EFFECT OF DISPERSION ON THE NONLINEAR EVOLUTION OF QUASI-MONOCROMATIC HELICAL WAVES IN A MAGNETOACTIVE PLASMA

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The concurrent manifestation of dispersion and nonlinear effects in the evolution of quasi-monochromatic helical waves in a magnetoactive plasma is investigated. (Implied here are the nonlinear effects due to cyclotron interaction between the wave and resonance particles.) It is shown that the role of dispersion becomes important at sufficiently large wave amplitudes. The stationary waves of the envelopes are investigated; in the case under consideration these waves have the form of shock waves with an oscillatory structure behind the wave front.

1. THE BASIC EQUATIONS

During the propagation of circularly polarized waves along a magnetic field, the dominant nonlinear effect (for not too large amplitudes) is the resonance cyclotron interaction of the wave with a group of particles whose velocity is close to the “resonance” velocity \( \nu_R = (\nu - \omega_c)/k \), where \( \omega_c \) is the cyclotron frequency (we consider, for definiteness, electron-cyclotron waves (“whistlers”); the extension of the corresponding results to the case of ion-cyclotron waves is apparent). If a packet of such waves has a sufficiently small spectral width, such that we can neglect dispersion, then its evolution is described by the theory developed in \([1]\).

If, however, the wave front is sufficiently steep (such a situation may, as has been shown in \([1]\), arise owing to the nonlinear twisting of the front due to the cyclotron interaction between the wave and the resonance particles), then the dispersive spreading of the front becomes important and should be taken into account together with the nonlinear effects. The present paper is devoted to the investigation of this problem.

For waves propagating along a magnetic field (the Z axis), Maxwell’s equations are easily reduced to a single equation which is conveniently written in the form

\[
\frac{\partial \mathbf{E}}{\partial t} - \frac{1}{c^2} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c^2} \mathbf{j},
\]  

(1)

where on the right-hand side stands the difference between the total current vector

\[
\mathbf{j} = -en \int f(t, Z, \mathbf{v}) d\mathbf{v}
\]  

(2)

and the same vector \( \mathbf{J}_L \) in the linear approximation, obtained by replacing the total distribution function \( f \) in the expression (2) by the solution \( f_L \), of the linearized equation \( \mathbf{L} \).

We require then, in addition, that the poles in \( \mathbf{J}_L \) be circled in the principal-value sense. Thus, \( \mathbf{J}_L \) determines the Hermitian part of the permittivity tensor, and, consequently, in the matter equation

\[
D_\alpha = \varepsilon_\alpha \mathbf{E} \beta \mathbf{E}_\beta \text{ we should take } \varepsilon_\alpha \mathbf{E} \text{ to mean a Hermitian tensor, as is henceforth assumed.}
\]

Taking into account the fact that the components of the field in a circularly polarized wave have the form

\[
\mathbf{E}_\alpha = A \cos(kZ - \omega t + \varphi); \quad \mathbf{E}_* = -A \sin(kZ - \omega t + \varphi),
\]

where \( A = A(Z, t) \) is a real amplitude and \( \varphi = \varphi(Z, t) \) is the slowly varying part of the phase which determines the correction to the frequency and the wave number, \( \delta\omega = -\partial\varphi/\partial t, \delta k = \partial\varphi/\partial Z \), it is convenient to introduce the following complex quantities:

\[
E = Ae^\varphi, \quad \mathbf{E}_* = -\mathbf{E}_* = E \exp(ikZ - i\omega t), \quad \mathbf{D}_* = i\mathbf{D}_* \cdot \mathbf{b}_t = \mathbf{b}_t = eA \int (v_t - iv_s) (f_L - f) d\mathbf{v}.
\]  

(3)

Then Eq. (1) can be easily reduced to the form

\[
\frac{\partial \mathbf{D}_*}{\partial t} = -i\frac{\partial \mathbf{E}}{\partial Z} - \frac{4\pi}{c^2} \mathbf{b}_t,
\]  

(4)

where \( \mathbf{N}(\omega) \) is the refractive index for the whistlers

\[
\mathbf{N}(\omega) = a s(\omega, -\omega).
\]  

(5)

The expansion

\[
-\frac{\partial^2 \mathbf{D}_*}{\partial \omega^2} = -\left\{ \frac{\partial}{\partial \omega}(\mathbf{N}(\omega)E) + \frac{\partial}{\partial \omega}(\mathbf{E}) + \frac{\partial}{\partial \omega} \mathbf{E} \right\} + \left\{ \frac{1}{2} \frac{\partial}{\partial \omega} \mathbf{E}^2 + \ldots \right\} \exp(ikZ - i\omega t),
\]  

(6)

where the dotted line stands for terms with third and higher-order derivatives with respect to the complex amplitude \( E(Z, t) \), may be used in the transformation of Eq. (4). We shall henceforth consider \( \partial \mathbf{E}/\partial t \) and \( \partial \mathbf{E}/\partial Z \) as terms of first order in smallness, and the second derivatives on the right-hand side of Eq. (4), as second-order terms. Then, the usual relation

\[
(\partial/\partial t + v_g \partial/\partial Z)E = 0,
\]

where \( v_g = d\omega/\partial k \) is the group velocity, follows, in the first approximation, from (4) and (6).

Retaining terms of second order in smallness, we obtain from (4) and (6) the following equation:

\[
\left( \frac{\partial E}{\partial t} + v_g \frac{\partial E}{\partial Z} \right) + \frac{1}{2} \frac{\partial}{\partial \omega} \mathbf{E}^2 = -2i\omega \frac{\mathbf{D}_*}{c^2} \mathbf{b}_t \exp(-ikZ - i\omega t),
\]  

(7)

where \( v_g' = dv_g/\partial k = d^2\omega/\partial k^2 \) (here, \( v_g' \mathbf{E}/c^2 \) has been substituted in place of \( \partial \mathbf{E}/\partial t \)).
It follows from Eq. (7) that \( \varphi(Z,t) \) and \( A(Z,t) \) satisfy the following system of equations:

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial z} \right) \varphi + \frac{v_r}{2} \left( \frac{\partial^2 \varphi}{\partial z^2} \right) &= -2\gamma \omega_l \Im \left( \frac{\mathcal{F}(\delta \varphi)}{\delta \varphi} \right), \\
\left( \frac{\partial}{\partial t} + v_r \frac{\partial}{\partial z} \right) A + \frac{v_r}{2} \left( \frac{\partial^2 A}{\partial z^2} \right) &= -2\gamma \omega_l \Re \left( \frac{\mathcal{F}(\delta \varphi)}{\delta \varphi} \right),
\end{align*}
\]

where \( \omega_l \) is the linear increment, \( \gamma \) is a constant, \( \mathcal{F}(\delta \varphi) \) is the function of the resonance particles in the plasma, and \( \Im \) and \( \Re \) denote the imaginary and real parts, respectively.

Equation (11) is a consequence of the energy-conservation law and the fact that on the average the wave with a constant-in-space amplitude, the quantity \( \Re(\mathcal{F}(\delta \varphi)/\delta \varphi) \) does not depend on the amplitude (and not on its derivatives, as this designation is connected with the fact that for a wave with a constant-in-space amplitude, the quantity \( \Delta \omega \) is proportional to the square root of the wave amplitude).

On the other hand, since we use the expression (12b), which was obtained in [11] in the adiabatic approximation, it is necessary that the length (17) be considerably greater than the nonlinear length \( \gamma_L \approx \omega_0 \tau_m \). This imposes the following limitation on the wave amplitude:

\[
\gamma_L \omega_0 \rightarrow \frac{1}{\omega_0} \left( 1 - \frac{4\omega_0}{\omega_1} \right).
\]

It can be seen from (14) and (18) that in order for the dispersion to manifest itself under the conditions of the adiabatic approximation, the wave should be sufficiently intense. In the opposite limiting case when \( l \ll \omega_0 \tau_m \), the dispersion effects are of little importance compared to the nonlinear modulation effects considered in [11] for a packet with a steep leading edge (see Sec. 6 of the paper cited).

Let us turn now to the analysis of Eq. (8). Let us set

\[
\Delta \omega = \frac{2\gamma \omega_l \Im \left( \frac{\mathcal{F}(\delta \varphi)}{\delta \varphi} \right)}{k c' \left| \delta \varphi \right|^2},
\]

and the function \( \mathcal{R}(y) \) is given parametrically:

\[
R = 1/K(K), \quad y = E(K)/K(K) \quad \text{and} \quad E(K) \quad \text{are complete elliptic integrals of the first and second kinds}.
\]

Equation (11) is a consequence of the energy-conservation law and the fact that on the average the wave does work on the resonance particles, which propagate with the average resonance velocity \( v_R = (\omega - \omega_c)/k \) (for the whistlers this velocity is always directed in the direction opposite to that of the motion of the packet, since \( \omega < \omega_c \)). The relation (11) can, of course, be obtained directly from the expression (2) for \( \delta \varphi \) and the kinetic equation for the resonance particles in the packet. Using (10) and (12a) and going over to a coordinate system moving with the group velocity, \( z = -Z - vt \) (the \( z \) axis in this system is directed into the packet, i.e., opposite the direction of its motion), we finally write Eq. (9) in the following form:
It can be seen from (20) that $\Delta \omega$ is, generally speaking, an extremely small quantity. More precisely, it follows from (6) and (20) that under the condition that
$$\nu_0 \tau_0 \gg \beta \ell,$$
which we shall henceforth assume to be fulfilled\(^3\), the right-hand side of Eq. (8) can be neglected.

2. HELICAL SHOCK WAVES

As the simplest solution to Eqs. (15) and (8) which gives some idea about the nature of the effects due to the concurrent manifestation of dispersion and nonlinear resonance interaction between the waves and the particles, let us consider a steady-state solution of the form
$$h = h(z - vt).$$
Substituting (22) into (8) and integrating, we obtain
$$\psi' = \left(\psi^2 - \nu_0 h(k + C)/\nu_k h^2\right),$$
where $C$ is the constant of integration and a prime denotes a derivative. Substituting (23) in (15), we shall have
$$h'' + \left(\nu_0 h - [\nu_0 h(k - C)]/\nu_k h^2\right) = 0.$$  
(24)

Let us choose the constant $C$ such that the resulting wave propagates into the unperturbed region, i.e.,
$h \to 0, \psi' \to 0$ as $z \to -\infty$ (we recall that our axis is directed against the motion of the packet). It should, however, be borne in mind here that the expression for $r(h)$ is, strictly speaking, not valid for very small $h$, since it was obtained in [11] under the condition that $\gamma L_0/m \ll 1$ (which is, in a sense, contrary to the linear-approximation condition). To overcome this difficulty, we replace the above-indicated boundary conditions by the following conditions:
$$h \to h_0, \quad \psi' \to 0 \quad \text{for} \quad z \to -\infty,$$
(25)
where $h_0$ is negligibly small compared to the maximum amplitude $h_m$, but, on the other hand, $\gamma L_0/m \ll 1$ ($\tau_0$ is the nonlinear time (14) in which $h_0$ has been substituted for $h_m$).

It follows from (23)-(25) that
$$C = -\nu_0 h + \nu_0 (h_0).$$
(26)
Substituting this in (24) and considering the region of the wave in front of the maximum, we must use the first of the expressions of (12b). Assuming that $h \gg h_0$, we obtain the following equation for the amplitude ($h_m = h_{\text{max}}$):
$$h'' + \left[\frac{\nu}{\nu_0} h - \frac{16\nu_0 \nu h_m}{0^2}\right] = 0.$$  
(27)
The solution of (27) has the form
$$h = \frac{16\nu_0 \nu h_m}{0^2} + H \cos \left(\frac{\nu}{\nu_0} \zeta\right),$$
where $\zeta = z - vt + \text{const}$, and $H$ is the constant of integration. Let us assume that the maximum value of the amplitude is attained at $\zeta = 0$. Then, it follows from (28) that

$$H = h_0 \left(1 - \frac{16\nu_0 \nu h_m}{0^2}\right) + O(h_0).$$
(29)

In order to satisfy the boundary condition (25), we should consider the solution to the more exact equation (24), where the constant is defined in (26). We can, however, proceed more simply by matching the approximate solution (28) with zero in such a way that the continuity of the first derivative is preserved. For this purpose we should require that $h(\xi) = h'(\xi) = 0$ when $\xi \leq \xi_0$. Hence, we obtain $\xi_0 = -\pi/\nu_0 |v|$, and
$$h = h_0 \left(1 - \cos \frac{\nu}{\nu_0} \xi\right), \quad \xi < \xi_0 < 0,$$
(30)
The approximate solution (30) is quite a good substitute for the exact solution, since it differs from the latter in a region of little importance where $\gamma L_T \gtrsim 1$.

The two different signs in expression (31) for $v'$, indicate that two types of waves—fast (whose velocity is higher than the group velocity by the amount $|v|$) and slow (whose velocity is smaller by the same amount)—can exist independently.

The length of the front part of the wave where $h$ changes from 0 to $h_m$ is, as was to be expected, equal to
$$\frac{\nu}{v} = \frac{3\pi}{16} \frac{\nu_0}{v} \sim l.$$  
(31)

Let us now investigate the profile of the wave behind the point where the amplitude is a maximum. In this region the second of the expressions in (12b) should be chosen for the function $r(h)$. Let us write Eq. (24) (where the constant $C$ can, as before, be neglected) in terms of the dimensionless variables $\eta = h/h_m$, $\lambda = v \xi/v', \rho(\eta) = 9r^2(h)/16a^2 h^2$ for $h_m^2$:
$$\frac{d^2 \rho}{d \lambda^2} - \frac{\nu_0}{\nu_0} \rho + \eta = 0.$$  
(32)

Equation (32) coincides in form with the equation for the motion of a particle in a force field with the potential energy:
$$P(\eta) = \frac{1}{2} \left[v' - 1 - \int_0^\eta \rho(\eta') d\eta'\right].$$
(33)
The arbitrary constant in $P(\eta)$ is chosen such that the potential is continuous at the point $\eta = 1$. The shape of $P(\eta)$ is shown in Fig. 1 (the integral in (33) was computed numerically). The amplitude of the envelope of the wave will behave in the same way as the coordinate of a “particle” oscillating in the potential shown; therefore, the profile of the wave is periodic, undamped oscillations with a spatial period $\sim l$. The resulting profile of the amplitude is shown in Fig. 2. It has the form of a shock wave with undamped oscillations behind the front, the characteristic period of these oscillations being the same as the width of the transition layer, i.e., of the order of the length (17) over which the nonlinear effects are comparable with the dispersion effects. The modulation index behind the wave front is quite high: it is equal to the distance $ab$ in Fig. 1, i.e., of the order of half the maximum ampli-

\(^3\) It follows from (21) and (18) that the amplitude of the wave should, in our case, satisfy the condition $l \gg \nu_0 \tau_0 \gg \beta \ell$.

\(^4\) The auxiliary constant $h_0$ then drops out of the solution.
FIG. 1. "Potential energy" $P(\eta)$: 1—before the first maximum of the envelope of the shock-wave amplitude; 2—behind the wave front after the maximum.

FIG. 2. Profile of the shock wave.

The periodic modulation behind the wave front leads to the appearance of an effective mean shift of the wave number by the amount $\delta k = -\delta \zeta$. It follows from formula (23) (where we can set $h \sim h_m$ and $C = 0$) that $\delta k \sim v/V_g \sim \pm 1/\lambda$. Correspondingly, the average frequency shift $\delta \omega = v_c \delta k \sim \omega_c \gamma L \tau_m$. This quantity is considerably higher than the nonlinear shift (20): $\delta \omega / \Delta \omega \sim (\tau_m \omega_c)/\beta \gg 1$. A frequency broadening of the spectrum $\Delta f \sim 1/\tau_m$ should be simultaneously observed.

Let us finally consider the results obtained from the point of view of the experiments on the observation of monochromatic whistlers emitted by ground-level transmitters with frequency $f = 15 \text{kHz}$ and propagating along the lines of force of the geomagnetic field. Reports of the experimental observation of this kind of amplitude modulations, which arise in the process of propagation and are apparently nonlinear in nature, have recently been published. The frequency broadening of the spectrum in such experiments is $\Delta f = 10^2 \text{Hz}$. We must consequently assume that $\tau \sim 10^{-2} \text{sec}$. The generation time for the satellites (which, according to [5], is of the order of $\gamma L^2$) is approximately $0.1 \text{sec}$.\(^5\)

Thus, it is reasonable to assume $\gamma L \tau_m \sim 0.1$. The cyclotron frequency in the equatorial region for the line of force along which the signal propagates (this region is roughly three times the earth’s radius away from the earth’s center), $f_c \sim 30 \text{kHz}$. The mean group velocity computed from the time lag of the signal, $V_g \sim 4 \times 10^8 \text{cm/sec}$. The length (17) computed from these data is $l \sim 2 \times 10^5 \text{cm}$. On the other hand, the nonlinear length $N = v_g \tau \sim 8 \times 10^7 \text{cm}$. Thus, $l \ll l_N$ in the experiments discussed and, consequently, the observed amplitude modulation apparently corresponds to the mechanism, considered in [1], in which dispersion is unimportant. The dispersion-nonlinear mechanism considered in the present paper appears in the case of waves of much higher intensity.

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\(^5\)This time can be estimated in the following way. Ground-level transmitters emitted dot-dash Morse-code type of signals. The trigger radiations were observed almost exclusively in the dash-type pulses (duration 150 usec); they were very rarely observed when the dots (duration ~ 50 usec) were transmitted. Hence it follows that the generation time for the trigger radiation is ~ usec.

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