SOVIET PHYSICS JETP VOLUME 34, NUMBER 3 MARCH, 1972

PARAMETRIC EXCITATION OF OSCILLATIONS IN A PLASMA BY THE FIELD OF A MODULATED MICROWAVE

Yu. M. ALIEV and D. ZYUNDER

P. N. Lebedev Physics Institute, USSR Academy of Sciences

Submitted April 2, 1971

Zh. Eksp. Teor. Fiz. 61, 1057–1064 (September, 1971)

It is well known[1] that the natural-oscillation spectrum of a plasma located in the field of a monochromatic wave depends on the amplitude of the applied microwave field. It is shown that by employing amplitude modulation of an external microwave field, to which the plasma is transparent, oscillations can be excited parametrically whose frequency spectrum is determined by the field strength at the carrier frequency. Cases of weak and strong modulation are considered. The frequencies and field strengths are found for which either high-frequency, low-frequency, or coupled potential high- and low-frequency oscillations are excited. Since the carrier frequency of the modulated field exceeds the Langmuir frequency, the field can easily penetrate deep into the plasma. Consequently, modulated microwave signals can be employed for volume heating and plasma diagnostics.

1. It was shown in[1] that a plasma placed in a homogeneous monochromatic electric field of high frequency has natural-oscillation spectra that depend on the amplitude of the applied field. A stabilizing effect of such a field on the plasma was also observed there. On the other hand, it is well known that if one of the parameters that determine the period of the oscillations of a linear system varies with a frequency that is an integer multiple of the Langmuir-oscillation frequency, then parametric resonance is produced and the oscillations in the system increase exponentially[2]. Thus, by amplitude-modulating a microwave field it is possible to excite parametrically natural plasma oscillations whose dispersion law is determined by the microwave field intensity at the carrier frequency. As shown in[1], the greatest change takes place in the spectrum of the low-frequency ion oscillations. The latter circumstance makes it possible to control the spectra of the excited oscillations by varying the intensity of the applied microwave field.

It is shown below that by a suitable choice of the frequencies and intensities of the microwave fields it is possible to excite either high-frequency Langmuir oscillations or low-frequency ion oscillations, or else both. The values of the threshold field at which excitation of the oscillations begins are determined together with the maximum values of the growth increments of the perturbations. Since the carrier frequency of the modulated microwave field greatly exceeds the Langmuir frequency $\omega_{Le} = (4\pi ne^2/m)^{1/2}$, such a field is not limited by the skin effect and can readily penetrate deep into the interior of the plasma. The latter circumstance makes it possible to use a modulated microwave field for volume heating of a plasma. On the other hand, the questions considered below can be of interest also for the diagnostics of a plasma situated in a strong high-frequency field.

2. Let us consider a fully ionized plasma interacting with a homogeneous electric field

$$\mathbf{E}(t) = \sum_{j=0}^{p} \mathbf{E}_j \sin \omega_j t.$$

To find the spectrum of the natural oscillations of the plasma we can use a method developed in[2,3]. It is easily shown that such a spectrum is obtained from the condition for the existence of solutions of the following system of equations for the Fourier components of the electron and ion charge-density perturbations:

$$\begin{aligned} \rho_e(\omega, \mathbf{k}) &= -R_e(\omega, \mathbf{k}) \sum_{m_0, m_1, \ldots, m_p=-\infty}^{\infty} \prod_{s=0}^{p} J_{-m_s}(a_s)\, \rho_i\Big(\omega + \sum_{s=0}^{p} m_s\omega_s, \mathbf{k}\Big), \\ \rho_i(\omega, \mathbf{k}) &= -R_i(\omega, \mathbf{k}) \sum_{m_0, m_1, \ldots, m_p=-\infty}^{\infty} \prod_{s=0}^{p} J_{m_s}(a_s)\, \rho_e\Big(\omega + \sum_{s=0}^{p} m_s\omega_s, \mathbf{k}\Big). \end{aligned} \tag{1}$$

Here $R_a(\omega, \mathbf{k}) \equiv \delta\epsilon_a(\omega, \mathbf{k})/[1 + \delta\epsilon_a(\omega, \mathbf{k})]$, $\delta\epsilon_a(\omega, \mathbf{k})$ is the contribution of the particles of sort a to the linear dielectric constant of the plasma in the absence of a microwave field, J_{l_j} is a Bessel function with index l_j and argument

$$a_j = e\mathbf{E}_j\mathbf{k} / m\omega_j^2 = \mathbf{k}\mathbf{r}_{Ej},$$

and e and m are the charge and mass of the electron. We note that the presence of collisions leads to heating of the plasma in the microwave field. We shall assume, however, that the processes considered by us evolve within times much shorter than the plasma-heating time in the unperturbed state[4].

We consider below the case when the microwave field is given by

$$\begin{gathered} \mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t + \mathbf{E}_1 \sin \omega_+ t + \mathbf{E}_2 \sin \omega_- t, \\ \omega_+ = \omega_0 + \omega_1, \quad \omega_- = \omega_0 - \omega_1, \quad \omega_1 \ll \omega_0. \end{gathered} \tag{2}$$

The carrier frequency ω_0 will be assumed to exceed greatly all the natural frequencies of the plasma oscillations ($\omega_0 \gg \omega_{Le}$).

In this case the system (1) takes the simpler form

$$\begin{aligned} \rho_e(\omega, \mathbf{k}) &= -R_e(\omega, \mathbf{k}) \sum_{s=-\infty}^{\infty} A_{-s}\rho_i(\omega + s\omega_1, \mathbf{k}), \\ \rho_i(\omega, \mathbf{k}) &= -R_i(\omega, \mathbf{k}) \sum_{s=-\infty}^{\infty} A_s\rho_e(\omega + s\omega_1, \mathbf{k}) \end{aligned} \tag{3}$$

A similar system of equations was investigated in detail in[2,3,5,6]. Unlike in the cited papers, the Bessel functions are replaced here by the quantities A_n, which are even functions of the index

$$A_n = A_{-n} = \sum_{k=-\infty}^{\infty} J_{-n-2k}(a_0) J_{n+k}(a_1) J_k(a_2). \quad (4)$$

We confine ourselves henceforth to an analysis of the two most interesting cases of the time dependence of the field (2). In the first case a strong field $\mathbf{E}_0 \sin \omega_0 t$ combines with two weaker signals with frequencies ω_+ and ω_- and with amplitudes $\mathbf{E}_1 = \mathbf{E}_2 = \alpha \mathbf{E}_0/2$, forming an amplitude-modulated microwave field

$$\mathbf{E}(t) = \mathbf{E}_0(1 + \alpha \cos \omega_1 t) \sin \omega_0 t. \quad (5)$$

The coefficients A_n are calculated in this case with the aid of formula (4), in which we must put $a_1 = a_2 = \alpha a_0/2$:

$$A_0 \approx J_0(a_0) + o(\alpha^2), \quad A_1 = A_{-1} = -\tfrac{1}{2}\alpha a_0 J_1(a_0) + o(\alpha^3), \quad A_2 = A_{-2} = o(\alpha^2), \quad (6)$$

etc. We consider also an example of strong modulation, when the time dependence of the microwave field is given by

$$\mathbf{E}(t) = \mathbf{E}_0 \sin \omega_0 t + \mathbf{E}_1 \sin(\omega_0 + \omega_1)t, \quad \omega_1 \ll \omega_0, \quad (7)$$

and the amplitudes of the fields $\mathbf{E}_0$ and $\mathbf{E}_1$ can be comparable in magnitude. The coefficients A_n for such a time dependence of the field are given by

$$A_n = J_{-n}(a_0) J_n(a_1). \quad (8)$$

3. Let us consider the excitation of high-frequency ($|\omega| \approx \omega_{Le}$) oscillations. It is seen from the system (3) that in the frequency region $\omega_1 \approx 2\omega_{Le}/p$, where p is an odd integer, only the Fourier components of the electron charge density $\rho_e(\omega)$ and $\rho_e(\omega - p\omega_1)$ corresponding to excitation of high-frequency Langmuir oscillations are not small. The condition for the solvability of the homogeneous system of equations for $\rho_e(\omega)$ and $\rho_e(\omega - p\omega_1)$ leads to the following dispersion equation (see[7]):

$$\left[1 - R_e(\omega) \sum_{m=-\infty}^{\infty} A_m^2 R_i^{(m)}\right]\left[1 - R_e(\omega - p\omega_1) \sum_{m=-\infty}^{\infty} A_{m+p}^2 R_i^{(m)}\right] = R_e(\omega) R_e(\omega - p\omega_1)\left[\sum_{m=-\infty} R_i^{(m)} A_m A_{m+p}\right]^2, \quad (9)$$

where

$$R_a^{(m)} \equiv R_a(\omega + m\omega_1, \mathbf{k}).$$

The right-hand side of (9) characterizes the coupling of two high-frequency fields in the field of the pump wave. It is easy to see that for a monochromatic wave having a frequency $2\omega_{Le}/p$, the right-hand side of (9) vanishes. In this case, as shown in[2], we obtain from (9) a spectrum of non-growing Langmuir plasma oscillations in an external monochromatic field. In our case, the right-hand side of (9) differs from zero. This fact is formally connected with the properties of the functions A_n, which, unlike Bessel functions, are even functions of the index. This property of Eq. (9) ensures the possibility of parametric excitation of high-frequency oscillations.

We consider first the case of weak amplitude modulation of the microwave field (5). We confine ourselves to the first resonant region $\omega_1 \approx 2\omega_{Le}$ (p = 1). Under these conditions, the excitation of the oscillations begins at the smallest modulation depth, and furthermore the maximum possible value of the increment is reached.

The threshold value of the depth of modulation α is determined by minimizing, with respect to the wave vectors, the right-hand side of the equation (see[7])

$$\alpha_{\text{thr}} = \min\left[\frac{m_i \delta\varepsilon_e''(\omega_{re})}{m_e a_0 J_0(a_0) J_1(a_0)}\right] \quad (10)$$

where

$$\delta\varepsilon_e''(\omega, \mathbf{k}) = \sqrt{\frac{\pi}{2}} \frac{\omega\omega_{Le}^2}{k^3 v_{Te}^3} e^{-\omega^2/2k^2 v_{Te}^2} + \frac{\omega_{Le}^2 \nu_{\text{thr}}}{\omega^3}$$

is the contribution of the electrons to the imaginary part of the linear dielectric constant,

$$\nu_{\text{eff}} = \frac{4\sqrt{2\pi}\, e^2 e_i^2 n_i}{3\sqrt{m_e}\, T_e^{3/2}} \ln \frac{r_{De}}{r_{min}}$$

is the frequency of the electron-ion collisions[1)],

$$\omega_{re}^2 = \omega_{Le}^2 + \omega_{Li}^2 J_0^2 + 3k^2 v_{Te}^2, \quad v_{Te}^2 = T_e/m_e$$

is the spectrum of the natural high-frequency oscillations of the plasma in the microwave field[1].

The threshold value of the depth of modulation is

$$\alpha_{\text{thr}} = \frac{1}{f_{max}} \frac{m_i}{m_e} \frac{\nu_{\text{eff}}}{\omega_{Le}}, \quad (11)$$

f_{max} is the maximum value of the function $a_0 J_0(a_0) J_1(a_0)$, which occurs at $a = a_{0,max}$. The first and largest maximum, equal to 0.43, is attained by this function at $a_{0,max} = 1.43$. The oscillations excited thereby have a wavelength of the order of

$$2\pi / k_0 \geqslant 2\pi r_{De} \sqrt{2 \ln \omega_{Le}/\nu_{\text{eff}}} \quad (12)$$

and the cosine of the angle between the propagation direction and $\mathbf{E}_0$ is close to

$$\cos\theta_0 = 1.43 / k_0 r_E. \quad (13)$$

It is necessary here to satisfy the condition[2)]

$$\omega_1 = 2\omega_{re}(x, k_0). \quad (14)$$

If the depth of modulation greatly exceeds (11) but remains less than unity, then the dissipative effects become negligible and the oscillation growth increment reaches the maximum possible value

$$\gamma_{max} = 0{,}7\alpha\omega_{Li}\sqrt{m_e/m_i}. \quad (15)$$

If the time dependence of the field is of the form (7), then the threshold value of the amplitude is $\mathbf{E}_1 = \alpha_{\text{thr}} \mathbf{E}_0$, where α_{thr} is given by (11). The oscillations

1)We neglect the influence of the microwave field on the particle-collision act. In a strong field ($v_E \gg v_{Te}$), however, such an influence may turn out to be appreciable [4].

2)The results remain valid in the geometrical-optics approximation if the characteristic dimension of the plasma inhomogeneity greatly exceeds the wavelength of the excited oscillations. In this case relation (14) defines the region of plasma density where buildup of oscillations takes place.

excited in this case in the region of the density values defined by (14) have a wavelength of the order of (12) and propagate at angles θ_0 to the direction of $\mathbf{E}_0$ (13). The maximum value of the increment is in this case

$$\gamma_{max} = \frac{\omega_{Li}}{2}\sqrt{\frac{m_e}{m_i}}\sum_{m=-\infty}^{\infty}\frac{J_m(a_0)J_m(a_1)J_{m+1}(a_0)J_{m+1}(a_1)}{(2m+1)^2}$$

4. Let us consider the excitation of low-frequency ($|\omega| \lesssim \omega_{Li}$) oscillations. The analysis of the excitation of low-frequency oscillations is analogous in many respects to that given above. Thus, for weak amplitude modulation of the microwave field, the equation for the low-frequency oscillations is ($p = 1$)

$$\chi(\omega)\chi(\omega-\omega_1) = (\alpha\beta)^2, \quad (16)$$

where

$$\chi(\omega) = 1 + \delta\varepsilon_e(\omega) + \delta\varepsilon_i(\omega) + [1 - J_0^2(a_0)]\delta\varepsilon_e(\omega)\delta\varepsilon_i(\omega),$$
$$\beta = a_0 J_0(a_0) J_1(a_0)[\delta\varepsilon_i(\omega)\delta\varepsilon_i(\omega-\omega_1)\delta\varepsilon_e(\omega)\delta\varepsilon_e(\omega-\omega_1)]^{1/2}.$$

In this case the ion oscillations whose spectrum was investigated in[1] are excited. For oscillations having a phase velocity higher than the thermal velocity of the electrons we have

$$\omega_{ri}^2 = \omega_{Li}^2[1 - J_0^2(a_0)] + 3k^2[v_{Ti}^2 + v_s^2 J_0^2], \quad v_s^2 = T_e/m_i. \quad (17)$$

For oscillations with a phase velocity exceeding v_{Ti} but smaller than v_{Te} we have

$$\omega_{ri}^2 = \omega_{Li}^2[1 - J_0^2(a_0)/(1 + k^2 r_{De}^2)] + 3k^2 v_{Ti}^2. \quad (18)$$

The threshold depth of modulation α_{thr} is obtained by minimizing with respect to the wave vectors, the right-hand side of the following equation

$$\alpha_{thr} \quad (19)$$
$$= \min\left|\frac{\delta\varepsilon_e''(\omega)[1 + (1 - J_0^2)\delta\varepsilon_i'] + \delta\varepsilon_i''(\omega)[1 + (1 - J_0^2)\delta\varepsilon_e''(\omega)]}{a_0 J_0(a_0) J_1(a_0)\delta\varepsilon_i'(\omega)\delta\varepsilon_e'(\omega)}\right|_{\omega\approx\omega_{ri}}.$$

For oscillations with a phase velocity larger than v_{Te}, the threshold value of the modulation is in this case

$$\alpha_{thr} = \min\left\{(a_0 j_0 j_1)^{-1}\left[\frac{8\nu_{eff}\,\omega_{Li}^2(1 + j_0^2)}{\omega_1^3}\right.\right. \quad (20)$$
$$\left.\left. + \sqrt{\frac{\pi}{8}}\frac{\omega_{Le}^2\omega_1}{(k_0 v_{Te})_3}\exp\left\{-\frac{\omega_1^2}{8k_0^2 v_{Te}^2}\right\}\right]\right\}.$$

It is necessary to satisfy here the condition $\omega_1 = 2\omega_{ri}(x, k_0)$.

With increasing depth of modulation, the increment of the growing oscillations increases, reaching in the limit its maximum value

$$\gamma_{max} = \frac{\alpha}{2}\varphi_{max}\omega_1. \quad (21)$$

The first maximum of the function $\varphi_{max} = 0.54$ is reached at $a_0 max = 1.19$.

Proceeding to consider the excitations of an oscillation with a phase velocity smaller than the thermal velocity of the electrons but larger than the thermal velocity of the ions, we confine ourselves to an analysis of the case when the amplitude of the oscillations of the electron in the pump-wave field r_E does not exceed the wavelength of the excited oscillations. Under these conditions[1]

$$\omega_{ri}^2 = \omega_{Li}^2\frac{a_0^2/2 + k^2 r_{De}^2}{(1 + k^2 r_{De}^2)} + 3k^2 v_{Ti}^2. \quad (22)$$

We confine ourselves further to the case of a nonisothermal plasma ($T_e \gg T_i$) and sufficiently weak microwave fields ($a_0 \lesssim k r_{De}$), when the frequency of the excited low-frequency oscillations does not exceed greatly the ion-acoustic frequency. Assuming, in addition, that the condition

$$\frac{T_e}{T_i} > \ln\left[\frac{m_i}{m_e}\left(\frac{T_e}{T_i}\right)^3\right] \quad (23)$$

is satisfied and neglecting the contribution made to the damping of the low-frequency waves by the Cerenkov effect on the ions, we obtain the following expression for the threshold modulation upon excitation of long-wave ($k r_{Dt} < 1$) low-frequency oscillations:

$$\alpha_{thr} = \frac{r_{De}^2}{r_E^2}\left[\sqrt{2\pi}\left(\frac{m_e}{m_i}\right)^{1/2} + \frac{16}{5}\frac{T_i}{T_e}\frac{\nu_{ii}}{\omega_1}\right]. \quad (24)$$

The maximum value of the increment at sufficiently large depth of modulation (but at $\alpha < 1$) turns out to be

$$\gamma_{max} = \frac{\alpha}{8}\left(\frac{r_E}{r_{De}}\right)^2\omega_1. \quad (25)$$

The wavelength of the oscillations that grow with this increment is determined from the condition

$$\omega_1 = 2\omega_{ri}(k_0). \quad (26)$$

With increasing depth of modulation, the picture of the oscillation excitation becomes more complicated. In the limit of weak pump fields ($r_E \ll r_{De}$), the analysis of parametric resonance at a large modulation depth is similar to that given above for excitation of high-frequency oscillations.

5. We consider, finally, the excitation of coupled low-frequency and high-frequency oscillations. In the case of a weakly-modulated microwave field, the parametric buildup of coupled oscillations of high and low frequency takes place at the modulation frequencies $\omega_1 \approx \omega_{Le}$. Just as in the case of a monochromatic pump wave[8], a periodic as well as an aperiodic instability can set in here. To excite periodic long-wave oscillations in a nonisothermal plasma (23) with not too high a collision frequency

$$\frac{\nu_{eff}}{\omega_{Le}} < \frac{5}{8}\sqrt{\frac{\pi}{2}}\left(\frac{m_e T_i}{m_i T_e}\right)^{1/2}\frac{1}{\sqrt{\ln \omega_{Le}/\nu_{eff}}} \quad (27)$$

the threshold depth of modulation α_{thr} is

$$\alpha_{thr}^2 = 16\left(\frac{1}{2} + \frac{r_{De}^2}{r_E^2}\right)^2\sqrt{2\pi}\left(\frac{m_e}{m_i}\right)^{1/2}\frac{\nu_{eff}}{\omega_{Le}}\ln\frac{\omega_{Le}}{\nu_{eff}}. \quad (28)$$

The wavelengths of the oscillations excited thereby are of the order of $2\pi/k_0 = 2\pi r_{De}\sqrt{2\ln(\omega_{Le}/\nu_{eff})}$. To find the maximum values of the increment we can use the following equation, which can readily be obtained from the system (3)[2,3]:

$$1 = R_i(\omega)\sum_{n=-\infty}^{\infty} A_n^2 R_e(\omega + n\omega_1). \quad (29)$$

At a low depth of modulation we find with the aid of (29) that the maximum increment

$$\gamma_{max} \approx \left(\frac{\alpha}{2}\right)^{2/3}\omega_{Le}\left(\frac{m_e}{m_i}\frac{\sqrt{27}}{32}\right)^{1/3} \quad (30)$$

is possessed by periodic oscillations with wave vector k_{max}[6]

$$k_{max}^2 r_{De}^2 = 2[\omega_1 - \omega_{Le}(1 + {}^3/_2 k_{max}^2 r_{De}^2) - \gamma_{max}]/3\omega_{Le}, \qquad (31)$$

propagating at an angle θ_0 to the direction of E_0:

$$\cos\theta_0 = 1.8 / k_{max} r_E. \qquad (32)$$

For pump fields in the form (7), the maximum increment is

$$\gamma_{max} = [J_1(a_0) J_1(a_1)]^{2/3} \omega_{Le} \left(\frac{m_e}{m_i} \frac{\sqrt{27}}{32}\right)^{1/3}. \qquad (33)$$

In the particular case of equal amplitudes of the fields E_0 and E_1, we obtain from (33)

$$\gamma_{max} = (0.58)^{4/3} \omega_{Le} \left(\frac{m_e}{m_i} \frac{\sqrt{27}}{32}\right)^{1/3}. \qquad (34)$$

Here, as in the derivation of (32), we took into the account the fact that the maximum of the Bessel function $J_1(x) \approx 0.58$ is attained at an argument equal to $x \approx 1.8$.

From a comparison of (30) and (34) we can conclude that the maximum increment increases with increasing depth of modulation, reaching the highest value (34). Finally, we note that the maximum increments for the development of aperiodic instability[2] are close to the values (24) and (28).

In conclusion, the authors are deeply grateful to V. P. Silin for constant interest in the work and valuable remarks. The authors are sincerely grateful to A. Yu. Kiriĭ for useful discussions, and also to N. E. Andreev, L. M. Gorbunov, I. S. Danilkin, and R. R. Ramazashvili for interest in the work.

[1] Yu. M. Aliev and V. P. Silin, Zh. Eksp. Teor. Fiz. 48, 901 (1965) [Sov. Phys.-JETP 21, 601 (1965)].
[2] V. P. Silin, ibid. 48, 1679 (1965) [21, 1127 (1965)].
[3] Yu. M. Aliev, V. P. Silin, and H. Watson, ibid. 50, 943 (1966) [23, 626 (1966)].
[4] V. P. Silin, ibid. 47, 2254 (1964) [20, 1510 (1965)].
[5] V. P. Silin, A survey of phenomena in ionized gases, IAEA, Vienna, 1968, p. 205–237.
[6] N. E. Andreev, A. Yu. Kiriĭ, and V. P. Silin, Izv. Vuzov, Radiofizika 13, 1321 (1970).
[7] Yu. M. Aliev and D. Zyunder, Zh. Eksp. Teor. Fiz. 57, 1324 (1969) [Sov. Phys.-JETP 30, 718 (1970)].
[8] N. E. Andreev, A. Yu. Kiriĭ, and V. P. Silin, ibid. 57, 1024 (1969) [30, 559 (1970)].

Translated by J. G. Adashko
106