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# METHODS OF PLASMA THEORY AND PROBLEMS OF GRAVITATIONAL STABILITY

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We investigate the stability of a spherically-symmetrical system of masses rotating along circular trajectories. The stability is analyzed with the aid of the collisionless Boltzmann-Vlasov kinetic equation. A method of solving this equation, similar to the known method of integration along the trajectories in plasma physics, is described. The stability of the system under consideration against arbitrary perturbations, which is proved in the article, may possibly explain the experimental fact that the age of globular star clusters is large compared with the age of spiral galaxies.

## 1. INTRODUCTION

MANY processes in the physics of hot plasma became understood when methods were developed for solving the Boltzmann-Vlasov kinetic equation. It is known, however, that this equation describes not only a system of electrically charged particles (electrons and ions), but any other system of particles with Coulomb interaction. Therefore the use of plasma methods may also be fruitful in the study of other systems with Coulomb interaction.

We demonstrate in this paper that plasma methods are effective in the dynamics of a gravitating medium. The need for a kinetic analysis of a gravitating system arises in the case when the motion of the particles of this system relative to one another becomes important. An example of such a system is the model proposed by Einstein[1] of a globular cluster of stars rotating on circular trajectories about a common mass center. The equilibrium state of the gravitating medium is then characterized by a distribution function that depends on the modulus of the velocity $v_\perp$, which is tangent to the radius in accordance with the law $f_0 \sim \delta(v_\perp - v_0)$, where $v_0$ is a certain function of the radius r and differs from zero at all $r \neq 0$. Such a distribution is of the "conical" type investigated in plasma theory.[2,3] A plasma with such a distribution function is unstable against a large class of perturbations. This raises the natural question of whether an instability of such a type can develop in Einstein's system of gravitating particles. This question is investigated in the present article. We consider the simplest case of a system of particles with uniform density and show that neither "conical" nor any other instabilities can develop in such a system. The stability deduced by us for the Einstein model may possibly explain the fact, known from observations, that the age of globular star clusters is large compared with the age of stars in the spiral arms of galaxies.

The larger the number of particles contained in a sphere of arbitrary radius, which in this case can be regarded as a Debye sphere,[4] the less the potential of the system under consideration deviates from spherically-symmetrical. If the number of particles in the Debye sphere is large, then we can neglect the paired interaction of the particles with one another.[4] Indeed, if only gravitational interaction takes place between the N particles in the system, then the pairwise interaction in the equilibrium system is, roughly speaking, weaker by a factor N (more accurately, N/ln N) than the collective interaction. Consequently, the problem reduces in the first approximation to the motion of particles in a collective self-consistent gravitational field.[1)] A collisionless motion of the particles is described by the kinetic Boltzmann-Vlasov equation. In Sec. 2 we describe a method of solving the kinetic equation, analogous to the well-known method of integration over the trajectories in plasma physics.[5] In Sec. 3 we give the spectrum of the natural frequencies of the system, which constitutes a discrete set of real numbers. A general discussion of the results is given in Sec. 4.

## 2. DERIVATION OF THE EQUATION FOR THE NATURAL OSCILLATIONS

Assume that in a unit interval of coordinate-velocity space there are f (r, v, t) particles. The function f satisfies the kinetic equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \nabla\Phi\frac{\partial f}{\partial \mathbf{v}} = 0, \qquad (1)$$

where Φ is the gravitational potential. The latter is connected in turn with the Poisson equation

$$\Delta\Phi = 4\pi G n, \qquad (2)$$

in which

$$n = \int f\, d\mathbf{v}, \qquad (3)$$

and G is the gravitational constant.

We use spherical coordinates r, θ, and φ and characterize the velocity by the quanties $v_r$ and $v_\perp = (v_\theta^2 + v_\varphi^2)^{1/2}$, where $\alpha = \tan^{-1}(v_\varphi/v_\theta)$. In terms of these variables, Eq. (1) takes the form

$$\hat{L}f + v_r\left(\frac{\partial f}{\partial r} - \frac{v_\perp}{r}\frac{\partial f}{\partial v_\perp}\right) - \left(\frac{v_\perp^2}{r} - \frac{\partial\Phi}{\partial r}\right)\frac{\partial f}{\partial v_r} - \nabla_\perp\Phi\frac{\partial f}{\partial \mathbf{v}_\perp} = 0, \qquad (4)$$

where

$$\hat{L} = \frac{\partial}{\partial t} + \frac{v_\perp}{r}\left[\cos\alpha\,\frac{\partial}{\partial\theta} + \frac{\sin\alpha}{\sin\theta}\frac{\partial}{\partial\varphi} - \sin\alpha\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\alpha}\right], \qquad (5)$$

[1)]This estimate does not hold if there are direct inelastic collisions between stars. We shall henceforth consider systems in which the inelastic collisions are so rare that they (as well as the elastic ones) can be neglected.

$$\nabla_\perp \Phi \frac{\partial}{\partial \mathbf{v}_\perp} = \frac{1}{r}\left(\cos\alpha \frac{\partial \Phi}{\partial \theta} + \frac{\sin\alpha}{\sin\theta}\frac{\partial \Phi}{\partial \varphi}\right)\frac{\partial}{\partial v_\perp} - \frac{1}{r v_\perp}\left(\sin\alpha \frac{\partial \Phi}{\partial \theta} - \frac{\cos\theta}{\sin\theta}\frac{\partial \Phi}{\partial \varphi}\right)\frac{\partial}{\partial \alpha}. \qquad (6)$$

We linearize Eqs. (2)–(4), marking the equilibrium quantities by the index 0 and the non-equilibrium quantities by the index 1. We assume that in the equilibrium state the potential depends only on the radius, $\Phi_0 = \Phi_0(r)$, the particles have no radial velocity, and the number at each point of the sphere of arbitrary radius r does not depend on $\theta$ and $\varphi$, and is symmetrical with respect to $\alpha$, $f_0 = \delta(v_r) F(r, v_\perp)$. For the function F we obtain from (4)

$$\left(\frac{v_\perp^2}{r} - \frac{\partial \Phi_0}{\partial r}\right) F = 0. \qquad (7)$$

This means that F has a $\delta$-function dependence on $v_\perp$. We obtain the normalization coefficient with the aid of (3)

$$F = \frac{n_0}{2\pi v_0}\delta(v_\perp - v_0), \qquad (8)$$

where $v_0 = (r\partial\Phi_0/\partial r)^{1/2}$, and $n_0$ is the equilibrium density. We regard the latter as independent of the radius; this, in accord with (2), is justified if the radial depencence of $\Phi_0$ is given by

$$\Phi_0 = {}^1/_2\Omega^2 r^2 + \text{const}, \qquad (9)$$

where

$$\Omega^2 = 4\pi G n_0 / 3, \qquad (10)$$

With $v_0 = \Omega r$.

In the linear approximation we obtain from (4)

$$\hat{L} f_1 + v_r\left(\frac{\partial f_1}{\partial r} - \frac{v_\perp}{r}\frac{\partial f_1}{\partial v_\perp}\right) + \left(\frac{v_\perp^2}{r} - \frac{\partial \Phi_0}{\partial r}\right)\frac{\partial f_1}{\partial v_r} = \frac{\partial \Phi_1}{\partial r}\frac{\partial f_0}{\partial v_r} + \frac{1}{r}\left(\cos\alpha\frac{\partial \Phi_1}{\partial \theta} + \frac{\sin\alpha}{\sin\theta}\frac{\partial \Phi_1}{\partial \varphi}\right)\frac{\partial f_0}{\partial v_\perp}. \qquad (11)$$

Recognizing that $f_0 \sim \delta(v_r)\delta(v_\perp - v_0)$, we find that (11) is satisfied if $f_1$ is of the form

$$f_1 = \delta(v_r)[A\delta(v_\perp - v_0) + B\delta'(v_\perp - v_0)] - C\delta'(v_r)\delta(v_\perp - v_0), \qquad (12)$$

where the prime denotes the derivative with respect to the argument. From (11) we find that the functions A, B, and C satisfy the equations

$$\hat{L}^0 A - \frac{1}{r\Omega}\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)B + \frac{1}{r}\frac{\partial}{\partial r}(rC) = 0, \qquad (13)$$

$$\hat{L}^0 B - 2\Omega C = \frac{n_0}{2\pi\Omega^2 r^2}\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)\Phi_1, \qquad (14)$$

$$\hat{L}^0 C + 2\Omega B = -\frac{n_0}{2\pi\Omega r}\frac{\partial \Phi_1}{\partial r}. \qquad (15)$$

The operator $\hat{L}^0$ differs from $\hat{L}$ in that $v_\perp/r$ is replaced by $\Omega$. According to (3), if $f_1$ is of the form (12), the density perturbation is equal to

$$n_1 = \int_0^{2\pi} (\Omega r A - B)\, d\alpha. \qquad (16)$$

The method of solving Eqs. (13)–(15) and finding $n_1$ is as follows. Multiplying both halves of (14) by the operator $\hat{L}^0$ and expressing $\hat{L}^0 C$ in terms of B and $\Phi_1$ with the aid of (15), we obtain

$$(\hat{L}^0 + 2i\Omega)(\hat{L}^0 - 2i\Omega)B = \frac{n_0}{2\pi\Omega^2 r^2}\left[\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)\hat{L}^0\Phi_1 + 2\Omega^2 r\frac{\partial \Phi_1}{\partial r}\right). \qquad (17)$$

Hence

$$B = \frac{n_0}{2\pi\Omega^2 r^2}(\hat{L}^0 - 2i\Omega)^{-1}(\hat{L}^0 + 2i\Omega)^{-1}\left[\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)\hat{L}^0\Phi_1 + 2\Omega^2 r\frac{\partial \Phi_1}{\partial r}\right] \qquad (18)$$

where the exponent $-1$ denotes the inverse operator, the action of which will be explained somewhat later.

The function A is expressed in terms of B and $\Phi_1$ with the aid of (13) and (15):

$$A = \frac{1}{r\Omega}(\hat{L}^0)^{-1}\left[\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)B - \frac{1}{2}\frac{\partial}{\partial r}(r\hat{L}^0 B) + \frac{n_0}{4\pi\Omega^2}\left(\hat{L}^0 - \frac{\partial}{\partial t}\right)\frac{\partial}{\partial r}\left(\frac{\Phi_1}{r}\right)\right]. \qquad (19)$$

We substitute this result in (16):

$$n_1 = -\int_0^{2\pi}\left\{\frac{\partial}{\partial t}\hat{L}_0^{-1}\left[B + \frac{n_0}{4\pi\Omega^2}\frac{\partial}{\partial r}\left(\frac{\Phi_1}{r}\right)\right] + \frac{1}{2}\frac{\partial}{\partial r}\left(rB - \frac{n_0}{2\pi r}\frac{\Phi_1}{\Omega^2}\right)\right\}d\alpha. \qquad (20)$$

Equation (20) together with (18) gives the sought connection between $n_1$ and $\Phi_1$; this connection is necessary for a self-consistent description of the perturbations by means of the equation

$$\Delta\Phi_1 = 4\pi G n_1(\Phi_1). \qquad (21)$$

We now present the form of the operator $(\hat{L}^0)^{-1}$. Let the function $X = X(t, \theta, \varphi, \alpha)$ satisfy the equation

$$\hat{L}^0 X = a, \qquad (22)$$

where a is a certain known function of the variables t, $\theta$, $\varphi$, $\alpha$, and vanishes at $t = -\infty$. The operator $\hat{L}^0$ can be represented in the form

$$\hat{L}^0 = \frac{\partial}{\partial t} + \frac{d\theta}{dt}\frac{\partial}{\partial \theta} + \frac{d\varphi}{dt}\frac{\partial}{\partial \varphi} + \frac{d\alpha}{dt}\frac{\partial}{\partial \alpha}, \qquad (23)$$

where the derivatives $d(\theta, \varphi, \alpha)/dt$ denote the rates of change of the angles $\theta$, $\varphi$, and $\alpha$ of a particle moving over a sphere of radius r with velocity $v_\perp = r\Omega$,

$$\frac{d\theta}{dt} = \Omega\cos\alpha, \quad \frac{d\varphi}{dt} = \Omega\frac{\sin\alpha}{\sin\theta}, \quad \frac{d\alpha}{dt} = -\Omega\sin\alpha\,\mathrm{ctg}\,\theta. \qquad (24)$$

We change over in (22) from the variables $\theta$, $\varphi$, $\alpha$ to the variables $\theta_0$, $\varphi_0$, $\alpha_0$, which are connected with $\theta$, $\varphi$, $\alpha$, t by the relations

$$\theta_0 = \theta - \Omega\int_{t_0}^{t}\cos\alpha\, dt, \quad \varphi_0 = \varphi - \Omega\int_{t_0}^{t}\frac{\sin\alpha}{\sin\theta}dt, \quad \alpha_0 = \alpha + \Omega\int_{t_0}^{t}\sin\alpha\,\mathrm{ctg}\,\theta\, dt. \qquad (25)$$

The quantities $\theta_0$, $\varphi_0$ and $\alpha_0$ denote the angles $\theta(t_0)$, $\varphi(t_0)$, and $\alpha(t_0)$ assumed at $t = t_0$ by a particle having at the instant t the angles $\theta(t) = \theta$, $\varphi(t) = \varphi$, $\alpha(t) = \alpha$. In terms of the new variables, Eq. (22) takes the form

$$\left(\frac{\partial X}{dt}\right)_{\theta_0, \varphi_0, \alpha_0} = a[t, \theta(t, \theta_0, \varphi_0, \alpha_0), \varphi(t, \theta_0, \varphi_0, \alpha_0), \alpha(t, \theta_0, \varphi_0, \alpha_0)]. \qquad (26)$$

Hence

$$X = \int_{-\infty}^{t} a[t', \theta(t'), \varphi(t'), \alpha(t')]\,dt'. \qquad (27)$$

Returning in this equation again to the variables $\theta$, $\varphi$, $\alpha$ and recalling that $X = (\hat{L}^0)^{-1}a$, we obtain the form of the inverse operator

$$(\hat{L}^0)^{-1}a = \int_{-\infty}^{t} a\{t', \theta[t', \theta_0(t, \theta, \varphi, \alpha), \ldots]\ldots\}\, dt', \qquad (28)$$

where the multiple dots stand for $\varphi_0$ and $\alpha_0$ expressed in terms of t, $\theta$, $\varphi$, and $\alpha$. Analogously, we obtain

$$(\hat{L}^0 \pm 2i\Omega)^{-1}a = \int_{-\infty}^{t} e^{\mp 2i\Omega(t-t')} a(t')\, dt'. \qquad (29)$$

We now describe the concrete way of calculating integrals of the type of (28) and (29). Let the perturbation of the potential be of the form[2)]

$$\Phi_1 = \chi_l(r)\Phi_1'(t, \theta, \varphi), \qquad (30)$$

where

$$\Phi_1' = e^{-i\omega t} Y_m^l(\varphi, \theta),$$
$$Y_m^l(\varphi, \theta) \equiv e^{-im(\pi/2-\varphi)} P_{m0}^l(\cos\theta), \qquad (31)$$

$\hat{P}_{m0}^l(\cos\theta)$ are functions that coincide, apart from coefficients, with the Legendre polynomials (see Vilenkin's book[6]). All the normalization coefficients are included in $\chi_l(r)$.

At a equal to the right-hand side of (31), Eqs. (28) and (29) are written in the form

$$(\hat{L}^0 + iq\Omega)^{-1}a = e^{-i\omega t}\int_0^\infty e^{i(\omega - q\Omega)\tau} Y_m^l[\varphi(t-\tau), \theta(t-\tau)]\, d\tau, \qquad (32)$$
$$q = 0, \pm 2,$$

The integration variable t′ is replaced by $\tau = t - t'$.

From the equations of motion (24) we get

$$\cos\theta(t) = \cos\gamma \sin\psi(t),$$
$$\operatorname{tg}(\tilde{\varphi}_0 - \varphi(t)) = \operatorname{ctg}\psi(t)/\sin\gamma, \qquad (33)$$
$$\operatorname{ctg}\alpha(t) = -\operatorname{ctg}\gamma\cos\psi(t),$$

where

$$\psi(t) = \psi_0 + \Omega(t - t_0). \qquad (34)$$

We have introduced here the constants $\tilde{\varphi}_0$, $\psi_0$, and $\gamma$, which can be expressed in terms of $\theta_0$, $\varphi_0$, and $\alpha_0$ by considering (33) at $t = t_0$.

Applying (33) and the addition theorem, we present $Y_m^l$ as a sum of triply indexed functions:[6]

$$Y_m^l = [\theta(t-\tau, \tilde{\varphi}_0, \gamma, \psi_0); \varphi(t-\tau, \tilde{\varphi}_0, \gamma, \psi_0)] = \sum_{s=-l}^{l} T_{ms}^l\Big[\frac{\pi}{2} - \tilde{\varphi}_0, \gamma - \frac{\pi}{2}, \frac{\pi}{2} - \psi(t)\Big] P_{s0}^l(0). \qquad (35)$$

Here the function

$$T_{ms}^l\Big[\frac{\pi}{2} - \varphi_1, \theta, \frac{\pi}{2} - \varphi_2\Big] = \exp\Big\{im\varphi_1 + is\varphi_2 - \frac{i\pi}{2}(m-s)\Big\} P_{ms}^l(\cos\theta). \qquad (36)$$

With the aid of (36) we transform the right-hand side of (32):

$$(\hat{L}^0 + iq\Omega)^{-1}a = e^{-i\omega t}\sum_{s=-l} T_{ms}^l\Big[\frac{\pi}{2} - \tilde{\varphi}_0, \gamma - \frac{\pi}{2}, \frac{\pi}{2} - \psi(t)\Big] P_{s0}^l(0) \times \int_0^\infty e^{i[\omega-(q+s)\Omega]\tau}\, d\tau. \qquad (37)$$

The integral with respect to $\tau$ is calculated by using the Landau circuiting rule ($\omega \to \omega + i\Delta$, $\Delta > 0$):

$$\int_0^\infty \exp\{i[\omega - (q+s)\Omega]\tau\}\, d\tau = i/[\omega - (q+s)\Omega]. \qquad (38)$$

Then, using the transformation (which, like (35), is a consequence of the addition theorem)

$$T_{ms}^l\Big[\frac{\pi}{2} - \tilde{\varphi}_0, \gamma - \frac{\pi}{2}, \frac{\pi}{2} - \psi(t)\Big] = \sum_{s'=-l}^{l} T_{ms'}^l\Big[\frac{\pi}{2} - \varphi(t), \theta(t), \frac{\pi}{2} - \alpha(t)\Big] e^{-is\pi} P_{s's}^l(0) \qquad (39)$$

and relations (25), we change over from the variables $\tilde{\varphi}_0$, $\gamma$, $\psi_0$ to the variables $\theta$, $\varphi$, $\alpha$. This completes the calculation of the function B.

We substitute the result in (20), integrate with respect to the angle $\alpha$, and obtain an expression for the perturbed density:

$$n_1 = e^{-i\omega t} Y_m^l(\varphi, \theta)\frac{n_0}{\Omega^2}\Big\{\Big[\frac{d^2\chi_l}{dr^2} + 2\frac{\partial}{\partial r}\frac{\chi}{r}\Big]\sum_{s=-l}^{l}|P_{s0}^l(0)|^2 \times \frac{-\Omega^2}{(\omega - s\Omega)^2 - 4\Omega^2} + \frac{\chi_l}{r^2}\sum_{s=-l}^{l}|P_{s0}^l(0)|^2 \frac{2\omega\Omega^2 + s\Omega\omega(s\Omega - \omega)}{(s\Omega - \omega)[(\omega - s\Omega)^2 - 4\Omega^2]}\Big\}, \qquad (40)$$

where

$$|P_{s0}^l(0)|^2 = (l+s)!(l-s)! \Big/ \Big[\Big(\frac{l+s}{2}\Big)!\Big(\frac{l-s}{2}\Big)!2^l\Big]^2. \qquad (41)$$

We transform the right-hand side of (40), expanding the functions of the frequency $\omega$ in partial fractions. The Poisson equation (21) is reduced thereby to the form

$$(1 + a_l)\Delta\chi_l(r) = 0. \qquad (42)$$

Here

$$\Delta = \partial^2/\partial r^2 + (2/r)\partial/\partial r - l(l+1)/r^2; \qquad (43)$$

$$a_0 = \frac{3\Omega^2}{(\omega^2 - 4\Omega^2)}, \quad a_1 = \frac{(\omega^2 - 3\Omega^2)3\Omega^2}{(\omega^2 - \Omega^2)(\omega^2 - 9\Omega^2)}, \qquad (44)$$
$$a_2 = \frac{3\Omega^2(\omega^2 - 7\Omega^2)}{(\omega^2 - 16\Omega^2)(\omega^2 - 4\Omega^2)};$$

$$a_l = \sum_{s=-(l+2)}^{l+2} \alpha_s^l \Big/ \Big(\frac{\omega}{\Omega} - s\Big), \quad l = 3, 4, 5, \ldots. \qquad (45)$$

The summation is over s in (45) under the condition that the number $(l + s)$ be even

$$\alpha_s^l = {}^3/_4\{|P_{s-2}^l(0)|^2 - |P_{s+2}^l(0)|^2\}, \ |s| \leqslant l - 2,$$
$$\alpha_{\pm l}^l = \mp {}^3/_4 |P_{l-2}^l(0)|^2, \qquad (46)$$
$$\alpha_{\pm(l+2)}^l = \mp {}^3/_4 |P_l^l(0)|^2.$$

Equation (42) is satisfied for an arbitrary radial dependence of $\Phi_1$, provided that

$$1 + a_l = 0. \qquad (47)$$

This is the sought dispersion equation for the natural oscillations of a homogeneous sphere. The case $\Delta\chi_l = 0$ corresponds to the absence of perturbations.

## 3. FREQUENCIES OF THE NATURAL OSCILLATIONS

Let us consider the consequences of the dispersion equation (47) at different orbital numbers $l$. The perturbations $l = 0$, corresponding to radial displacements of the sphere, have a frequency

$$\omega^2 = \Omega^2. \qquad (48)$$

The case $l = 1$, corresponding to dipole perturbation,

[2)]The idea of using spherical harmonics is due to A. Z. Patashinskiĭ.

is singled out in the sense that it requires an additional condition, namely the conservation of the total momentum of the system[3)]

$$\int_0^R \rho_1 r^3 dr = 0. \quad (49)$$

The condition (49), as can be readily shown, is equivalent to $\Delta\chi = 0$ (only at $l = 1$), i.e., to the absence of perturbations $\rho_\perp = 0$.

Stability takes place at all other $l$. In the case $l = 2$, the squares of the natural frequencies are

$$\omega_{1,2}^2 = {}^1/_2(17 \pm \sqrt{117})\,\Omega^2. \quad (50)$$

At $l = 3$ we get from (45)

$$a_3 = \frac{3\Omega^2(\omega^2 - 13\Omega^2)\,\omega^2}{(\omega^2 - \Omega^2)(\omega^2 - 9\Omega^2)(\omega^2 - 25\Omega^2)}. \quad (51)$$

Taking this into account, we get from (47) the natural frequencies

$$\omega_1^2 = 1.24\,\Omega^2; \quad \omega_2^2 = 8.1\,\Omega^2; \quad \omega_3^2 = 22.7\,\Omega^2. \quad (52)$$

With increasing $l$, the coefficients $\alpha_s^l$ in (45) decrease like $1/l$. They are numerically small already at $l = 4$. Consequently, for $l \geq 4$ we can assume that

$$\omega^{(l)} = \Omega(s - \alpha_s^l), \quad (53)$$

where $|s| \leq l + 2$ and $s + l$ is even. The use of (53) at $l = 3$ leads to a result close to (52).

## 4. DISCUSSION OF RESULTS

We have thus investigated the influence of small perturbations on the stability of a spherically-symmetrical system of rotating particles, assuming that the particles move on circles, and that their average mass density does not depend on the radius. Under these assumptions we have shown, first, that there is a certain spectrum of natural oscillations in the gravitating medium, and, second, that all oscillations of this spectrum have real frequencies. Neither of the results is self-evident, although, as shown by the following considerations, they are perfectly reasonable from the physical point of view.

The absence of a radial velocity component of the equilibrium motion of the particles makes it possible to regard the radial perturbed motion of our medium as that of a "cold" medium. Therefore the question of the natural oscillations of the gravitating system can be set in correspondence with the question that arises in the investigation of the oscillations of a cold plasma. It is known that the natural oscillations of the cold plasma exist only when its density is homogeneous. Otherwise it is impossible to construct any initial perturbation whose amplitude would satisfy after the lapse of a certain time the relation $\exp(-i\omega t)$ with a real or complex frequency.

All perturbations of a cold inhomogeneous plasma are subject to non-exponential (power-law) damping. In our example of a gravitating medium, not all the stationary parameters are spatially homogeneous: the particle velocities and the potential depend on the radius. In the final differential equation for the perturbed potential, however, the coefficients of the equation are constants proportional to the particle angular rotation frequency $\Omega$ = const. Owing to the homogeneity of the density and of the angular velocity, the problem of the initial perturbations reduces to a problem of natural oscillations and of finding the spectrum of the natural frequencies. On the basis of the analogy with plasma problems, we can expect that in the case of an inhomogeneous density there will be no spectrum of the natural oscillations of the gravitational medium.

We now estimate qualitatively the stability of our system against different types of perturbations. We start with a discussion of the case of radial perturbations ($l = 0$). Such perturbations can also be investigated with the aid of the energy principle developed for rotating gravitational systems in [7], as was indeed done in [8]. It turns out here that the radial perturbations of a homogeneous medium lead to an increase of the potential energy of the system; this, as is well known, is evidence of stability.

Perturbations with $l \neq 0$, in which an important role is played by the relative motion of the particles along $\gamma$ and $\theta$, are analogous to cyclotron oscillations of a magnetized plasma with a $\delta$-function particle-velocity distribution. The problem of plasma oscillations involves two characteristic frequencies—cyclotron and plasma. The so-called cone instability occurs if the plasma frequency greatly exceeds the cyclotron frequency. The analog of the plasma frequency in the problem of a gravitational medium is the Jeans frequency $\omega_0 \equiv \sqrt{3}\Omega$, and the role of the cyclotron frequency $\omega_B$ is played by the angular frequency of particle rotation $\Omega$. The quantities $\omega_0$ and $\Omega$ are of the same order, so that on the basis of the analogy with the plasma it is natural to expect the absence of instability; this is in accord with the calculation presented above.

Finally, it is necessary to explain one more result, which is paradoxical at first glance, namely the absence of an instability connected with the fact that the square of the "Langmuir" frequency $\omega_L^2 \equiv \omega_0^2$ is negative. (In plasma physics an instability of this type is known as the negative-mass instability, and in astrophysics it is known as the Jeans instability that leads to collapse.) Such an instability is suppressed as a result of the sufficiently intense rotational motion of the particles. This is simplest to explain with perturbations with $l = 0$ as an example. Let us write down the dispersion equation of such perturbations in a form corresponding to cyclotron oscillations of the plasma in a magnetic field:

$$1 - \frac{\omega_L^2}{\omega^2 - \omega_B^2} = 0. \quad (54)$$

For $|\omega_L^2| \gg \Omega^2$ this would lead to $\omega^2 < 0$, meaning instability. However, the equilibrium condition leads to $\omega_L^2 = -3\Omega^2$ ($\omega_B^2 \equiv 4\Omega^2$), so that $\omega^2 > 0$.

Let us discuss now the extent to which our results can have a bearing on observable objects. In really observed globular clusters, the density always decreases towards the edge quite sharply, in proportion to $1/r^2$–$1/r^3$. Energy estimates permit us to assume that systems analogous to that considered here, with a density decreasing towards the edge, have a large stability margin compared with a homogeneous mass. It is also reasonable to assume that the case of non-circular or-

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[3)] For all the remaining $l$, the conservation of the total momentum is automatically satisfied by the angular dependence of $\rho_1(\theta)$.

bits will not change the conclusion that the system is stable.

It is shown in [7,9] that the evolution of stellar systems of a more complicated type, containing two or more subsystems, leads to the formation of spiral arms. Our conclusion that a globular cluster of stars is stable, together with the results of [7,9], favors Oort's hypothesis [10] of galaxy evolution. Oort's hypothesis, unlike Hubble's, regards different forms of galaxies not as successive stages of evolution, but as a result of differences in the initial conditions of their occurrence (depending on the total angular momentum of the system, etc.).

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186