POSSIBLE EXISTENCE OF SUBSTANCES INTERMEDIATE BETWEEN METALS AND DIELECTRICS

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The question of the possible existence of substances having an electron spectrum without any energy gap and, at the same time, not possessing a Fermi surface is investigated. First of all the question of the possibility of contact of the conduction band and the valence band at a single point is investigated within the framework of the one-electron problem. It is shown that the symmetry conditions for the crystal admit of such a possibility. A complete investigation is carried out for points in reciprocal lattice space with a little group which is equivalent to a point group, and an example of a more complicated little group is considered. It is shown that in the neighborhood of the point of contact the spectrum may be linear as well as quadratic.

The role of the Coulomb interaction is considered for both types of spectra. In the case of a linear dispersion law a slowly varying (logarithmic) factor appears in the spectrum. In the case of a quadratic spectrum the effective interaction becomes strong for small momenta, and the concept of the one-particle spectrum turns out to be inapplicable. The behavior of the Green's functions is determined by similarity laws analogous to those obtained in field theory with strong coupling and in the neighborhood of a phase transition point of the second kind (scaling). Hence follow power laws for the electronic heat capacity and for the momentum distribution of the electrons.

1. INTRODUCTION

ONE of the basic assumptions of the Landau theory of a Fermi liquid is the relationship, according to which the limiting momentum of the excitations in an isotropic Fermi liquid is determined in the same way as in a gas, by the density of the atoms in the liquid, i.e., \( p_0 = (3\pi n)^{1/3} \) (where \( n \) is the density of particles). According to the work of Luttinger and Ward, with a certain amount of alteration this theorem is also applicable to the electronic liquid in metals. Namely, it turns out that the total volume, bounded by all Fermi surfaces, determines the density of the electrons:

\[
n = \frac{2(2\pi)^3}{V_r} \left( p V_c + V_F \right),
\]

where \( V_r \) denotes the volume of an elementary cell of the reciprocal lattice, \( p \) is an integer, and \( V_F \) denotes the total volume inside all of the Fermi surfaces (from the side of smaller energies), referred to a single cell of the reciprocal lattice.

It is not difficult to see that the cited relation is exactly the same as for a noninteracting Fermi system in a periodic field. In view of the fact that \( V_F = (2\pi)^3V \), where \( V \) is the volume of an elementary cell of the crystal, the substance can be dielectric only when the number of electrons per elementary cell of the crystal is even. If in this case the substance nevertheless is a metal, then the number of holes must necessarily be equal to the number of electrons. For metals with an odd number of electrons per cell, on the other hand, such compensation cannot occur.

In the case of interest to us, the maximum of the valence band must coincide with the minimum of the conduction band. In general an increased symmetry of the appropriate point of the reciprocal lattice favors degeneracy of the energy levels, but it may also occur at accidental points. However, here somewhat more is required, namely, that this point simultaneously corresponds to extrema of the two bands. The occurrence of such a situation at an accidental point is an improbable special case. In view of this we shall consider only "regular" cases, i.e., points of high symmetry. A complete analysis is given in the case when the little group of the corresponding point is equivalent to the point group, and an example of a more complicated group is considered. It turns out that a "Fermi point" is possible in a whole series of different cases, where in the most typical cases the dispersion law in the vicinity of such a point will either be linear or quadratic.

However, in general a model of noninteracting electrons does not correspond to a real electronic liquid. In the case of an isotropic liquid with short-range inter-
action forces, Landau was able to show (see 42) that the energy spectrum of the quasi-particles retains the same form as for a Fermi gas. This proof was then generalized to a Fermi liquid in a crystal 43 and to a Fermi liquid with long-range Coulomb forces 37. In the last case the fact that the presence of “free electrons” leads to a screening of the Coulomb field and converts it into a short-range field is extremely important. Close-lying excited states play the most important role in the integral which determines the screening. Therefore, in any case the screening is absent if there is a gap in the spectrum. On the other hand, in a real metal the screening radius is of the order of interatomic distances.

From this point of view it is necessary to investigate what kind of influence the Coulomb field has in the case of a “Fermi point.” The absence of any gap in the spectrum must necessarily lead to a weakening of the Coulomb potential at large distances. At the same time the replacement of the Fermi surface by a point leads to a change in the nature of the screening, and it is not obvious beforehand how this will affect the energy spectrum.

The usual way to prove the presence of a Fermi spectrum in the theory of a Fermi liquid (see 42) consists in the following. It is assumed that the spectrum has the appropriate form and, starting from this, the Green’s function is written down. After this it is proved that the constant coefficients appearing in it are actually constants which depend on the interaction of the particles. This method of proof is universal for that case when one can foresee the result beforehand. However, in the case under consideration this is by no means so. Therefore, let us proceed according to a different method. Let us consider the Coulomb interaction, which we shall assume to be weak and, using the Green’s function of noninteracting particles as the zero-order approximation, let us find the Green’s function and the other characteristic quantities associated with the presence of the interaction.

The supposition about the weakness of the interaction may be justified by the following consideration. In the cases under consideration there exists a large range of momenta for which the valence band is close to the conduction band. As we shall see below, in the very neighborhood of the point of contact between the bands a non-trivial situation arises which, in particular, leads to a change in the nature of the interaction at small values of the momenta. However, the nearness of the bands is preserved even at a certain distance from this neighborhood. If we shall consider the electronic transitions outside the “dangerous neighborhood,” then they lead to the appearance of the ordinary dielectric constant \( \varepsilon_0 \) which, in view of the proximity of the bands, should be rather large (as, for example, in Ge and Si). In view of this the bare Coulomb interaction \( e^2/\varepsilon_0 \) or, more precisely, the characteristic dimensionless constant \( e^2/\varepsilon_0 \), where \( v \approx 10^{10} \) cm/sec, must be small. Of course, in addition the case is possible when the dielectric constant is not large and consequently the interaction will not be weak which, in turn, leads to an increase of the “dangerous neighborhood” up to dimensions which are comparable with the period of the reciprocal lattice. However, to judge from the existing data on \( \alpha-Sn \) and \( HgTe \), this case scarcely occurs. But even if \( \varepsilon_0 \) is not overly large then, in any case, with the aid of a study of the weakened interaction one can formulate some idea about what happens in the neighborhood of the Fermi point.

We have considered the two most typical cases: linear and quadratic spectra in the neighborhood of the contact point. It turns out that both types of spectra lead to quite different consequences upon taking the interaction into account. One can assume that these two characteristic types of behavior exhaust all possibilities.

2. CONTACT IN THE GROUPS \( D_n \)

First of all let us consider a point \( k_0 \) in reciprocal lattice space with a little group which is equivalent to one of the point groups. Since the question involves the contact of bands, then degeneracy of the levels must occur at the point \( k_0 \) which is lifted (if only partially) upon a change of \( k \) in an arbitrary direction. This means that the little group of \( k_0 \) must have a multidimensional representation, i.e., contain axes of the third, fourth, or sixth orders. In addition, it is obvious that an intersection of the axes and planes of symmetry must occur at the point \( k_0 \). If the spin-orbit coupling is small enough so that one can neglect it, then it is necessary to consider single-valued irreducible representations. Upon taking account of the spin-orbit coupling it is necessary to consider double-valued representations.

Let us start with simple groups. Let us demonstrate that contact is possible in the groups \( D_3 \) and \( D_6 \). As is shown in article 15, in order to determine the dispersion law it is necessary to construct an invariant form \( Q(k; r, r') \) containing the components of the vector \( k \) and which is bilinear in the basis functions \( u_2(r) \) and \( u_3(r') \). Let us consider single-valued representations. In the groups under consideration, one invariant exists which is linear in \( k \): \( u_2(r)u_3(r') - u_3(r)u_2(r') \). The presence of such an invariant is very important because it precisely this property which guarantees the possibility of a “Fermi point”, where the functions \( u_2 \) transform like \( x \pm iy \). With the components \( k_x \) and \( k_y \) one can form only the quadratic invariants

\[
u_0(r)u_0^*(r')k_x^2 + u_0(r)u_0^*(r')k_y^2 \]

and

\[
u_1(r)u_0^*(r')k_x^2 + u_1(r)u_0^*(r')k_y^2 \]

The latter combination of \( u_0 \) is an invariant by itself, then one can also multiply it by the invariant \( k_z \) finally the invariant

\[
u_2(r)u_0^*(r')k_x^2 + u_2(r)u_0^*(r')k_y^2 \]

still exists in the group \( D_6 \). Collecting everything together one can write down the following invariant form:

\[
Q(k; r, r') = \left[ \nu_0(r)u_0^*(r') + u_0(r)u_0^*(r') \right] (ak_x^2 + \beta k_y^2^2) + \left[ \nu_1(r)u_1^*(r') + u_1(r)u_1^*(r') \right] k_x^2 + \left[ \nu_2(r)u_2^*(r') + u_2(r)u_2^*(r') \right] k_y^2.
\]

(1)

In the groups \( D_3 \) and \( D_6 \) the coefficient \( \delta = 0 \). Putting together the secular equation, we obtain the following result for the energy levels:

\[
v_{1,2} = \pm ak_x^2 + \beta k_y^2 \mp \sqrt{[v_0^2 + (\gamma + \delta)]k_z^2 - 8\delta k_x^2k_y^2}. \]

(2)

It is not difficult to see that ranges of values of the...
coefficients exist for which one of the branches is positive for all directions k and the other is negative, i.e., a point Fermi surface is possible. It is of interest to note that the behavior of the branches of the spectrum is strongly anisotropic. In a direction outside the basis plane \( \epsilon_{1,2} \) changes basically according to a linear law, but in the basis plane it changes according to a quadratic law.

In the presence of strong spin-orbit coupling it is necessary to use double-valued representations. Two double-valued representations exist in the group \( D_3 \) (see (22)). With the aid of the basis functions for the representation \( E'_1 \) which transform like the components of a spinor, one can set up invariants which are linear in \( k_z \) and \( k_z \). The general invariant form has the form

\[
Q(k; r, r') = a\{u_i(r)u'_i(r') - u_i(r)u'_i(r')\} k_i + \beta\{u_i(r)u'_i(r') k_i + u_i(r)u'_i(r') k_i\}.
\]

The solution of the corresponding secular equation has the form

\[
\epsilon_{i,1,2} = \pm(k^2 + \beta k^4),
\]

(4)

As to the other double-valued representation \( E'_2 \) (which in the present case is a combination of two complex-conjugate one-dimensional representations), then from the corresponding basis functions one can construct only an invariant or a combination which gives an invariant together with \( k_z \). Thus, for the energy spectrum we obtain

\[
\epsilon_{1,2} = \pm k^2 \pm \beta k^4 \pm \gamma k_i.
\]

(5)

In this connection a "Fermi point" cannot appear.

In the case of the group \( D_4 \) the representations \( E'_1 \) and \( E'_2 \) give a situation analogous to \( E_2 \) in \( D_3 \), i.e. to formula (4), and the representation \( E'_3 \) is analogous to \( E'_1 \) in \( D_3 \), i.e., it does not correspond to a "Fermi point." Finally, in the group \( D_4 \) both double-valued two-dimensional representations are analogous to \( E'_2 \) in the group \( D_3 \) and give formula (4).

In all of the described cases of the occurrence of a "Fermi point" the fact that the basis functions permitted the construction of an invariant linear in \( k_z \) was very important. This does not hold in other point groups with the need to use double-valued representations. Two-dimensional representations exist in the group \( D_3 \) and give formula (4).

Having taken \( k \) in the \([111]\) direction we obtain \( \epsilon_{1,2} = \pm k^2 \), from which it follows that although degeneracy occurs at the point \( k = 0 \) it does not simultaneously correspond to a minimum of one of the functions \( \epsilon_{1,2}(k) \) and \( \epsilon_{1,2}(k) \)

Let us go on to the three-dimensional representations. Here it is easiest of all to follow the method indicated in (8) (see also (9)). The bilinear combinations of interest to us are \( u'_j(r') \{ f(\tilde{J}) \} u_j(r) \), where \( f(\tilde{J}) \) is some kind of product of the spin projection operators (in the present case \( J = 1 \)). These combinations transform in the same way as the operators \( f(\tilde{J}) \) themselves, and therefore it is necessary for us in turn to construct invariants out of the components of \( k \) and of the operators \( J \). Since \( J = 1 \) the invariants may be either linear or quadratic in \( J \). From the requirement of invariance under time reversal it follows that the total power of \( k \) and \( J \) should be even.

If the little group \( k \) is equivalent to the group \( T \) or \( O_2 \), then an invariant exists which is linear in \( k \), namely, the product \( k \cdot J \). Confining our attention to this approximation we obtain

\[
\epsilon_{1,2} = \pm \mu k, \quad 0, \quad -\mu k.
\]

(6)

For the final solution of the question about whether the point \( k = 0 \) possesses the necessary properties, it is necessary to determine whether it corresponds to an extremum of the function \( \epsilon_{1,2}(k) \). For this it is necessary to find the next approximation for \( \epsilon_{1,2}(k) \). Among the invariants is \( \pm k^2 \), and one can assume that it is larger than all remaining invariants. In this connection, the function \( \epsilon_{1,2}(k) \) will have the same sign for all values of \( k \). Therefore, in the present case the point \( k = 0 \) may correspond to contact of the conduction band with the valence band.

In the case when the little group is equivalent to \( T_2 \) or \( O_2 \), i.e., when it contains a center of inversion, the invariants must necessarily contain an even number of \( k \) and an even number of \( J \). The same pertains in the present case to the group \( T_d \), although it does not contain a center of inversion. This is related to the fact that the scalar product \( k \cdot J \) is not an invariant for this group, but the invariants which are linear in \( k \) and cubic in \( J \) are absent for \( J = 1 \). One can show that in all three groups there is one and the same common invariant:

\[
\tilde{D} = ak^2 + \beta \{(l^2 + \beta k^2 + \gamma J^2)(k_x^2 + \theta k_y^2 + \theta k_z^2) + (l^2 + \beta k^2 + \gamma J^2)(k_y^2 + \theta k_x^2 + \theta k_z^2) + \gamma(k_x k_y, l, J) + k_y k_x, l, J) + k_z k_x, l, J)\}.
\]

(7)

where \( \{J_x, J_y\} = J_x J_y + J_y J_x \). The corresponding secular equation has the form

\[
\epsilon_{1,2} = \pm (k^2 + \beta k^4 + \gamma J^2).
\]
Let us demonstrate that the coefficients \( \alpha, \beta, \) and \( \gamma \) can be chosen in such a way that the eigenvalues \( \xi(k) \) will have different signs, where each of them will retain its sign for all directions \( k \). The determinant of the matrix \( D_{\text{jk}}(k) \) given by Eq. (9) is the product of the three functions \( E_{\gamma}(k) \) have different values, i.e., for example, \( E_{1}(k)E_{2}(k)E_{3}(k) \neq 0 \). For this it is obviously necessary that all extrema of \( E_{\gamma}(k) \), then the determinant must vanish at this point. Hence one of the conditions is \( ||D_{\text{jk}}(k)|| 
eq 0 \). For this it is obviously necessary that in one of the indicated directions \( k \) do not vanish and have one and the same sign. But the extremum of the determinant correspond to the most symmetric directions, namely \([100],[110], \) and \([111]\). Therefore, it is sufficient to establish the conditions under which, let us say, \( ||D_{\text{jk}}|| < 0 \) in these three directions. In addition, it is necessary that in one of the indicated directions the functions \( E_{\gamma}(k) \) have different values, i.e., for example, \( ||D_{\text{jk}}|| < 0 \) the values \( E_{1} < 0 \) and \( E_{2}, E_{3} > 0 \). Then the condition that be one negative function and two positive functions \( E_{\gamma}(k) \) for all three symmetric directions of \( k \) is obviously equivalent to these requirements.

For such directions the functions \( E_{\gamma}(k) \) can easily be found:

\[
\begin{align*}
[100]: & \quad E_{1} = (\alpha - 2\beta)k, \quad E_{2} = (\alpha + \beta)k, \\
[110]: & \quad E_{1} = (\alpha - \beta)(\beta + y)k, \quad E_{2} = (\alpha + \beta)(\beta - y)k, \quad E_{3} = (\alpha + \beta)k, \\
[111]: & \quad E_{1} = (\alpha - \beta)(\beta + y)k, \quad E_{2} = (\alpha + \beta)(\beta - y)k, \quad E_{3} = (\alpha + \beta)k.
\end{align*}
\]
\[
\text{(11)}
\]

It is not difficult to show that the ranges of the values \( \alpha, \beta, \) and \( \gamma \), in which \( E_{1} < 0 \) and \( E_{2}, E_{3} > 0 \). Thus, contact of the conduction band with the valence band is possible in the groups \( T_{d}, T_{h}, \) and \( O_{h} \).

4. CUBIC GROUPS. DOUBLE-VALUED REPRESENTATIONS

Let us go on to double-valued representations. There are two kinds of such representations: two-dimensional corresponding to \( J = 1/2 \) and four-dimensional corresponding to \( J = 3/2 \). If the little group \( k = 0 \) is equivalent to \( T \) or \( O \) then again the invariant \( k \cdot J \) exists. For two-dimensional representations we obtain

\[
\begin{align*}
& E_{1} = \pm \gamma k, \quad E_{2} = -\pm \gamma k. \\
& \text{(12)}
\end{align*}
\]

One cannot consider two-dimensional representations of the remaining groups. In fact, in the presence of a center of inversion in the lattice, the two-fold degeneracy is retained for arbitrary \( k \), which corresponds to the fact that the energy does not depend on the direction of the spin. As for the group \( T_{d} \), then here, just as for the single-valued three-dimensional representations, the necessary invariant which is linear in \( k \) is absent. Therefore, in the present case the fundamental invariant will simply be \( \pm \gamma k^{2} \), i.e., the two functions \( E_{1}, E_{2} \) coincide in this approximation. This degeneracy is lifted only by small terms which are cubic in \( k \), from which it is clear that in the present case the point \( k = 0 \) does not possess the required properties.

Let us go on to four-dimensional representations. In the absence of a center of inversion (\( T, T_{d}, \) or \( O \)) the admissible invariants can be linear in \( k \) and cubic in \( J \). For the group \( T \) the general form of the invariant is given by

\[
D = a(kJ_{z} + \beta(kJ_{x} + kJ_{y} + kJ_{z})) + [k(J_{z} (I_{z} - I_{2}^2)) + k(J_{x} (I_{x} - I_{2}^2)) + k(J_{y} (I_{y} - I_{2}^2))].
\]
\[
\text{(13)}
\]

where \( \{A, B\} = AB + BA. \)

In the group \( O \) the last invariant is absent, i.e., \( \gamma = 0 \), and in the group \( T_{d} \alpha = \beta = 0. \) The possible existence of a "Fermi point" in the groups \( T \) and \( O \) is determined by the following simple consideration. Let \( \alpha \neq \beta, \gamma \). In this case \( E_{1} = -(3/2)\alpha k, E_{2} = -(1/2)\alpha k, E_{3} = (1/2)\alpha k, \) and \( E_{4} = (3/2)\alpha k. \) Here \( k = 0 \) possesses the required properties. Hence it follows that ranges of values of the coefficients exist for which the point \( k = 0 \) retains these properties.

The group \( T_{d} \) is in a special position. The solution of the secular equation leads to the following eigenvalues:

\[
E_{1,2} = \pm \gamma(k^{2} + 9\beta^{2})^{1/2} - (\gamma \mp 3\beta^{2})k,
\]
\[
E_{3,4} = (\alpha \pm 2\beta)k.
\]
\[
\text{(14)}
\]

where \( n \) denotes a unit vector in the direction \( k \). In the direction \([111]\) two of the roots vanish in this approximation. In analogy to the preceding discussion, by adding the invariant \( \pm \beta k^{3} \) one can also verify here that the point \( k = 0 \) can possess the necessary properties. In the case when the little group contains a center of inversion (\( T_{h} \) or \( O_{h} \)), the invariant must be quadratic in \( k \) and \( J \). The general form of the invariant is again given by formula (9). Here, however, the matrix \( J_{1} \) corresponds to \( J = 3/2 \). Solving the secular equation we obtain the well-known formula

\[
E_{1,2} = \pm \beta(k^{2} + 3\gamma k^{2} + 3\gamma^{2})k,
\]
\[
E_{3,4} = (\alpha \pm 2\beta)k.
\]
\[
\text{(15)}
\]

Each level is doubly degenerate. For the symmetric directions we obtain

\[
[100]: \quad E_{1} = (\alpha \pm 3\beta)k^{2},
\]
\[
[110]: \quad E_{3} = (\alpha \pm 3\beta)k^{2},
\]
\[
[111]: \quad E_{3} = (\alpha \pm 3\beta)k^{2}.
\]
\[
\text{(16)}
\]

From here it follows that contact of the valence band and the conduction band can actually occur at the point \( k = 0 \). We note that precisely this representation is realized in \( \alpha \)-Sn (usually denoted by \( \Gamma_{s} \)).

5. MORE COMPLICATED SYMMETRY

The analysis carried out above exhausts all cases involving the appearance of a "Fermi point" with a little group equivalent to a point group. We shall not give such a complete analysis of the cases of more complicated symmetry, and we only show that here "Fermi points" can also appear. The appearance of nontrivial screw axes and glide planes frequently increases the multiplicity of the degeneracy of the energy levels. In the literature this phenomena is sometimes called coalescence of the energy bands. It is essential that in the majority of the cases the bands coincide on intact pieces of the surface of the Brillouin zone, and consequently there cannot be any "Fermi points" here. However, in a number of cubic groups (namely \( O \) and \( O^{6} \), and also \( O^{6} \), using the notation from \( \Omega^{6} \)) the little groups of the vectors \( k_{0} \) with a point at a corner of the cubic Brillouin zone have four-dimensional and six-dimensional single-valued representations.

Let us present the invariant for the four-dimensional representations:

\[
Q(k; r, r') = \omega([u_{0}(r)u_{0}'(r') - u_{0}(r)u_{0}'(r')]k + [u_{1}(r)u_{1}'(r') - u_{1}(r)u_{1}'(r')]k_{1} + [u_{2}(r)u_{2}'(r') - u_{2}(r)u_{2}'(r')]k_{2} + [u_{3}(r)u_{3}'(r') - u_{3}(r)u_{3}'(r')]k_{3}.
\]
The matrix \( \mathbf{D} \) may be written in the form \( \mathbf{D}(k) = \omega(\mathbf{k}) \), where the matrices \( \mathbf{Q} \) possess the commutation relations of the Dirac matrices. In our approximation the levels remain doubly degenerate and \( \epsilon_{1,2}(k) = \pm \nu k \).

The six-dimensional representations also admit realization of a point Fermi surface but, as in the case of three-dimensional representations of the simple cubic groups, part of the energy bands begins with terms which are quadratic in \( k \).

Thus, we have shown that cases when contact of the valence band with the conduction band occurs in the electron spectrum are not unusual, and consequently such substances may be encountered in nature. Of course, we did not exhaust all possible cases involving complicated symmetry, but here it is apparently reasonable to consider specific examples for which the experimental data will indicate the existence of a "Fermi point."

Our conclusions have been reached in the one-electron approximation, i.e., we have not taken the interaction of the electrons into consideration. This problem merits very serious attention because it is quite possible that it may turn out that the interaction leads to instability of the obtained spectrum.

6. COULOMB INTERACTION ASSOCIATED WITH A LINEAR SPECTRUM

We consider, first of all, the simplest case of two energy branches which are described by formula (12). In this case the matrix \( \mathbf{D} \) may be written in the form

\[
\mathbf{D} = \omega(\mathbf{k}),
\]

where the \( \mathbf{Q} \) are the Pauli matrices. In accordance with this the Green's function for noninteracting electrons may be written in the form

\[
\mathbf{G}_n = [\omega - \nu(\mathbf{k}) + i\delta \mathop{\text{sign}}\omega]\cdot.
\]

Now let us introduce the Coulomb interaction between the electrons, which we shall write in the usual form. Let us consider how screening occurs. The screening interaction has the form

\[
\Gamma(\mathbf{k}, \omega) = \frac{4\pi e^2}{\varepsilon \omega^2 - 4\pi e^2 \Pi(\mathbf{k}, \omega)},
\]

where \( \Pi(\mathbf{k}, \omega) \) denotes the sum of all possible electron loops. In the first approximation

\[
\Pi(\mathbf{k}, \omega) = -i \mathop{\text{Sp}} \int \mathbf{G}_n(\omega, k, l) \mathbf{G}_n(\omega + \omega, k + k, l) \frac{d\omega, d^2k}{(2\pi)^2}.
\]

Substituting here the function \( \mathbf{G}_0 \), which is more convenient to write in the form

\[
\mathbf{G}_0 = \frac{\omega + \nu(\mathbf{k})}{\omega^2 - (\nu(\mathbf{k}) + \nu(-\mathbf{k}))^2},
\]

we obtain

\[
\Pi = -\frac{k^4}{12\pi^2} \ln \frac{k_{\max}}{\omega^2 - \omega^2/4},
\]

where \( k_{\max} \) is the limit of integration, which is of the order of the period of the reciprocal lattice.

Thus, in the case under consideration the Coulomb interaction is practically not screened. As we see below, the evaluation of the self-energy corrections to the electronic spectrum with the aid of formula (20) also leads to a logarithmic integral. From here it follows that at the point \( \mathbf{k} = 0 \) the energy spectrum has a singularity, and the matrix \( \mathbf{D} \) cannot be expanded in a series in powers of \( k \). However, in view of the fact that the singularity is weak, we may, without violating the symmetry conditions, assume that the matrix \( \mathbf{D} \) is multiplied by a slowly varying invariant function, i.e., by a function which depends on the absolute value of \( k \) and on \( \omega \). To find this function we apply the method of article[10].

The equation for the Green's function is written in the form

\[
G^{-1} = G_n^{-1} - \Sigma = \omega - \nu(\mathbf{k}, \omega) - \Sigma(\mathbf{k}, \omega).
\]

We shall seek the complete Green's function in the form

\[
G = \frac{\omega + \nu(\mathbf{k})f(\mathbf{k}, \omega)}{\omega^2 - [\nu(\mathbf{k}, \omega) - \delta F]^2},
\]

where \( f \) is a slowly varying function of \( k \) and \( \omega \). The first approximation for the polarization operator, evaluated with the aid of this function instead of (19'), is

\[
\Pi(\mathbf{k}, \omega) = -\frac{k^4}{12\pi^2} \int \frac{d\omega}{f(\mathbf{k})},
\]

where \( \xi = \ln \frac{k_{\max}}{\omega(v)\mathop{\text{max}}(\omega)} \), \( \eta = \ln \frac{k_{\max}}{k_1} \).

(24)

Thus, we have shown that cases when contact of the valence band with the conduction band occurs in the electron spectrum are not unusual, and consequently such substances may be encountered in nature.

Our conclusions have been reached in the one-electron approximation, i.e., we have not taken the interaction of the electrons into consideration. This problem merits very serious attention because it is quite possible that it may turn out that the interaction leads to instability of the obtained spectrum.

In general, besides the simplest loops of two Green's functions, more complicated diagrams (Fig. 1) enter into \( \Pi \). One can join all of them together into a diagram of the first type, but in this connection one of the triple-vertices is replaced by the appropriate total vertex. For simplicity we make the assumption: \( k \gg \omega/v \). The logarithmic integral in \( \Pi \) goes over the region where \( \omega/v \sim k_1 \gg k \). Consequently, according to (20),

\[
\Gamma(\mathbf{k}, \omega) = \frac{4\pi e^2}{\varepsilon_0 v(\mathbf{k})} d(k, \omega), \quad d(k, \omega) = \left[ 1 + \frac{\xi^2 + \eta^2}{3\omega v} \right] \frac{d\omega}{f(\mathbf{k})}.
\]

The equation for the Green's function is written in the form

\[
G^{-1} = G_n^{-1} - \Sigma = \omega - \nu(\mathbf{k}, \omega) - \Sigma(\mathbf{k}, \omega).
\]

(22)

We shall seek the complete Green's function in the form

\[
G = \frac{\omega + \nu(\mathbf{k})f(\mathbf{k}, \omega)}{\omega^2 - [\nu(\mathbf{k}, \omega) - \delta F]^2},
\]

(23)

where \( f \) is a slowly varying function of \( k \) and \( \omega \). The first approximation for the polarization operator, evaluated with the aid of this function instead of (19'), is

\[
\Pi(\mathbf{k}, \omega) = -\frac{k^4}{12\pi^2} \int \frac{d\omega}{f(\mathbf{k})},
\]

(24)

In general, besides the simplest loops of two Green's functions, more complicated diagrams (Fig. 1) enter into \( \Pi \). One can join all of them together into a diagram of the first type, but in this connection one of the triple-vertices is replaced by the appropriate total vertex. For simplicity we make the assumption: \( k \gg \omega/v \). The logarithmic integral in \( \Pi \) goes over the region where \( \omega/v \sim k_1 \gg k \). Consequently, according to (20),

\[
\Gamma(\mathbf{k}, \omega) = \frac{4\pi e^2}{\varepsilon_0 v(\mathbf{k})} d(k, \omega), \quad d(k, \omega) = \left[ 1 + \frac{\xi^2 + \eta^2}{3\omega v} \right] \frac{d\omega}{f(\mathbf{k})}.
\]

(25)

In general, besides the simplest loops of two Green's functions, more complicated diagrams (Fig. 1) enter into \( \Pi \). One can join all of them together into a diagram of the first type, but in this connection one of the triple-vertices is replaced by the appropriate total vertex. For simplicity we make the assumption: \( k \gg \omega/v \). The logarithmic integral in \( \Pi \) goes over the region where \( \omega/v \sim k_1 \gg k \). Consequently, according to (20),

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\]

(25)
Now let us go on to the self-energy of an electron. In the first approximation we have
\[ \Sigma(\omega, k) = \frac{i}{\omega} \int \mathcal{G}(\omega + \omega_1, k + k) \frac{d^3k_1}{(2\pi)^3}. \]
By means of a renormalization of the chemical potential we can always eliminate an arbitrary constant from \( \Sigma \).
Therefore we obtain
\[ \Sigma(\omega, k) = \frac{i}{\omega} \int \mathcal{G}(\omega + \omega_1, k + k) \frac{d^3k_1}{(2\pi)^3}. \]
(26)

Aside from this term, the total self-energy contains terms corresponding to more complicated diagrams (see Fig. 2), but one can show that they give a small contribution. In fact, let us consider the next correction containing two \( \Gamma \)-lines. In order for this term to be important, it must contain two logarithmic integrals with respect to the momenta of the \( \Gamma \)-lines: \( k_1 \) and \( k_2 \). Consequently there should be two logarithmic regions, \( k_1 \gg k_2 \gg k \) and \( k_2 \gg k_1 \), but as already mentioned earlier the vertex \( \gamma \) may contain logarithmic integrals only in that case if its arguments differ appreciably in magnitude, where these integrals correspond to the regions between the values of the arguments. In view of this, in the region \( k_1 \gg k_2 \gg k \) the logarithmic integral with respect to \( k_1 \) is not present, but in the second region there is no logarithmic integral with respect to \( k_2 \). Thus, we may confine our attention to expression (26). Substituting expressions (23) and (25) here, we find
\[ \Sigma = \frac{2e^f}{3n_{\text{vo}}} \int \frac{d^3k_1}{(2\pi)^3} \frac{d(\xi)}{d(\xi)}. \]
(27)

Here \( \xi_1 = \ln \left( \frac{k_{\text{max}}}{k_1} \right) \), \( \xi = \ln \left( \frac{k_{\text{max}}}{k} \right) \), and it is assumed that \( \nu k \ll \omega \). In the case when \( \nu k \ll \omega \),
\[ \Sigma = \frac{2e^f}{3n_{\text{vo}}} \int \frac{d^3k_1}{(2\pi)^3} \frac{d(\xi)}{d(\xi)} \left( \xi - \eta \right), \]
(27')

where \( \eta = \ln \left( \frac{k_{\text{max}}}{\nu k} \right) \) and \( \xi = \ln \left( \frac{k_{\text{max}}}{k} \right) \). The most interesting case is \( \omega \sim \nu k \), i.e. formula (27). It is necessary to explicitly specify the case \( \omega \approx \pm \nu k \).
Strictly speaking, at such values of \( \omega \) a singularity must appear in \( \Sigma \) because the spectrum linear in \( k \) is unstable with respect to the decay of quasi-particles into many other particles traveling in the same direction. Such singularities actually exist, but they appear only in the next approximation, in terms proportional to
\[ \frac{\nu^f}{\nu^f} \ln \left( \frac{\nu k_{\text{max}}}{\nu k} \right). \]
In those terms which we are considering, the lower limit of the integrals is simply \( k \), and the proximity of \( \omega \) to \( \pm \nu k \) does not manifest itself in any way. Thus, in the approximation under consideration the case \( \omega \approx \pm \nu k \) is described by formula (27).

If formulas (22), (23), (25), and (27) are compared, then we obtain an equation for the function \( f(\xi) \) (we assume \( \nu k \ll \omega \))
\[ f(\xi) = 1 + \frac{e^f}{3n_{\text{vo}}} \int \frac{d^3k_1}{(2\pi)^3} \frac{d(\xi)}{d(\xi)}. \]
By means of two differentiations this equation reduces to a differential equation. Solving the latter equation we obtain
\[ f(\xi) = \left( 1 + \frac{e^f}{3n_{\text{vo}}} \right)^{-\nu k}. \]
(29)

and substituting into the expression for \( d(\xi) \) we find
\[ d(\xi) = \left( 1 + \frac{e^f}{3n_{\text{vo}}} \right)^{-\nu k}. \]
(30)

Thus, for the case under consideration one obtains the result that \( G \) has a pole for \( \omega = \pm \nu k f(\xi) \). The imaginary part which determines the attenuation is small in comparison with the real part \( \text{Im} G^{-1} / \text{Re} G^{-1} \sim e^f \nu k \). For momenta of the order of or smaller than \( k_{\text{max}} \), the energy spectrum differs from a purely linear spectrum by a logarithmic factor.

We note that the sign in front of the logarithm \( f(\xi) \) is positive, which corresponds to a weakening of the interaction in the region of small momenta and frequencies. This situation is analogous to quantum electrodynamics and may be called the "zero-charge" situation (in electrodynamics the situation involves, of course, not a decrease of the momentum but a decrease of the radius over which the interaction is smeared out, i.e., an increase of the upper limit on the logarithmically divergent integrals).

It is of interest to clarify whether this situation is common for all types of linear spectra or if it is unique only for the model under consideration. In general a sum of the invariants of \( k \) and \( J \) with arbitrary coefficients appears in the matrix \( \mathcal{D} \). Under the influence of the interaction each of these coefficients changes into a slowly varying function. Ultimately one must obtain a system of integral equations for these functions. Let us confine our attention to such a generalization of the case under consideration, for which the matrix \( \mathcal{D} \) contains only one invariant of \( k \) and \( J \), i.e., there is only one unknown function. An examination of the problem with several arbitrary invariants would be of significant interest.

In a noninteracting model the matrix \( \mathcal{D} \) has the form \( \mathcal{D} = \nu k \). From symmetry conditions it follows that the Green's function of the electrons has the form
\[ G = \left( \omega - \nu k f(\xi) + i\epsilon \frac{\text{sign} \omega}{2} \right)^{-1}, \]
(31)
where \( f(\xi, \omega) \) is an invariant function which, in analogy with the preceding discussion, we assume to be logarithmic (this is confirmed by the following calculation). Since factors of the order of unity are not important inside the logarithm, then one can assume that the invariant function \( f \) always depends only on \( k^2 \).

Let us evaluate the polarization operator \( \Pi \). It is given by
\[ \Pi(k, \omega) = -i\mathcal{P} \int \mathcal{G}(\xi, \omega) \mathcal{G}(k + \xi, \omega + \nu k) \frac{d^3k_1}{(2\pi)^3}. \]

Let us assume that the integral runs over the region \( k_1 \approx k \). In the first approximation one could neglect \( k \). However, if this is done, then the integral turns out to be equal to zero. In fact, if the trace is taken with
respective to the eigenfunctions of the operator $\hat{D}(k_1)$, then we obtain the integral
\[
\sum_{\nu} [\omega - \epsilon(k) + i\delta \text{ sign } \epsilon]^{-1} \text{ d}k,
\]
which is equal to zero, since the poles always lie on one side.

In view of this fact, it is necessary to use the next term of the expansion in powers of $k$. A nonvanishing contribution can appear only from terms of the second order in $k$. It has the form
\[
\int \left[ \sum_{\nu} [\omega - \epsilon(k) + i\delta \text{ sign } \epsilon]^{-1} \text{ d}k \right] \text{ d}(\omega),
\]
and taking the trace on both sides. In this connection we obtain
\[
\sum_{\nu} \int \left[ \hat{G}(k_1, \omega) \hat{G}(k_1, \omega + \omega) \hat{v}_k \right] \text{ d}(\omega) = \int \left[ \hat{G}(k_1, \omega) \hat{G}(k_1, \omega + \omega) \right] \text{ d}(\omega),
\]
where $\hat{v}_k = \partial D(k_1)/\partial k_1$. Having taken the trace with respect to the eigenfunctions of the operator $\hat{D}(k_1)$, we obtain
\[
\int \left[ \sum_{\nu} [\omega - \epsilon(k) + i\delta \text{ sign } \epsilon]^{-1} \text{ d}k \right] \text{ d}(\omega).
\]

From considerations of symmetry it follows that $\sum \sim \hat{v} \cdot k$. According to formula (31), $\hat{v} = \hat{v} \cdot k(\Gamma - 1)$. Having substituted this expression, we multiply by $\hat{v}_k$ and take the trace on both sides. In this connection we obtain
In each of the terms of the sum, the poles with respect to \( \omega_2 \) lie on the same side of the real axis. Therefore the corresponding integral is equal to zero. A finite contribution appears only in first order with respect to \( k \). This introduces an extra factor \( k/k^2 \) into the integrand. After this one can easily see that the integral will converge, and in the region \( k < m^2/\epsilon_0 \) it turns out that \( \gamma^{(1)} \approx 1 \). One can verify that this is valid for the corrections to \( \gamma \) of arbitrary order.

Here we encounter a new situation: at small momenta all of the terms of the perturbation-theoretic series become of the same order, i.e., weak coupling changes into strong coupling. Such a situation also arises in certain versions of field theory, and also in the problem of a phase transition of the second kind. This situation is determined by the self-energy parts. But the corresponding integrals only depend on \( k \) if \( k \) is sufficiently large, and they only depend on \( \omega \) if \( \omega \) dominates. This is guaranteed only in that case when in (39)

\[
G = c_k^{-2} q(\omega / sk^2),
\gamma = c_k^{y} q(\omega / sk^2),
\Gamma = c_k^{-2} d(\omega / sk^2),
\]

and we choose the powers \( \alpha, \beta, \delta, \nu \) and the constants \( c_1, c_2, c_3, c_4 \) in order to satisfy all the relations of the theory (here a symmetric \( \gamma \) is to be understood, i.e., having arguments of the same order).

First of all, on the basis of the previous discussion one can conclude that for small momenta \( 4\pi k^2 \gg \gamma \), \( \epsilon_0 \kappa^2 \) and \( \Sigma \gg G^\dagger \), i.e., the behavior of the \( G \)- and \( \Gamma \)-functions is determined by the self-energy parts. But the corresponding integrals only depend on \( k \) if \( k \) is sufficiently large, and they only depend on \( \omega \) if \( \omega \) dominates. This is guaranteed only in that case when in (39)

\[
g(0) = 1, \quad g(x \gg 1) \sim x^{-\alpha},
g(q) = 1, \quad q(x \gg 1) \sim x^{\beta},
g(d) = 1, \quad d(x \gg 1) \sim x^{-\delta},
\]

(40)

(41)

(42)

\( \nu, \alpha, \beta \) (the values at zero are normalized by the appropriate choice of the constant).

Let us consider the diagrams of higher orders for \( \gamma \). The transition to the next order means the appearance in the diagram of two additional \( G \)-lines, two \( \gamma \) vertices, one \( \Gamma \)-line, and integration over \( \d k \d w \). We require that this operation preserve the form of \( \gamma \). For this it is necessary that

\[
\int G \gamma^{\dagger} \Gamma \, d\d k \, d\d w, \quad \sim 1
\]

Substituting (22) and proceeding to the integration with respect to \( \varepsilon_1 = \omega / sk^2 \), from here we obtain

\[
c_1 c_2 c_3 c_4 k^{-2+2\gamma+\delta+\nu} \sim 1
\]

Hence it follows that

\[-2a + 3b = 0, \quad c_1 c_2 c_3 c_4 \sim 1\]

Here it is necessary to add a condition in order that the integral in the correction for \( \gamma \) should converge. Considering the first correction, but with complete lines and vertices, we obtain

\[-2a + 3b = 0, \quad \sim 1\]

This guarantees the convergence of the integral over \( k \). It is not difficult to see that here the integral over \( x \) will automatically converge.

Then let us consider the \( \Gamma \)-function. First of all we note that it satisfies the so-called Ward identity:

\[
\partial \gamma^\dagger / \partial \omega = \gamma.
\]

Substituting formula (39) here, we obtain

\[
\sim 1.
\]

from where it follows that

\[-2a + 3b = 0, \quad \sim 1\]

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Now we introduce a new function¹ \( \gamma_2 \) which we define as

\[
\gamma_2 = \partial \gamma^\dagger / \partial k^2.
\]

We assume that this function has a form analogous to (39), that is,

\[
\gamma_2 = c_k^{y} q(\omega / sk^2).
\]

Substituting (39) and (48) into relation (47), we obtain

\[
\sim 1, \quad \sim 1, \quad \sim 1.
\]

Now let us consider \( \Sigma(k, \omega) - \Sigma(0, 0) \). In the expansion in terms of \( k \) in the integral for the self-energy, the contribution of second order in \( k \) is essential. Taking into consideration that according to our condition all of the diagrams give a contribution of the same order of magnitude, it is sufficient to consider only the first diagram (Fig. 4). From it we obtain (\( \omega \ll k^2 \))

\[
\int G \gamma^{\dagger} \Gamma \, d\d k \, d\d w, \quad \sim 1
\]

If this expression is assumed to be of order \( \gamma^{-1} \), then conditions (41) and (42) are again obtained here together with (49) and (50). From this point of view the introduction of the quantity \( \gamma_2 \) does not give anything new. It is necessary, however, so that the integral \( \Sigma(k, \omega) - \Sigma(0, 0) \) converges. Hence follows the requirement

\[
3 + \mu + 2\beta = 0
\]

(51)

Let us go on to the polarization operator. In the previous Section it was shown that in the first approximation \( \Pi = 0 \) for \( k = 0 \). In the general case it is not difficult to see that \( \Pi(0, 0) = 0 \). This follows from the
fact that $\Pi(0,0)$ determines the change of the number of particles due to the action of the electrostatic potential, which does not depend on the coordinates. Mathematically this is written as:

$$\Pi(0,0) = -\int \text{Sp} G'(\omega_k, \omega_l) \frac{d\omega_k}{2\pi i} = \int \frac{dG}{\partial \omega_k} \frac{d\omega_k}{2\pi i} = 0.$$  

Expanding the integrand up to second order in $k$, we obtain for $\omega \ll sk^\nu$ (see Fig. 5)

$$\Pi \sim k^\nu \int d^dk \, d\omega \, G'(\omega_k)\chi'_{\nu}.$$  

Assuming $\Gamma \sim 1/\Pi$ and using (39) and (48), we again obtain the old relations for the powers and for the coefficients. However, the condition for convergence of the right hand side gives a new restriction:

$$3 + \mu + 2\beta + \nu - 3\alpha < 0. \quad (52)$$

Relations (41), (45), and (49) give the possibility to express all of the coefficients $\alpha$, $\beta$, $\delta$, $\mu$, and $\nu$ in terms of any two of them, for example:

$$\delta = 3 - \nu, \quad \beta = \alpha - \nu, \quad \mu = \alpha - 2. \quad (53)$$

From conditions (43), (51), and (52) with the aid of (53) we obtain the following restrictions on the possible values of the coefficients:

$$\nu > 1, \quad \alpha < 2, \quad \beta < 2, \quad \beta < 0, \quad \mu < 0. \quad (54)$$

The question may arise whether new restrictions cannot appear if we enclose several vertices of type $\gamma_2$ and $\gamma$ by Coulomb lines. If we have $n_1$ vertices of type $\gamma_2$ and $n_2$ vertices of type $\gamma$, then the condition for convergence is written in the form

$$n_1 + n_2 - a(n_1 + n_2 - 1) < 0.$$  

Substituting (45) and (49), we obtain

$$a < 2n_1 + n_2.$$  

This condition is ensured by the inequalities (54).

It is of interest that upon fulfillment of the restrictions (54), the behavior of $G$ and $\Gamma$ for small momenta and frequencies actually is determined by the self-energy parts. For the constants $c_1$, $c_2$, $c_3$, and $s$ we have, according to (42) and (46),

$$c_1 c_2 c_3 \sim 1, \quad c_1 c_2 c_3 \sim 1. \quad (55)$$

The orders of magnitude of these constants are obtained from the conditions for joining with the region of large momenta. According to the previous discussion, joining should occur for $k \sim me^3/\epsilon_0^2$. In the region of large momenta the electron spectrum is given by $\omega \sim k^7/m$.

Therefore, with regard to $\omega$, joining should occur for $\omega \sim me^3/\epsilon_0^2$. It is natural to expect that in the joining region $\omega \sim sk^\nu$. From here it follows that

$$s \sim (e^3/\epsilon_0)^{1/2} k^{-1-\nu}.$$  

In the joining region the function $\Gamma$ is of the order of $(e^3/\epsilon_0 k^2) \sim (\epsilon_0/m^2) e^3$. Comparing with expression (39) we have

$$c_1 \sim (e^3/\epsilon_0)^{1/2} k^{-1-\nu} \sim (e^3/\epsilon_0 k^2)^{1/2}.$$  

In the joining region the function $G$ is of the order of $1/\omega$ or $(me^3/\epsilon_0^2)^{1/2}$. Hence it follows that

$$c_1 \sim (e^3/\epsilon_0 k^2)^{1/2}.$$  

Finally, for $k \sim me^3/\epsilon_0$ $\gamma$ should be of the order of unity. Hence

$$c_1 \sim (me^3/\epsilon_0)^{1/2}.$$  

The values (56)–(59) satisfy the relations (55).

From the obtained results it follows that in the region $k \sim me^3/\epsilon_0$ the electron's Green's function does not have such a simple form as for large momenta. Its poles, if such exist in general, may be complex, where in general their imaginary part is of the order of the real part. Therefore, in this region it is impossible to talk about quasi-particles in the usual sense of the word.

However, the value of the Green's function makes it possible to determine various physical effects. Of course, it is necessary to bear in mind that the theory developed above does not give the exact value of the constant $\Gamma$, and also of the functions $g$, $q$, and $d$ in the region $x \sim 1$. However, even what is known makes it possible to make a number of predictions.

Let us consider the momentum distribution of the particles. It is expressed by the integral

$$n(k) = -\int \frac{G(\omega_k)}{s} \frac{d\omega}{2\pi i}.$$  

But as $x \rightarrow \mp \infty$ the quantity $G \sim x^{-1/2}$ and since $\alpha < \nu$ according to what was said previously, the integral diverges. Hence for small momenta $n(k)$ is actually determined by the value of $G$ in the region of large momenta, and we do not have the right to use formula (22) directly. One can, however, evaluate the difference

$$n(k) - n(0) = \int \frac{G(\omega_k)}{s} \frac{d\omega}{2\pi i}.$$  

In this integral the region of large values of $\omega$ is contracted, and the integral converges for $x \sim 1$. In this connection we obtain

$$n(k) - n(0) \sim c_1 sk^{1-\nu} \sim (me^3/\epsilon_0)^{1-\nu}. \quad (60)$$

According to Eq. (37) the power $\nu - \alpha$ is positive. In the region of large momenta we again obtain a constant.

Now let us consider the heat capacity at low temperatures. According to (8) we have

$$\Omega(T) - \Omega(0) = 2 \left( T \sum \frac{1}{2\pi i} \int \frac{d\omega}{2\pi i} \int \frac{dk}{2\pi i} \, e^{i\omega T} \ln G(\omega, k), \right.$$  

It is necessary to take $G(\omega)$ for $T = 0$. Replacing the summation over $\omega$ by an integration over real frequencies and differentiating with respect to $T$, we obtain the entropy

$$S(T) = \frac{1}{2} \int \frac{d\omega}{2\pi i} \int \frac{dk}{2\pi i} \, e^{\omega T} \ln G(\omega, k).$$  

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Therefore, with regard to $\omega$, joining should occur for $\omega \sim me^3/\epsilon_0^2$. It is natural to expect that in the joining
where $n_F$ denotes the Fermi function. It is impossible to evaluate this integral by the same method as in the presence of the Fermi surface. However, we obtain a result correct to within a constant coefficient if we assume that only small values of $\omega$ are essential. Taking into account the evenness of $\frac{\partial G^{-1}(\omega, k)}{\partial \omega}$ we obtain

$$S(T) \sim T \int \frac{d^3k}{(2\pi)^3} \left[ \frac{\partial}{\partial \omega} \ln G_\omega(\omega, k) \right]_{\omega = 0}.$$  

Regarding the imaginary part of $G$ to be of the order of the real part, we may remove the $\text{Im}$ sign and write the derivative inside the square brackets as

$$G(\omega, k) \frac{\partial G^{-1}(\omega, k)}{\partial \omega} = G_V.$$  

Substituting $G$ according to formula (22) we obtain

$$S \sim C \sim c_3 \sqrt{I} \int d^3k \frac{1}{k^2} \text{tr} \left( \frac{\partial}{\partial k^2} \right) \text{tr} \left( \frac{T}{k^2} \right).$$

Changing to the variable $x = T/k^2$, we verify that the integral over $x$ is convergent. Substituting the values of the coefficients $c_1$, $c_2$, and $s$, we finally obtain

$$C \sim \left( \frac{me^2}{\epsilon^2} \right)^{3/2} \left( \frac{T_\epsilon^2}{\epsilon^2} \right)^{1/2}.$$  

Taking the restriction (37) into account, we find the power of the temperature:

$$3 > 3/\nu > 0.$$  

(63)

we note that one can also obtain the temperature dependence of the heat capacity from the following simple argument. The heat capacity is proportional to $T_\rho(T)$, where $\rho$ is the density of the electronic states. The polarization operator $\Pi$ is proportional to $\rho$. Therefore

$$\rho(T) \sim k^2(T),$$

where $k(T)$ denotes the momenta characteristic for a given temperature. It is obvious that $k(T) \sim T^{1/\nu}$. Also keeping in mind that $\delta = 3 - \nu$, we obtain

$$C \sim T^{3/\nu}.$$  

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