$

where

$$
\sum_{\nu_{0}}(r, r') = \int \frac{d\nu_{0}}{2\pi} e^{2\nu_{0}} q_{\nu_{0}}(r, r')
$$

is the impurity self-energy part, we can easily obtain the conditions satisfied by the function $g^{(a)}_{\xi}(A)(r, r')$. We introduce the vector $n = (r - r')/|r - r'|$; then

$$
\left[ \bar{g}_{\nu_{0}}(r, r') + \bar{g}_{\nu_{0}}(r, r') \right] d\nu_{0} = 1,
$$

(6)

$$
\pm \omega_{o} \nu_{0} \nu_{1} \sum_{\nu_{0}}(r, r') = \frac{1}{2\pi} e^{2\nu_{0}} q_{\nu_{0}}(r, r') + \frac{\Delta_{\nu_{0}}}{\omega_{o} \nu_{0}} \sum_{\nu_{0}}(r, r') = \pm \frac{1}{\Delta_{\nu_{0}}},
$$

(7)

where $\nu_{0} = \pm \omega_{o} \nu_{0} \nu_{1} \sum_{\nu_{0}}(r, r')$. Indeed, the functions $g^{(a)}_{\xi}(A)(r, r')$ with coinciding points $(r' - r)$ are connected with the functions

$$
g^{(a)}_{\xi}(A)(r, r') = \int g^{(a)}_{\xi}(A)(r, r) d\nu_{0},
$$

(8)

which are integrated with respect to the energy variable $\xi = \nu_{0}e(\nu_{0}p_{0})$, by the relations

$$
\int g^{(a)}_{\xi}(A)(r, r') d\nu_{0} = \int f^{(a)}_{\xi}(A) \nu_{0} e(\nu_{0}p_{0}) d\nu_{0},
$$

(9)

satisfy the equations derived by Eilenberger [19] and the corresponding boundary conditions at infinity:

$$
-\nu_{0} \frac{\partial}{\partial \xi} \bar{g}_{\nu_{0}}(r, r') + \frac{1}{2\pi} \left( f^{(a)}_{\xi}(A) - f^{(a)}_{\xi}(A) \right) = 0,
$$

(10)

$$
\left[ 2e - \nu_{0} \frac{\partial}{\partial \xi} \left( \bar{v} + \frac{2e}{c} \right) \right] f^{(a)}_{\xi}(A) - \nu_{0} \frac{\partial}{\partial \xi} \bar{g}_{\nu_{0}}(r, r') f^{(a)}_{\xi}(A) = 0,
$$

(11)

$$
\left[ 2e - \nu_{0} \frac{\partial}{\partial \xi} \left( \bar{v} + \frac{2e}{c} \right) \right] f^{(a)}_{\xi}(A) - \nu_{0} \frac{\partial}{\partial \xi} \bar{g}_{\nu_{0}}(r, r') f^{(a)}_{\xi}(A) = 0,
$$

(12)

The subscript 0 designates throughout functions averaged over the direction of $\nu_{0}$. It follows from (8) that

$$
g^{(a)}_{\xi}(A)(r, r) = \int g^{(a)}_{\xi}(A)(r, r) d\nu_{0},
$$

(13)

We have used here the fact that as $\nu_{0} \rightarrow \infty$ we get

$$
g^{(a)}_{\xi}(A)(r, r) = \frac{\nu_{0}}{\nu_{0}^{2}} \int f^{(a)}_{\xi}(A) d\nu_{0} = \frac{\Delta_{\nu_{0}}}{\nu_{0}^{2}},
$$

(14)

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We write down the equations that are satisfied by the functions $g^{(0)(v, k)} = \int_{\mathbb{R}^3} g^{(0)}(v, p - k) \, dp$, $\xi = v_0 - (p - \mu)$,

\[
\begin{align*}
&\left( -A_i - A_j \right) \cdot \xi + \Sigma(r) g^{(0)}(v, r) - g^{(0)}(v, r) \Sigma(r) \\
&\quad + i(2\epsilon)^{-1} \varepsilon(V) g^{(0)}(v, r) - g^{(0)}(v, r) \chi(r) \
&\quad - n_l \frac{e}{2\hbar} \frac{2m}{2\hbar} \left( n_l g^{(0)}_a(r, \xi) - g^{(0)}_a(r, \xi) n_l \right),
\end{align*}
\]

where

\[
\begin{align*}
\{ A_i \} &\sim -i \left( \xi \cdot \nabla - \epsilon \right) \left( g^{(0)}_a(v, r) - f^{(0)}(v, r) \right), \\
\{ A_j \} &\sim -i \left( \xi \cdot \nabla - \epsilon \right) \left( f^{(0)}(v, r) - g^{(0)}_a(v, r) \right), \\
\{ A_k \} &\sim -i \left( \xi \cdot \nabla - \epsilon \right) \left( f^{(0)}(v, r) - g^{(0)}_a(v, r) \right) \
&\quad + \Delta \left( g^{(0)}_a(v, r) - f^{(0)}(v, r) \right).
\end{align*}
\]

For the sake of brevity, we shall henceforth drop the subscripts $c$ and $\epsilon - \omega$, bearing in mind that $g^{(0)}$ refers to the frequency $\epsilon$, $g^{(0)}$ to the frequency $\epsilon - \omega$, and $g^{a}$ to both frequencies.

Averaging (10) over the directions of the vector $v_0$, we obtain

\[
\begin{align*}
&\langle f^{(0)}(v, r) \rangle \sim -i \left( \xi \cdot \nabla - \epsilon \right) \left( g^{(0)}_a(v, r) - f^{(0)}(v, r) \right), \\
&\langle g^{(0)}_a(v, r) \rangle \sim -i \left( \xi \cdot \nabla - \epsilon \right) \left( f^{(0)}(v, r) - g^{(0)}_a(v, r) \right) \
&\quad + \Delta \left( g^{(0)}_a(v, r) - f^{(0)}(v, r) \right).
\end{align*}
\]
with the aid of the conditions (8) (see [13]):

\[ t = - \frac{1}{2 \Delta} \ln \left( j_0 / j_1 \right), \quad r = - \frac{1}{2 \Delta} \ln \left( j_1 / j_0 \right), \]

\[ \lambda = \frac{\Delta}{\lambda_{\text{static}}}, \quad \lambda_s = \frac{\Delta_s}{\lambda_{s_{\text{static}}}}, \quad \lambda = \frac{\epsilon}{\Delta}, \quad \lambda_s = \frac{\epsilon_s}{\Delta_s}, \]

where \( D = \nu_s k/3 \) is the diffusion coefficients. For the sake of brevity, the superscripts \( R \) and \( A \) to the functions and we readily obtain from (19)

\[ \frac{\partial \phi}{\partial t} \left( \frac{(\gamma_1 + \gamma_2)^2}{\Delta} \right) + D \frac{\partial^2 \phi}{\partial t^2} \left( \frac{\gamma_1 + \gamma_2}{\Delta} \right) = \frac{2 \gamma_1}{T} \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_1} - \frac{2 \gamma_2}{T} \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_2}. \]  

We have put

\[ f = \frac{1}{\epsilon_0} \frac{\epsilon}{(\epsilon^2 - \Delta^2)^{5/2}}, \quad \epsilon > \Delta, \]

\[ f = \frac{1}{\epsilon_0} \frac{\epsilon}{(\epsilon^2 - \Delta^2)^{5/2}}, \quad \epsilon < \Delta. \]

We see that owing to the smallness of \( D \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_1} \) we get

\[ \gamma_1 + \gamma_2 = - \frac{\pi}{3} \pi \frac{1}{T} \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_1} \frac{\partial \phi}{\partial \gamma_2}. \]

This expression coincides with Eq. (8) of [13] at \( \epsilon < \Delta \). Unlike (20), it does not contain the small factor \( D \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_1} \) in the left-hand side, and therefore in the region \( \epsilon < \Delta \) the quantity \( (\gamma_2 + \gamma_2) \) is anomalously large in the region \( \epsilon > \Delta \).

We note here the fact that at \( \epsilon > \Delta \) the integral

\[ \int \frac{\gamma_1 + \gamma_2}{d\epsilon/\partial t} \frac{d\epsilon}{\partial t} \]

is determined by the frequencies \( \epsilon \sim \Delta \ll T \).

At \( \epsilon < \Delta \), the terms of first order in \( \epsilon \) in the right and left sides of (19) vanish. In this case it is necessary to take into account in the right-hand side the difference between \( f^R_{0\epsilon} \) and \( f^A_{0\epsilon} \) which is due to \( \omega \). As a result we get

\[ \gamma_1 + \gamma_2 = - \frac{\pi}{3} \pi \frac{1}{T} \frac{\partial^2 \phi}{\partial \gamma_1 \partial \gamma_2} \frac{\partial \phi}{\partial \gamma_1} \frac{\partial \phi}{\partial \gamma_2}. \]

As to Eq. (4b), we shall now show that it is not independent, but follows from the expressions (4c, d) for the current and the continuity equation

\[ \nabla \times H = \frac{4 \pi}{c} \frac{\partial \phi}{\partial t} = 0, \]

which is the charge density. The continuity equation in a superconductor, in turn, just as in any metal, means \( \nabla \times \mathbf{J} = 0 \), since no charges accumulate, owing to the strong Coulomb interaction, and \( \frac{\partial \phi}{\partial t} = 0 \) (the quasi-neutrality condition).

We note that the last equation in our model is satisfied identically; namely, in slow nonstationary processes the deviation of the charge density from the equilibrium value is proportional to the frequency, so that \( \partial \phi / \partial t \) is of second order of smallness and should be taken equal to zero. In other words, Eq. (4b) is the consequence of Maxwell's equation

\[ \nabla \times H = \frac{4 \pi}{c} \frac{\partial \phi}{\partial t} = 0, \]

and of the current for the electric field. Indeed, it is seen from (11) that, accurate to terms of first order in the
frequency
\[ \nu \mathbf{V}(\mathbf{r}_i - \mathbf{r}_f) = 2\Delta(\mathbf{r}_i - \mathbf{r}_f). \]

Multiplying this equation by \((\nu / 2\pi)^2(d\mathbf{e}/4\pi n_I)\) and integrating with respect to \(\mathbf{r}_i\), we obtain with the aid of (4d)

\[
\text{div } \mathbf{j} = -\frac{m_p e^2}{2\pi c} D_i \text{div } \left[ \frac{\mathbf{A}}{T} \right] = -\frac{2\pi m_p e^2}{2\pi c} D_i \text{div } \left[ \frac{\mathbf{A}}{T} \right] - \int \frac{d\mathbf{e}}{4\pi n_I} \Delta(\mathbf{r}_i - \mathbf{r}_f). \]

Comparing this with (4c) we obtain immediately (4b).

Thus, it suffices to write only the equations that determine \(j(\mathbf{a})\). To find \(j_1 - j_T\), it is necessary in general to solve the whole set of equations (11). We can verify, however, that the contribution made to the energy dissipation in the vortex by the anomalous part of the current \(j(\mathbf{a})\) is small in comparison with the contribution of \(j_2 + j_T\).

Indeed, the expression for \(j_1 - j_T\) at \(|\mathbf{e}| > \Delta\) is

\[
(j_1 - j_T) = \frac{1}{4\pi} \int \left( -\mathbf{g} \mathbf{V} \left( \mathbf{r}_i + \mathbf{r}_f \right) + \frac{2\pi m_p e^2}{2\pi c} \mathbf{g} \mathbf{V} \left( \mathbf{r}_i + \mathbf{r}_f \right) \right) \frac{d\mathbf{e}}{4\pi n_I}. \]

For \(j_1 + j_T\) we have in turn from (11) and (13)

\[
\mathbf{V}^*(\mathbf{r}_i + \mathbf{r}_f) = \mathbf{V}^* \left( \mathbf{r}_i + \mathbf{r}_f \right) - \frac{2\pi m_p e^2}{2\pi c} \mathbf{g} \mathbf{V} \left( \mathbf{r}_i + \mathbf{r}_f \right). \]

Retaining in the integral

\[
\int (\mathbf{r}_i - \mathbf{r}_f) d\mathbf{e}/4\pi n_I \]

only the principal terms in \(\Delta / T\), we obtain

\[
\mathbf{j} = -\sigma \left\{ -\frac{1}{4\pi} \frac{\partial \mathbf{Q}}{\partial t} + \frac{1}{2\pi} \mathbf{V} \left( \mathbf{r}_i + \mathbf{r}_f \right) \frac{d\mathbf{e}}{4\pi n_I} \right\} \mathbf{V}^*(\mathbf{r}_i + \mathbf{r}_f), \]

where \(\sigma = \frac{p^2 e^2}{3\pi^2 m}\) is the conductivity of the normal metal. Thus, the divergence of the normal current is equal to zero. As applied to the motion of vortex filaments, the last equation gives a stronger result. Namely, the vector potential \(\mathbf{Q}\) at the center of the filament varies like \(\mathbf{r}^{-1}\), where \(\mathbf{r}\) is the distance to the filament axis (15).

Stipulating that the normal current be finite in the center of the vortex \(\rho \ll \xi\), we obtain for the principal terms in \(\Delta / T\)

\[
\mathbf{V}(\mathbf{r}_i + \mathbf{r}_f) = \frac{2\pi m_p e^2}{2\pi c} \frac{d\mathbf{e}}{4\pi n_I}. \]

That is to say, the normal current is equal to zero in this case. The corrections to \(j(\mathbf{a})\) in the next higher approximation in \(\Delta / T\), as seen from the equation for \(j_1 + j_T\), are of the order of \(\Delta / T\) relative to \(c^{-2} \partial \mathbf{Q} / \partial t\) and, consequently, make a small contribution to the energy dissipated by the moving vortex.

3. MOTION OF VORTEX FILAMENTS IN AN ALLOY AT \(T_c - T < T_c\)

Just as in the earlier work (14), we assume a magnetic field \(H_0 \ll H_2\). This means that the vortex-current density \(B_0 / \Phi_0\) is small (\(\Phi_0 = \Phi_0 / \Phi_0\) is the flux quantum and \(B_0\) is the induction), i.e., the distance between filaments is much larger than the dimension \(\xi (T)\) of the core of the filament. This condition enables us to reduce the problem of the motion of a lattice of vortices to the problem of the single filament. Namely, assume that a transport current \(j_T\) perpendicular to the magnetic field \(H_0\) flows through the sample.

As shown by us in (15), the current produced by an individual vortex at distances \(\rho, \xi \ll \rho, \xi\) from the center is equal to the transport current. It is thus necessary to establish the connection between the velocity of a single filament and the current at large distances from the filament center. Knowing the vortex velocity, we can determine from the formula \(\mathbf{E} = -c^{-2} \mathbf{B} / \partial t\) the intensity of the average electric field

\[
\mathbf{E} = \frac{B}{c} [\mathbf{n}_H \times \mathbf{u}]. \tag{21} \]

\(\mathbf{n}_H\) is a unit vector in the magnetic-field direction and \(\mathbf{u}\) is the velocity of the vortex filament., and consequently determine the resistance of the sample. We shall henceforth assume that the Ginzburg-Landau parameter is \(\kappa = 0 / \xi \gg 1\).

The function \(\Delta(\mathbf{r})\) for an immobile vortex was obtained by Abrikosov (15). We shall label the corresponding functions \(\Delta\) and \(\mathbf{Q}\) by the index 0. At low velocities it can be assumed that in first approximation the vortex moves as a unit, i.e., \(\Delta\) and \(\mathbf{Q}\) are the functions \(\Delta_0\) and \(\mathbf{Q}_0\) of \(\mathbf{r} - \mathbf{u}\), and the deviations due to the "deceleration" term are small corrections, \(\Delta_1\) and \(\mathbf{Q}_1\). We choose also a cylindrical coordinate system \((\rho, \varphi, z)\) in which the \(z\) axis coincides at the instant of time with the filament axis. We have

\[
\Delta = \Delta_0(r - ut) + \Delta_1, \quad \mathbf{Q} = \mathbf{Q}_0(r - ut) + \mathbf{Q}_1. \tag{22} \]

We recall that Eqs. (4) without the anomalous term and the time derivative are static equations that are invariant relative to any displacement of the origin. The functions \(\Delta_0(\mathbf{r} + \mathbf{a}) = \Delta_0(\mathbf{r} + \mathbf{a})_0\), where \(\mathbf{a}\) is an arbitrary constant vector, are therefore also solutions of (4). We put \(\Delta_0^2 = \langle \mathbf{a} \cdot \mathbf{V} \rangle \Delta_0\) and \(\mathbf{Q}_0^2 = \langle \mathbf{a} \cdot \mathbf{V} \rangle \mathbf{Q}_0\). Then \(\Delta_1^2\) and \(\mathbf{Q}_1^2\) also satisfy the static equations linearized relative to the small deviations \(\Delta_1\) and \(\mathbf{Q}_1\).

We substitute the expressions of (22) in the equations of (4). Linearizing them and recalling that \(\Delta_0 = t(\mathbf{u} \cdot \mathbf{V}) \Delta_0\) and \(\mathbf{Q}_0 - t(\mathbf{u} \cdot \mathbf{V}) \mathbf{Q}_0\) are static solutions, we obtain

\[
\frac{\pi}{8T} D_i \left[ \mathbf{V} \Delta_0 - \frac{4e^2}{c} \mathbf{Q}_0 \Delta_0 - \frac{8e^2}{c} \mathbf{Q}_0 \mathbf{Q}_0 \Delta_0 \right] + \frac{T}{T_c} - 3\Delta_1 \Delta_0 \left( \frac{8e^2}{c} \mathbf{Q}_0^2 \right) \Delta_1 = -\Delta_0^2, \tag{23'} \]

\[
\text{where} \quad j_T + \frac{m_p e^2 D_i}{2\pi c T} [\Delta_0 \mathbf{Q} + 2\Delta_0 \mathbf{Q}_0] = 0; \tag{23''} \]

\(\Delta_1^0 = \int \frac{d\mathbf{e}}{4\pi n_I} (\mathbf{r}_i + \mathbf{r}_f). \]

The quantity \(\gamma_2 + \gamma_3\) is determined from (20) where \(\Delta_0 / \partial t\) is replaced by \(-\mathbf{u} \cdot \mathbf{V} \Delta_0\) and \(\Delta = \Delta_0\).

As already noted, \(\Delta (\mathbf{a})\) is large, and we can therefore omit from (23') the term \(1 / \kappa T^2 \Delta_0 \partial / \partial t\) in the right-hand side. As to \(j(\mathbf{a})\), this quantity, as already mentioned, is of the order of \(\kappa c^{-2} \mathbf{B} / \partial t / \partial t\).

For a moving vortex we have therefore

\[
\mathbf{j} = \frac{\Delta_0}{\Phi_0} \mathbf{u} - \frac{\Delta_0 \mathbf{H}_0 \mathbf{u}}{T}, \tag{23'''} \]

We shall show below (see (28)) that this makes a smaller contribution (by a factor \(\kappa^2 / T^2\)) to the conductive than the anomalous term \(\Delta (\mathbf{a})\), so that \(j(\mathbf{a})\) can be left out from the right-hand side of (23'').

To determine the current \(j_T\) at large distances \(\rho \gg \xi (T)\) from the center of the filament, we use the procedure employed by us earlier (15) to find the integral.
of the system of equations for $\Delta_1$ and $Q_\mu$. We need also the equation stemming from the superconductor electroneutrality condition:

$$\text{div} [\Delta Q_\mu + 2\Delta Q_{\Delta_1}] = 0.$$  \hspace{1cm} (24)

We are not interested here in the bending of the vortex filaments, and therefore leave out the corresponding terms in (23) and (24). The results can be easily generalized to this case.

Let the velocity of the vortex $u$ be parallel to the $x$ axis. Then  

$$\frac{\partial \Delta}{\partial t} = -(uv)\Delta_0 = -\frac{\partial \Delta}{\partial p} \cos \varphi.$$  

It is seen from (20) that $\Delta(a)$ is also proportional to $\cos \varphi$. If we express (23) and (24) in cylindrical coordinates and separate the dependence on the angle, then we can obtain, using the same procedure as in [8],

$$[a_{\mu} a]_{\mu} - [a_{\nu} a]_{\nu} = -\frac{m \rho}{\pi} \int \Delta^\alpha \Delta p \, dp \, dq.$$  \hspace{1cm} (25)

We now calculate $\Delta(a)$. We can neglect the term with $\partial A$ in the left-hand side of (20), since the vortex velocity is low. We express (20) in cylindrical coordinates, putting

$$\gamma_1 + \gamma_2^* = u/(\rho)w(p) \cos \varphi.$$  

Since $\Delta^\alpha = (a \cdot \nabla) \Delta_0$, the integral in (25) is proportional to the scalar product $u \cdot a$:

$$[a_{\mu} a]_{\mu} = \frac{m \rho u}{\pi} \int \rho \, dp \, \frac{\partial \Delta}{\partial p} \int 4\pi \rho \, dp \, \frac{\Delta}{(\epsilon - \Delta^\alpha)^n}.$$  \hspace{1cm} (26)

and $w$ is obtained from the equation

$$\frac{d}{dp} \left[ \frac{1}{p} \frac{d}{dp} (\rho w) \right] = \frac{2\pi}{T} \frac{d}{dp} \left( (\epsilon - \Delta^\alpha)^n \right), \quad |\epsilon| > \Delta(p).$$  \hspace{1cm} (27)

Equation (27) has a solution that ensures that the functions $\gamma_1 - \gamma_1$ and $\gamma_2 + \gamma_2^*$ are finite at $p = 0$. This solution is

$$w = \frac{2\pi}{T} \frac{1}{p} \left[ (\epsilon - \Delta^\alpha)^n - C \right] p \, dp.$$  

To determine the constant $C$ we recall that in our approximation $\gamma_2 + \gamma_2^*$ should be regarded as different from zero only if $|\epsilon| > \Delta_0$. The condition that $\gamma_2 + \gamma_2^*$ decrease at large distances therefore determines $C$ only when $\epsilon > \Delta_0$, then there exists a certain value $\rho = \rho_\epsilon$ at which $\epsilon = \Delta_0$. Thus, $\gamma_2 + \gamma_2^*$ differs from zero when $\rho < \rho_\epsilon$ and equals zero when $\rho > \rho_\epsilon$.

In the approximation in which the retarded and advanced functions $g_{\tau R} (\overline{A})$ and $g_{\tau R} (\overline{A})$ (16) vary slowly, the rigorous boundary condition for the values of $\gamma_1$ at $\rho = \rho_\epsilon$ necessitates that the solutions be matched together in this region. We consider it natural, however, to impose on the real quantities $\gamma_2 + \gamma_2^* / 4\pi t$ etc. the requirement that they be continuous at the point $\rho_\epsilon$. Putting $w(\rho_\epsilon) = 0$, we get

$$C - C_1 = \frac{2}{\rho_\epsilon} \left( (\epsilon - \Delta_0)^n \right) p \, dp, \quad \epsilon < \Delta_0.$$  

Thus

$$C = \left( (\epsilon - \Delta_0)^n \right) p \, dp, \quad \epsilon < \Delta_0.$$  

To calculate the integral in (26), it is convenient to integrate first with respect to $\rho$ by parts, and then again interchange the order of integration. We introduce the dimensionless quantities

$$\rho = \epsilon \delta, \quad \Delta = \Delta_0 \delta, \quad \epsilon = \Delta_0 \delta,$$

where, as seen from (4)

$$\xi = (\kappa T D / T \gamma (3) \Delta_0)^n, \quad \Delta_0 = (8\kappa T (T_\gamma / T) / T \gamma (3))^{1/3},$$

and $\Delta$ satisfies the equation (we recall that $\rho \ll \delta$, where $\delta$ is the field penetration depth):

$$\left( \frac{d^2}{dp^2} + \frac{d}{dp} - \frac{1}{\rho} \right) \Delta + (1 - \Delta') \Delta = 0$$

with boundary conditions $\Delta \to 1, \, \delta \to \pm \infty$ and $\Delta(0) = 0$. From (26) we have

$$J_0 = (\frac{\pi^4}{T \gamma (3)} \frac{1}{c} \frac{\partial H_\alpha}{\partial \Delta} \delta y'(pT) [a_{\mu} a]_{\mu}.$$  \hspace{1cm} (28)

$$= \frac{1}{2} \frac{\partial K p \rho}{\partial \rho}.$$  

Thus, the constant $\alpha$ we have approximated $\Delta(\delta)$ by the function $\Delta(1 + \delta^2)$, which has the correct asymptotic form as $\rho \to \infty$ and duplicates $\Delta(\rho)$ quite well in the remaining range of $\rho$. All the integrals can then be easily calculated and yield $\alpha = 0.26$.

We see from (28) that in our case there is no velocity perpendicular to the current, i.e., there is no Hall effect in our approximation. Comparing (28) with expression (21) for the electric field, we obtain the effective conductivity of the superconductor with vortex filaments:

$$\sigma_{\text{eff}} = \frac{\pi^4}{T \gamma (3)} \frac{1}{c} \frac{\partial H_\alpha}{\partial \Delta} \delta y'(pT)$$

$$= \frac{1}{2} \left( \frac{1 - T}{T_0} \right)^{-\kappa} \frac{\partial H_\alpha}{\partial \Delta} \delta y'(pT).$$

We can also write down an expression for the viscosity coefficient $\eta$. The energy dissipation per unit volume and per vortex of unit length is $\sigma_{\text{eff}}^2 \Phi_0 / B_0$. Equating it to $\gamma_1$ we obtain

$$\eta = a \left( \frac{\pi^4}{T \gamma (3)} \right) \frac{1}{c} \frac{\partial H_\alpha}{\partial \Delta} \delta y'(pT).$$

For alloys with $l \ll \xi$ we have, in particular ($\gamma(pT) = 1$):

$$\sigma_{\text{eff}} = \frac{\pi^4}{T \gamma (3)} \frac{1}{c} \frac{\partial H_\alpha}{\partial \Delta} = 1,1 \left( \frac{1 - T}{T_0} \right)^{-\kappa} \frac{\partial H_\alpha}{\partial \Delta}.$$  

We call attention to the fact that the temperature dependence of the conductivity in ordinary alloys differs from the dependence of $\sigma_{\text{eff}}$ in alloys with paramagnetic impurities, obtained earlier in [9], owing to the presence of the additional factor $T/\Delta_0$.

The last circumstance reflects the slowness of the relaxation processes near $T_C$. The existence of such a temperature dependence is indicated by recent measurements of $\sigma_{\text{eff}}$ near $T_C$.

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1) Since $\kappa \sim 1$ in real pure superconductors ($T_C \gg 1$), the condition $\kappa \gg 1$ is actually satisfied in experiments only in the limit of "dirty" alloys, where $l \ll \xi$.

2) The vanishing of the normal current evidently allows us to disregard the normal energy loss at the center of the vortex [$^{10,11}$].
3) A temperature dependence of the same kind can be obtained also from our results [9], by cutting off, in order of magnitude, the large parameter $r_s T_c \sim 0$ at the value $T/\Delta$ (see [10]).

4 L. P. Gor'kov and G. M. Eliashberg, ibid. 56, 1297 (1969) [29, 698 (1969)].
15 A. A. Abrikosov, ibid. 32, 1442 (1957) [5, 1174 (1957)].

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