ION ACCELERATION UPON EXPANSION OF A RAREFIED PLASMA

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Acceleration of impurity ions of various masses and charges upon expansion of a rarefied plasma into vacuum is investigated. Acceleration of the ions is produced by a self-consistent electric field arising during expansion of the plasma. It is shown that as a result of acceleration a considerable part of the impurity ions may acquire energies of the order of \(10^2\) to \(10^3\) \(T_e\), where \(T_e\) is the plasma electron temperature. Excitation of self-similar and ion-sound waves is investigated. Results of numerical calculations are presented.

1. INTRODUCTION

When a rarefied (collisionless) plasma expands in vacuum, the electrons strive to overtake the ions. This results in an uncompensated space charge and in an electric field that decelerates the electrons and accelerates the ions. The plasma expansion process was considered by us earlier\(^1\), and we have shown that it is scale-invariant (self-similar) and is described by a nonlinear equation for the ion distribution function:

\[
\left( u_i - \frac{\partial g_i}{\partial \tau} \right) - \frac{1}{2} \frac{\partial g_i}{\partial u_i} \frac{\partial \psi}{\partial u_i} = 0, \tag{1}
\]

where \(u_i = v_i(2T_e Z_i/M_i)^{-1/2}\) is the dimensionless velocity, \(v_i = \sqrt{\frac{T_e}{M_i}}\), \(\tau = x^2 (2T_e Z_i/M_i)^{-1/2}\), \(T_e\) is the electron temperature, \(M_i\) is the ion mass, \(Z_i\) is the ion charge, and \(x\) is the direction in which the plasma expands. Further, \(g_i = \frac{1}{\pi^{3/2}} F(x, v_i, t)\), where \(F\) is the distribution function of the ions with respect to the velocity \(v_i\). By virtue of the scale invariance of the problem, the function \(g\) depends only on the ratio \(x/\tau\). In the derivation of (1) and (2) it is also assumed that the characteristic dimension of the inhomogeneity is much larger than the Debye radius in the plasma,

\[
R_D = \frac{1}{\sqrt{\pi}} D, \tag{3}
\]

and the ion velocity considered are small in comparison with the electron thermal velocity:

\[
|u_i| \ll \left( \frac{M_i}{m_e} \right)^{1/2}. \tag{4}
\]

When these conditions are satisfied, the potential \(\psi\) of the electric field in the plasma is connected with the ion concentration by the simple relation (2); here

\[
\psi = \psi_0 / Z_i. \tag{5}
\]

The boundary conditions of (1) and (2), corresponding to expansion in vacuum of a half-space filled with a plasma, are

\[
g_i \rightarrow \exp(-\beta_1 u_i) \quad \text{as} \quad \tau \rightarrow -\infty, \tag{6}
\]

\[
g_i \rightarrow 0 \quad \text{as} \quad \tau \rightarrow +\infty,
\]

where \(\beta_1 = T_e / T_{11}\), and \(T_{11}\) is the ion temperature in the unperturbed plasma \((\tau \rightarrow -\infty)\).

The previously obtained\(^1\) solution of (1) and (2) shows that part of the ions in the strongly rarefied region is noticeably accelerated by the action of the electric field. It is obvious, however, that the acceleration of the ions by the electric field depends strongly on their mass and charge. One can expect, for example, multiply charged impurity ions to acquire the highest energy when the plasma expands. The present paper is devoted to an analysis of the problem of impurity-ion acceleration in a plasma.

2. ACCELERATION OF IMPURITY IONS

We assume that in addition to the main ions, of mass \(M_1\) and charge \(Z_1\), the plasma contains also a small admixture of ions of mass \(M_2\) and charge \(Z_2\). The impurity-ion distribution function \(g_2(\tau, \psi)\) is described by the equation

\[
\left( u_2 - \frac{\partial g_2}{\partial \tau} \right) - \frac{1}{2} \frac{\partial g_2}{\partial u_2} \frac{\partial \psi}{\partial u_2} = 0. \tag{7}
\]

Here \(u_2 = v_2(2T_e Z_2/M_2)^{-1/2}\), and \(v_2 = v_2 x\) is the velocity of the impurity ions. The dimensionless potential \(\psi(\tau)\) of the electric field is determined by relations (1) and (2). The boundary conditions for Eq. (7), in the case of expansion of a half-space in vacuum, are

\[
g_2 = \alpha \exp(-\beta_2 u_2 M_2 / M_1) \quad \text{as} \quad \tau \rightarrow -\infty, \quad g_2 \rightarrow 0 \quad \text{as} \quad \tau \rightarrow +\infty. \tag{8}
\]

Here \(\beta_2 = T_e / T_{12}\), and \(\alpha\) is a normalization constant proportional to the concentration of the impurity ions.

Equation (7) is linear. Its solution can be easily obtained by integrating the equation of the characteristics \(u_2(\tau)\)

\[
u_2(\tau) = \nu_2^0 + \int_{\nu_2^0}^{\nu_2(\tau)} \frac{\partial \psi}{\partial \tau} d\tau.
\]

The values of the distribution function \(g_2\) are conserved on the characteristics; they are determined by formula (8). Equations (9) were integrated numerically for 300 characteristics. The characteristics in the \((u, \tau)\) plane for \(M_2Z_2/M_1Z_1 = 16\) are shown by way of example in Fig. 1. The same figure shows the characteristics of (1). We see that the impurity ions are much more energetically accelerated than the ions of the main gas. The distribution functions of the impurity ions are shown in Fig. 2, while Fig. 3 shows their concentration \(N_2\), the flux \(J_2 = N_2u_2\), and the mean energy \(\epsilon = T_e Z_2^2 M_2/M_1\). The impurity-ion concentration decreases quite slowly.
with increasing $\tau$, while the changes of the flux $j_2$ are very small. This means that the greater part of the flux of impurity ions is captured by the field and is accelerated to high energies. For example, the flux at $\epsilon \sim 500$ $T_0$ $Z_1$ is $j_2 \sim 0.4 j_{20}$, where $j_{20} = j_2$ ($\tau = 0$).

At large values of $\tau$, the translational velocity of the ions is high. The thermal spread of the velocities is then of little importance, since it is convenient to analyze the behavior of the solution by using the hydrodynamic equations. Assuming in (7) a distribution function of the form:

$$g_1 = N_0 \delta(u_1 - u_0(\tau)) \text{e}^{\xi},$$

we obtain:

$$(u_1 - \xi) \frac{dN_1}{dt} + N_1 \frac{du_1}{dt} = 0,$$  \hspace{1cm} (10)

$$(u_1 - \xi) \frac{du_1}{dt} + Z M \frac{d\phi}{2Z M \frac{du_1}{dt}} = 0.$$  \hspace{1cm} (11)

We recognize, in addition, that in the hydrodynamic approximation:

$$\frac{d\phi}{dt} = \begin{cases} 0 & \tau < -2^{-1/2}, \\ -2^{-1/2} & \tau \geq -2^{-1/2}. \end{cases}$$  \hspace{1cm} (12)

It follows therefore that the plasma is unperturbed at $\tau < -2^{-1/2}$. The boundary conditions for (10) and (11) are therefore specified at $\tau = -2^{-1/2}$:

$$N(\tau = -2^{-1/2}) = N_{0u}, \quad u_1(\tau = -2^{-1/2}) = 0.$$  \hspace{1cm} (13)

Integrating (11) and (12), we have:

$$u_1 = \tau + z - \alpha_1(\tau), \quad z = Z M \frac{1}{2Z M} \frac{d\phi}{du_1}.$$  \hspace{1cm} (14)

where $\alpha_1(\tau)$ is defined by:

$$a_1 = z \ln a_1 + C = \tau.$$  \hspace{1cm} (15)

Using the boundary condition (13), we determine the constant $C$:

$$C = z \ln (z - 2^{-1/2}) + z.$$  \hspace{1cm} (16)

It is assumed here that $z > 2^{1/2}$, i.e., $Z_2 M_1 / Z_1 M_2 > 1$. Substituting equations (14) and (15) for $u_1(\tau)$ in (10) and integrating the latter, we get:

$$N_0(\tau) = \frac{\alpha_1(\tau) N_{0u}}{(2^{1/2} - 1) (2^{1/2} - \alpha_1(\tau))}.$$  \hspace{1cm} (17)

The same expressions hold also when $z < 2^{1/2}$, but in this case $\alpha_1 < 0$ and therefore in $\alpha_1$ and $\ln (z - 2^{1/2})$ are replaced by $\ln (-\alpha_1)$ and $\ln (2^{1/2} - z)$, respectively. A plot of $\alpha_1(\tau, z)$ determined from formulas (15) and (16) is shown in Fig. 4. Here:

$$z = \frac{\tau}{z} + 1 + \ln \left( \frac{z}{z - 2^{1/2}} \right).$$

As $x \to 1$ we have:

$$a_0 / z = 1 - [2(z - 1)]^z,$$

and at $x \gg 1$,

$$a_0 / z = (x^2 - 1)^{-1}.$$

Knowing $\alpha_1$, we can obtain from formulas (14) and (17) the average velocity, the concentration, the flux $j_2 = N_0 u_2$, and the average energy $\epsilon = M_2 Z_1 T_0 u_2^2 / M_1$ of the impurity ions. At large values of $\tau$ we have $\alpha_1 \to 0$. It then follows from (17) that:
It is easy to obtain the energy distribution of the accelerated impurity ions. Indeed, let \( n \) be the total number of impurity ions passing through a unit surface at the point \( x_0 \), namely, \( n = \int J \, dt \). Then

\[
\frac{dn}{dE} = \frac{1}{M_i} \frac{dJ_i}{dt} \left( \frac{Z_i M_i}{M_e} \right)^n \left( \frac{dE}{dt} \right)^{-1}.
\]

Recognizing that

\[
\frac{dJ_i}{dt} = \frac{M_i}{M} \frac{dz_i}{dt},
\]

and expressing \( N_i \) and \( \tau \) in terms of \( \epsilon \) with the aid of formulas (14)–(17), we find at \( \epsilon \geq \epsilon_k \)

\[
\frac{dn}{dE} = \frac{z_i N_i}{T_e} \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right) - 1 + \left(1 - \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right)\right)^{n-1},
\]

and at \( \epsilon \leq \epsilon_k \)

\[
\frac{dn}{dE} = 0.
\]

Here \( \rho \) and \( \epsilon_k \) are defined by the relations

\[
\rho = \frac{M Z_i^2}{2 M Z_0}, \quad \epsilon_k = \frac{E}{\rho T_e} = \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right) - 1 + \left(1 - \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right)\right)^{n-1}.
\]

At \( \epsilon = \epsilon_k \) the density \( dn/dE \) becomes infinite. This is not surprising, since the energy \( \epsilon \geq \epsilon_k \) corresponds to \( \tau = 0 \), i.e., particles with such an energy appear at the point \( x_0 \) only if \( t \to \infty \). For any finite value of \( t \) we always have \( \epsilon > \epsilon_k \).

It is seen from (19) that the energy distribution of the accelerated impurity ions does not depend on the observation point \( x_0 \). At sufficiently high energies, \( \epsilon/\rho T_e \gg 1 \), it turns out to be, furthermore, similar for ions with different charges and masses:

\[
\frac{dn}{dE} = \frac{z_i N_i}{T_e} \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right) \left[\left(1 - \frac{Z_i M_i}{Z_0 M_0} \exp\left(-\frac{\epsilon}{\epsilon_k}\right)\right)^n - 1\right].
\]

The scaling parameter is \( \rho = M Z_i^2 / 2 M Z_0 \), where \( M_i \) and \( Z_0 \) are the mass and charge of the impurity ions, and \( M_i \) and \( Z_i \) are the mass and charge of the main ions of the plasma. If \( \rho \gg 1 \), then the impurity ions are accelerated much more energetically than the main ions. It is also important that the \( dn/dE \) distribution decreases relatively slowly with increasing particle energy \( \epsilon \).

The total number of ions \( n(\epsilon) \) that acquire an energy higher than \( \epsilon_0 \) during the acceleration, is given at \( \epsilon_0 \gg \rho T_e \) by the expression

\[
n_{>\epsilon_0} = 2 N_i z_i \frac{M_i}{M_0} \left(\frac{\epsilon_0}{\epsilon_k}\right)^n \exp\left(-\frac{\epsilon_0}{\epsilon_k}\right) - 1.
\]

It follows therefore, for example, that 0.1% of the total number of impurity ions \( N_{i0} x_0 \) acquires an energy \( \epsilon \geq 50 \rho T_e \).

Acceleration of multiply-charged ions in expansion of a rarefied plasma was observed by Bykovskii et al.\(^{[3]}\) Theoretical estimates based on formula (2) agree with the results of these experiments\(^{[1]}\).

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\(^{[1]}\) Acceleration of the ions of the main plasma was observed by Plyutto et al.\(^{[4]}\)

\(^{[2]}\) This agrees with the results of a numerical calculation \(^{[4]}\). We note that the maximum velocity of the main ions, \( u_m = \tau_m \), increases with increasing ratio \( R_p/D_o \). In the calculation of Widner et al.\(^{[4]}\) this ratio was assumed to be small (\( R_p/D_o \sim 10^2 \)). This was apparently the cause of the relatively low value of the maximum ion velocity. Under real conditions \( R_p/D_o \) can equal \( 10^3 - 10^7 \). In the foregoing estimates of the impurity-ion acceleration it was assumed that \( \tau_m = 5 \); this holds true if \( R_p/D_o \gtrsim 10 \).
celerated most energetically, their concentration $N_2$ decreases with increasing $\tau$ much more slowly than the concentration of the main ions $N_1$. At a certain value of $\tau$, the concentrations $N_2$ and $N_1$ become comparable. Then the entire plasma motion is significantly altered by the impurity ions. These phenomena will be analyzed in the next section.

3. EXPANSION OF AN IMPURITY-ION-CONTAINING PLASMA IN VACUUM

It was assumed above that the concentration of the impurity ions is low in comparison with the concentration of the main ions of the plasma, so that the influence of the impurity ions on the motion of the main ions was neglected. We assume now that these concentrations are comparable, i.e., the plasma contains a mixture of ions of two kinds. The expansion, in vacuum, of a plasma containing a mixture of ions with $M_1$, $Z_1$ and $M_2$, $Z_2$ is described as before by the kinetic equations (1) and (7) with boundary conditions (6) and (8). All that changes is the expression for the dimensionless potential

$$\psi(\tau) = \ln(Z_1N_{10} + Z_2N_{20}) - \ln \left[ Z_1 \frac{1}{M_1} \int_{-\infty}^{\infty} g \, du_1 \right] + Z_2 \left( \frac{M_2}{M_1} \right)^{\frac{1}{2}} \int_{-\infty}^{\infty} g \, du_2. \tag{23}$$

This formula follows from the quasineutrality condition $N_0 = Z_1N_{10} + Z_2N_{20}$. If the impurity-ion concentration $N_{20}$ is negligibly small, then (23) goes over into (2).

Owing to the dependence of $\psi$ on $N_{10}$ and $N_{20}$, Eqs. (1) and (7) are coupled.

The result of the numerical integration of (1), (7), and (23) is shown in Figs. 5–7. Figure 5 gives the characteristics of Eqs. (1) and (7) in the $(u, \tau)$ plane for $Z_2/Z_1 = 16$ and $M_1 = M_2$. Figures 6 and 7 show the distribution functions and the concentrations $N_1(\tau)$ and $N_2(\tau)$. Figures 5–7 represent in fact the same case as Figs. 1–3 in the preceding section, except that in Sec. 1 it was assumed that $N_2 \ll N_1$ and here we assume a finite value of $N_2$. We assume also that in the unperturbed plasma $N_{20}Z_2/N_{10}Z_1 = 0.1$. Comparing Figs. 1–3 with Figs. 5–7 we see that in the region $\tau \geq 3.5$, where $N_2Z_2$ becomes comparable with $N_1Z_1$ and then exceeds it, the characteristics and the distributions $N(\tau)$ differ significantly in the different cases. We see first that the concentration $N_1(\tau)$ of the main ions decreases sharply, practically to zero, at values of $\tau$ exceeding a certain $\tau_K$. The concentration of the impurity ions, on the contrary decreases slowly at $\tau > \tau_K$, and the function $N_2(\tau)$ even has an appreciable plateau at $\tau \geq 5$. At the same values of $\tau$, all the characteristics of the impurity ions also flatten out (see Fig. 5). The distribution function hardly varies with changing $\tau$ in the plateau region. A characteristic plateau region occurs also in other cases in which a plasma containing a mixture of ions is expanded in vacuum.

To analyze the character of these features and their dependence on the parameters it is natural to use, as before, the hydrodynamic equations, which in the case of a mixture of two sorts of ions have the following form:

$$\begin{align*}
(u_1 - \tau) \frac{dN_1}{d\tau} + N_1 \frac{du_1}{d\tau} &= 0, \\
(u_2 - \tau) \frac{dN_2}{d\tau} + N_2 \frac{du_2}{d\tau} &= 0,
\end{align*} \tag{24}$$

$$\begin{align*}
(u_1 - \tau) \frac{dM_1}{d\tau} + M_1 \frac{du_1}{d\tau} &= 0, \\
(u_2 - \tau) \frac{dM_2}{d\tau} + M_2 \frac{du_2}{d\tau} &= 0.
\end{align*} \tag{25}$$

Here $N_1$ and $N_2$ are the concentrations and $u_1$ and $u_2$ the mean velocities of the ions. The equations (24) and (25) possess an integral

$$Z_1N_1(u_1 - \tau)^2 \left[ (u_1 - \tau)^2 - \frac{1}{2} \right] + Z_2N_2(u_2 - \tau)^2 \left[ (u_2 - \tau)^2 - \frac{M_2}{M_1Z_2} \right] = 0. \tag{26}$$

We consider the case of low impurity-ion concentration in the unperturbed plasma: $Z_2N_{20} \ll Z_1N_{10}$. In the
first approximation, the concentration $N_2$ can be neglected in the expression for the force, and the solution of (24) and (25) coincides with that considered in the preceding section. Then, as is clear from (18), if $M_1Z_2/M_2Z_1 < 1$, then the concentration of the impurity ions decreases more rapidly than that of the main ions with increasing $\tau$, so that the condition $Z_2N_2 \ll Z_1N_1$ is always satisfied. If, to the contrary,

$$M_1Z_1/M_2Z_1 > 1,$$

(27)

then the concentration of the impurity ions decreases with increasing $\tau$ more slowly than that of the main ions. Their relative concentration then increases, and at a certain value of $\tau$ the quantity $Z_2N_2$ becomes comparable with $Z_1N_1$. The character of the motion then changes strongly. To examine the solution in the region where $Z_2N_2 \sim Z_1N_1$, we use the integral (26). We recognize that, according to (18),

$$u_1 - \tau = 2^{-\alpha}M_1Z_1/M_2Z_1.$$

It follows then from (26) that

$$u_1 - \tau = -\frac{Z_1N_1}{2[Z_1N_1(1-M_1Z_1/M_2Z_1)+Z_2N_2]}.$$  

(28)

Substituting this expression in the first equation of (24), we get

$$\frac{dn_1}{d\tau} + n_1(1 - (2\tau^2)^{-1}) + \left(1 - (2\tau^2)^{-1}\right)n_1^2 \frac{dn_1}{d\tau} = -1,$$

$$n_2 = Z_1N_1,$$  

(29)

If the concentration of the impurity ions is negligibly small ($n_2 \to 0$), then the solution of (29) coincides with that obtained earlier (15):

$$n_1 = C e^{-2\tau^2},$$

(30)

We recognize that, according to (18), if $z > 1$ the concentration $n_1$ changes much more slowly than $n_2$. Neglecting therefore the variation of $n_1$ in (29) and integrating the latter, we obtain

$$2 \ln [n_2 + (n_2 + (1 - 2z^2)^{-1})^{1/2}] + 2 \left(n_2 + (1 - 2z^2)^{-1}\right)^{1/2} = -\tau + C + 2 \ln (Z_1N_1)^2 + (Z_1N_1 - n_2Z_2 (1 - 2z^2)^{-1})^2$$

$$-n_2Z_2(1 - 2z^2)^{-1} - 2z^2.$$  

(31)

At $n_1 \ll n_2$, the solution of (31) coincides with (30). In the region of $\tau$ where $n_1 \lesssim n_2 (1 - 2z^2)^{-1}$, the course of the solution becomes distorted, and the concentration begins to decrease much more rapidly.

At $\tau = \tau_k$, where

$$\tau_k = C - \ln [n_2 + (1 - 2z^2)^{-1}] = 2 \left(n_2 - n_2Z_2(1 - 2z^2)^{-1}\right)^{1/2}$$

$$-2z^2 \ln (n_2Z_2)^2 + (n_2Z_2 + n_2Z_2(1 - 2z^2)^{-1})^2$$

$$- (n_2Z_2(1 - 2z^2)^{-1})^2,$$  

(32)

the concentration $n_1(\tau)$ vanishes. In other words, at $\tau > \tau_k$ there are no ions of the main gas ($N_1 = 0$).

Equations (25) for the impurity ions assume at $\tau > \tau_k$ the form

$$u_1 - \tau = -\frac{M_1Z_1}{2M_2Z_1N_1} \frac{dn_1}{d\tau} = 0,$$

$$u_1 - \tau = -\frac{M_1Z_1}{2M_2Z_1N_1} \frac{dn_1}{d\tau} = 0.$$  

(33)

According to (18), the boundary conditions at $\tau = \tau_k$ are

$$n_1 = N_{n_1} = N_2 \exp (-\left[1 + \tau_1/\tau_k\right]),$$

$$u_1 = u_{n_1} = \tau + 1.$$  

(34)

The solution of (33) with the boundary condition (34) is

$$n_1 = N_{n_1},$$

$$u_1 = u_{n_1} = \tau + 1,$$  

(35)

as $\tau \gg \tau_k$,

where

$$\tau_1 = \tau + 1 - 2z^2.$$  

We see therefore that at $\tau_k < \tau < \tau_k$, a plateau region is indeed produced. The width of this region is

$$\Delta \tau = z(1 - 2z^2)^{-1} = \frac{Z_1M_1}{Z_2M_2} \left(1 - \left(\frac{M_1Z_1}{M_2Z_2}\right)^{1/2}\right).$$  

(36)

By virtue of condition (27) we always have $\Delta \tau > 0$; at $z \gg 1$, the width of the plateau region $\Delta \tau$ is much larger than unity. The concentration and velocity of the particles are constant in the plateau region, and there is no electric field. At the points $\tau_k$ and $\tau_k$, the obtained solution has weak discontinuities (discontinuities of the derivatives $dn_1/d\tau$ and $dn_2/d\tau$). The presented solution of the problem of expansion, in vacuum of a plasma containing an ion mixture can be used, in particular, in the analysis of the structure of the perturbed zone in the vicinity of bodies moving in the ionosphere, such as rockets and satellites. The point is that at altitudes $h \sim 500-1200$ km the ionosphere plasma contains a mixture of ions, mainly of atomic oxygen ($M_o = 16$) and hydrogen ($M_h = 1$). The relative hydrogen content $n_{H^+}/(n_{O^+} + n_{O^+})$ ranges from $1-2\%$ at $h = 500$ km to $100\%$ at $h = 1200-1500$ km. As shown earlier (15), the structure of the perturbed zone near the moving body is determined to a considerable degree by the self-similar solution considered here. A plot of $N(\tau) = n_{H^+} + n_{O^+}$ for different values of $n_{H^+}$, obtained by numerically integrating Eqs. (1), (7), and (23) for $Z_1 = Z_2 = 1, M_2 = 16, M_1 = 1, \beta = T_0/T_1 = 1$, is shown in Fig. 8.

Using the formulas obtained in (8) we can now compare the results of the calculations and measurements in the ionosphere. Figure 9 shows by way of example the variation of $N$ behind the body ($\beta = 180^\circ$) as a function of the relative hydrogen content $n_{H^+}$. The points on the figure are the results of measurements made in the ionosphere by Samir and Wrenn (6) with the satellite Explorer-31, and the solid curve is the result of the calculation. A detailed discussion of the results and a comparison with theory is contained in (8). We note that in that reference they used an approximate expression for the summary ion concentration:

$$N(\tau) = \frac{1}{2} n_{H^+} [1 + \Phi \left(\frac{M_1}{M_2}\right)^{1/2} \tau] + \frac{1}{2} n_{O^+} [1 + \Phi(\tau)].$$  

(37)

in which the influence of the electric field on the ion motion was neglected. The result of the calculation with the approximate formula (37) is shown dashed in Fig. 9. We note that the hydrogen ions exert a definite influence on the structure of the perturbed zone behind the moving body, near its surface, even if their relative concentration in the plasma is very small. They are appreciably accelerated by the electric field: the average energy of the $H^+$ ions on the boundary of the quasineutral zone be-
hind the body, at \( n_{H^+} \approx 0.1 \), is \((5-8)T_{eo}\), i.e., of the order of 1-1.5 eV. The hydrogen-ion acceleration becomes stronger when their relative concentration is decreased.

4. SCALE-INVARIANT WAVES. ION-SOUND INSTABILITY

Figure 8b shows the concentration \( N(\tau) \) for an expanding plasma containing a mixture of oxygen and hydrogen ions. It is seen from the figure that behind the plateau region the plot of \( N \) against \( \tau \) shows characteristic oscillations. These oscillations become more clearly pronounced if one considers the dimensionless force \( F(\tau) = \frac{1}{2}N^2 dN/d\tau \) (see Fig. 10), the force \( F(\tau) \) is proportional to the electric field intensity \( E = F(2T_eM)^{1/2} \). The amplitude of the oscillations, as seen from Fig. 10, increases with increasing initial concentration of the heavy oxygen ions \( (N_{O^+} = 1 - N_{H^+}) \). At the start of the plateau region, the heavy ions vanish almost completely. Consequently, the oscillations in question propagate already in a pure hydrogen plasma. The onset of these oscillations is the result of excitation of unique scale-invariant waves in the expanding plasma.

Let us consider a weak perturbation of the scale-invariant distribution function

\[ f = g_0(u, \tau) + \delta g(u, \tau), \quad \delta g \ll g_0. \]  

Substituting (38) in (1) and (2) and linearizing the latter, we arrive at the following equation for \( \delta g \):

\[ (u - v) \frac{\partial \delta g}{\partial \tau} \frac{1}{2N_e} \frac{dN_e}{du} \frac{\partial \delta g}{\partial u} + \frac{1}{2N_e} \frac{dN_e}{du} \frac{\partial \delta g}{\partial u} \left( \frac{dN_e}{du} \frac{1}{N_e} \frac{dN_e}{du} \frac{\partial \delta N}{\partial u} \right) = 0, \]  

\[ \delta N = \frac{1}{N_e} \int \delta g \, du. \]  

If the dimension of the perturbations in \((u, \tau)\) space is small in comparison with the characteristic dimension of the variation of the main quantities \( N_0(\tau) \) and \( g_0(u, \tau) \), then the solution of (39) can be sought in the quasiclassical approximation by expanding in a Fourier integral

\[ \delta g = \delta g_n e^{i\mathbf{q} \cdot \mathbf{r}} \exp \left[ \left( i \int \rho \, dr - \mathbf{q} \cdot \mathbf{r} \right) \right], \]

\[ \delta N = \delta N_n \exp \left( i \int F \, dr \right), \]

\[ \delta N_n = \delta g_n \int \frac{\delta g_n}{N_e} e^{i\mathbf{q} \cdot \mathbf{r}} \, d\mathbf{r} = \delta g_n N_0(\mathbf{q}). \]

Substituting (40) in (39) we obtain a dispersion equation that defines the parameter \( p \):

\[ p = -\frac{dN_0}{du} \frac{\partial}{\partial \tau} \left( \frac{dN_0}{du} - \eta(q) e^{i\mathbf{q} \cdot \mathbf{r}} \right) \frac{\partial g_0}{\partial u} - \frac{\eta(q)}{N_e} \frac{\partial g_0}{\partial u} e^{i\mathbf{q} \cdot \mathbf{r}} du. \]  

\[ \frac{\partial}{\partial \tau} \frac{\partial g_0}{\partial u} + \eta(q) = \frac{n_{H^+}}{N_e} \frac{\partial g_0}{\partial u} e^{i\mathbf{q} \cdot \mathbf{r}} du. \]

We see therefore that at sufficiently large values of \( q \) the real part of \( p \) becomes predominant, meaning the presence of oscillatory solutions. Formula (41) is the dispersion relation for the scale-invariant waves.

It is quite important that there are no scale-invariant waves in hydrodynamics. This can be easily verified by considering small perturbations of the hydrodynamic equations (33). These are specifically kinetic waves, due to the presence of a particle-velocity distribution, as can be seen also directly from the relations (40) and (41). The profile of such a wave depends on the ratio \( x/t \), i.e., different points of the wave move with different velocities. The wavelength increases with time, and the frequency decreases. It is easy to understand the mechanism whereby scale-invariant waves are excited in the case of expansion, in vacuum of a plasma containing an admixture of heavy ions \( (N_{O^+} = 1 - N_{H^+} \ll 1) \). Indeed, the heavy ions have a small thermal velocity spread; they vanish rapidly at values \( T - 0 \). Therefore the change of \( N(\tau) \) at \( \tau - 0 \) becomes noticeably accelerated, and an additional force, \( F_{O^+} \sim dN_{O^+}/d\tau \), appears and perturbs the distribution of the hydrogen ions. It is this perturbation which propagates further in the form of scale-invariant waves traveling in an expanding hydrogen plasma.

We have confined ourselves above to an analysis of self-similar perturbations. Let us consider now arbitrary ion-sound waves. The dispersion equation for ion-
sound waves in a plasma containing two sorts of ions is
\[ 1 + (kD)^2 = \frac{T}{N} \int \frac{dv}{v-\nu_{ph}+i\gamma/k} \left( \frac{Z_1}{M_1} \frac{df_1}{dv} + \frac{Z_2}{M_2} \frac{df_2}{dv} \right). \] (42)

Here \( f_1 \) and \( f_2 \) are the distribution functions of the ions with masses \( M_1 \) and \( M_2 \), respectively; \( N = N_1Z_1 + N_2Z_2 \); \( \nu_{ph} = \omega/k \) is the phase velocity of the wave; \( \gamma \) is its damping; \( k \) is the wave vector; \( D = (T_e / 4\pi e^2 n) \) is the Debye radius.

The plasma stability limit with respect to ion-sound waves is given by the condition \( \gamma = 0 \). In our case the functions \( f_1 \) and \( f_2 \) are scale-invariant. Condition (42) with \( \gamma = 0 \) is then rewritten in terms of the dimensionless variables \( u \) and \( g \) in the form
\[ 1 + (kD)^2 = \frac{1}{2N_1n_1^2} \int \left( \frac{dg}{du} \right)^2 du, \] (43)

The function \( g(u) \) is shown in Fig. 11 for different values of \( \tau \) (at \( Z_1 = Z_2 = 1, M_2/M_1 = 16, n_1/n_2 = 0.2 \)). We see that at large negative values of \( \tau \) the function \( g(u) \) has a single hump. Equation (43) is then satisfied only at \( k \approx 0 \) for \( \nu_{ph} \approx 1 \). This is the undamped ion-sound branch that appears in all scale-invariant solutions. Near the point \( u = 1 \), the distribution function \( g(u) \) vanishes identically. There are no unstable waves in the vicinity of this root, just as in the usual problem of plasma expansion in vacuum. There appears, however, a point of horizontal inflection (\( u_0 = 0.014 \)) for the function \( g(u) \). At \( \tau > \tau_0 \), the function \( g(\gamma) \) already has two humps. The reason is that the function \( g(u) \) is made up of the distribution functions \( g_1 \) and \( g_2 \) of the light and heavy ions. The light ions accelerated by the electric field acquire an appreciable translational velocity. They overtake the heavy ions and form, as it were, a second rapid stream among the heavy ions. This is seen from Fig. 11. It is important that the scale-invariant functions \( g(u) \) decrease extremely rapidly near the separatrix (see \( kD \)). The function \( g_1(u) \) has near the separatrix a front close to a step function. Consequently, Eq. (43) is satisfied at \( \tau = \tau_0 = 0.263 \) not only at \( \nu_{ph} = \nu_0 \), but also directly at the point \( \nu_{ph} = \nu_0 = 0.014 \). A new branch of undamped ion sound with a large wave vector \( k_0 \) \( (k_0D > 10^9) \) appears in this case. When \( \tau > \tau_0 \) there appears a double-hump distribution function, and simultaneously also unstable (growing) ion-sound waves for \( k \leq k_0 \). At \( \tau = \tau_1 = 0.251 \), the integral in the right-hand side of (43) is equal to unity for the point \( u_m = 0.043 \) at which the function \( g(u) \) has a minimum. This means that already at \( \tau = \tau_1 \) all the waves with wave vector \( k_0 \) \( (k_0 < 0) \) are unstable. The width of the unstable region, in terms of the phase velocities of the waves, is
\[ \Delta \nu_{ph} = (u_m - u_0)(2\pi / \nu_{ph}) = 0.029(2\pi / \nu_{ph}). \]

We note that the ion-sound instability considered here, in a plasma containing a mixture of ions, can be the cause of oscillations in an ionosphere perturbed by a moving body, as observed by Boyd et al. with the satellite Ariel-1. It must also be emphasized that this instability apparently limits the possibilities of the acceleration mechanism considered here at appreciable concentrations of the impurity ions.


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3 For example, at \( \tau_0 = 0.263 \) the distribution \( g_1(u) \) is discontinuous at the point \( u_0 = 0.014 \): accurate to nine significant figures, it changes at \( u = u_0 \) from \( g_1 = 0 \) to \( g_1 = 0.170 \).
ION ACCELERATION UPON EXPANSION OF A RAREFIED PLASMA


Translated by J. G. Adashko