POSITRON PRODUCTION IN A COULOMB FIELD WITH $Z > 137$

V. S. POPOV

Submitted April 14, 1970

The spontaneous production of positrons in a strong Coulomb field with $Z > Z_{cr}$ is considered, where $Z_{cr}$ is the “critical” value of the nuclear charge (at $Z = Z_{cr}$ the energy of the 1S$_{1/2}$ ground state passes below the boundary of the lower continuum $\epsilon = -mc^2$; the value $Z_{cr}$ is always $> 137$ and depends on the cut-off radius $R$ and, to a lesser degree, on the electric charge distribution over the nuclear volume). It is shown that at threshold [i.e., for $(Z - Z_{cr}) \ll Z_{cr}$] the probability for spontaneous production of positrons vanishes exponentially. For the entire analysis it is very important that the electron state at $\epsilon = -mc^2$ remains localized within a distance of order $\hbar/mc$ from the nucleus ($m$ is the electron mass); this is in sharp contrast to the usual nonrelativistic situation at $\epsilon \rightarrow +mc^2$. This property is characteristic for the relativistic Coulomb problem and can be understood already on the example of the Klein-Gordon equation. Spontaneous pair production in a short-range potential of square-well type is also considered.

1. INTRODUCTION

In a recent paper Gershtein and Zel’’dovich have posed themselves the interesting question of the spontaneous production of electron-positron pairs in the Coulomb field as the strength of the potential goes slowly (adiabatically) through a critical value. This situation arises when two bare nuclei with charge $Z > Z_{cr}/2$ approach each other up to a distance $r \lessapprox \hbar/mc = 3.86 \times 10^{-13}$ cm, and it also obtains in a single nucleus with charge $Z > Z_{cr}$. Here $Z_{cr}$ is the critical value of the nuclear charge for which the energy of the ground state of the electron drops below the boundary of the lower continuum $\epsilon = -mc^2$. The value of $Z_{cr}$ is mainly determined by the charge radius $R$, but depends also on the character of the charge distribution over the nuclear volume; in all cases $Z_{cr} > 137$.

It should be noted that the relativistic Coulomb problem for $Z > 137$ has a number of specific features. Thus, the Dirac equation with a potential corresponding to a point charge $Ze$ is not correct mathematically for $Z > 137$; here the “falling into the center” known from quantum mechanics occurs (for a state with angular momentum $j = \frac{3}{2}$). In order to obtain physically reasonable results for $Z > 137$ one is therefore forced to take account of the finite dimensions of the nucleus. This approach to the problem is due to Pomeranchuk and Smorodinskii, who gave a (qualitatively) correct description of the phenomenon at $Z > 137$. However, their approximation in deriving an equation for $Z_{cr}$ is too crude, so that the values for $Z_{cr}$ quoted in\textsuperscript{2} are too high.\textsuperscript{4} A detailed calculation of $Z_{cr}$ for different model charge distributions of the nucleus has been carried out in\textsuperscript{3}.

The study of the phenomena for $Z > 137$ is closely connected with the question of the polarization of the vacuum in a supercritical Coulomb field. In particular, the production of pairs for $Z > Z_{cr}$ gives rise to appearance of an imaginary part in the polarization operator. Although the parameter $\alpha = Ze^2/\hbar c = Z/137$ of perturbation theory exceeds unity in this case, the problem can be solved if the exact relativistic wave functions of the electron in a Coulomb field are used (the necessity of making such a calculation has been emphasized in\textsuperscript{1}).

Let us now give a synopsis of the present paper. In Sec. 2 we consider the limiting case of a very small cut-off radius for the Coulomb field $R$, where $\ln[\hbar/(mcR)] = \eta^{-1} \gg 1$. In this case the small parameter $\eta$ enters in the problem, so that the solution can be found in analytic form. In Sec. 3 these calculations are generalized to the realistic case of heavy nuclei, where the nuclear radius $R \ll \hbar/mc$, but $\eta$ is not yet very small compared to unity. The formulas (26), (38), and (38) for the probability of spontaneous pair production $w$ are obtained. The general conclusion of\textsuperscript{1} about the production of pairs for $Z > Z_{cr}$ is confirmed, but the details of this process are not the same as assumed in\textsuperscript{1}. In particular, the state of the electron at the boundary of the lower continuum $\epsilon = -mc^2$ remains localized, owing to the specifics of the relativistic Coulomb problem (cf.\textsuperscript{14}, and also Sec. 2). Therefore the probability $w$ has a characteristic threshold behavior (28) for $Z = Z_{cr}$ and the pair production process sets in slowly for $Z > Z_{cr}$.

2. THE LIMITING CASE $R \rightarrow 0$

When $Z > Z_{cr}$ a number of fundamental questions arise (the rearrangement of the vacuum because of pair production, the applicability of the single-particle approximation, etc.), which are answered most simply if one has a solution of the problem in analytical form. We therefore begin with the limiting case $R \rightarrow 0$ (more precisely, we assume that not only $R \ll 1$ but

\textsuperscript{1} Cf. also\textsuperscript{[1–3]}. As communicated to the author by Ya. A. Smorodinskii, the question of the pair production in the collision of two near-critical nuclei has been considered by I. Ya. Pomeranchuk already in 1945 (unpublished).

\textsuperscript{2} Thus the radius $R = 1.2 \times 10^{-12}$ cm corresponds to $\hbar/(mcR) = 32$ and $s = 0.3$; therefore the approximation $\eta \ll 1$ is very inaccurate. For example, the asymptotic formula (11) gives here $\eta_{cr}/2$ with an error of a factor of two.
In the theory:
\[ \eta = \left( \ln \frac{1}{R} \right)^{\gamma}. \tag{1} \]

For simplicity we begin with the case of a scalar particle. As long as \( \alpha < \frac{1}{\gamma} \), the Coulomb problem for spin \( s = 0 \) is meaningful also for a point charge. The energy of the lowest 1S level is
\[ \epsilon = \left( \frac{1}{2} + \sqrt{\frac{1}{4} - \alpha^2} \right)^2. \tag{2} \]
For \( \alpha > \frac{1}{\gamma} \) the quantum \( \gamma \) becomes complex, corresponding to the "falling into the center." Therefore the Coulomb potential must be cut off at small distances:
\[ \psi (r) = \begin{cases} -\alpha r / R & \text{for } r > R, \\ -\alpha / R \left( 1 - r / R \right) & \text{for } r < R. \end{cases} \tag{3} \]
The form of the cut-off function \( \psi (r / R) \) depends on the electric charge distribution over the nuclear volume. Solving the Klein-Gordon equation with the potential (2) we obtain for \( l = 0 \)
\[ \chi_0 (r) = W_{\alpha, l}(2x) \quad (r > R), \tag{4} \]
where \( x = \gamma / \alpha \) is the nuclear boundary and for \( r < R \) the Whittaker function for arbitrary angular momentum \( l \) the solution also has the form (4) with \( g = \sqrt{\alpha^2 - \left( l + 1 / 2 \right)^2} \). For \( 1 < \gamma > -1 \) this wave function corresponds to a bound state and for \( \gamma > 1 \) and \( \gamma < -1 \) it has an asymptotic form of the type of a divergent wave:
\[ \chi_0 (r) \propto \left\{ \begin{array}{ll} r^{\alpha r / \kappa} & \text{for } r < 1, \\ \exp \left( \Gamma (1 + \gamma / 2 - p) / 2 \right) r^{-\alpha r / \kappa} & \text{for } r > 1, \end{array} \right. \tag{5} \]
where \( \gamma = \sqrt{1 - \alpha^2} \) is almost independent of \( \alpha \). If the logarithmic derivative at the nuclear boundary \( \gamma = 1 \) \( \chi_0 (r) \) is determined by the quantity \( \alpha \) and the form of the cut-off function \( \psi (r / R) \). For example, for the simple cut-off \( \psi (r / R) \) \( \equiv 1 \) we have \( \gamma = \alpha \cot \alpha \). The equation for the energy of the level \( \epsilon \) takes the form
\[ \epsilon W_{\psi^1}(x) = \gamma W_{\psi^0}(x), \quad x = 2x. \tag{6} \]
Since \( x \ll 1 \), we use the asymptotic form
\[ W_{\psi^0}(x) \approx \sqrt{2 \pi} \Gamma (1 / 2 + i \beta - p) / \Gamma (1 / 2 - p) \left( e^{2 \pi i} - e^{-2 \pi i} \right) \tag{7} \]
with the help of which we can simplify Eq. (6) to
\[ \epsilon W_{\psi^0}(x) = 2 \pi / 4 \left( \Gamma (1 / 2 + i \beta - p) / \Gamma (1 / 2 - p) \right) \left( e^{2 \pi i} - e^{-2 \pi i} \right) \tag{8} \]
(below \( \nu = 2 \pi / 4 \sqrt{\alpha^2 - 1} \)), For \( R \to 0 \) the value of \( \alpha \) is close to \( \gamma / 2 \) and \( \nu \to 0 \); the value of \( \alpha \) is small for \( \nu \to 0 \) \( \left( \frac{\pi / 2}{\gamma / 2} \right) \)
\[ \psi (z) \text{ is the logarithmic derivative of } \Gamma \text{ function, Eq. (6) is transformed to the final form} \]
\[ \ln 2nR + \psi \left( \frac{\lambda - e}{2 \gamma} \right) - 2 \psi (1) - \psi (2) / 2 \gamma - 1 = \frac{\pi}{\gamma} \tag{10} \]
[the integer \( n \) appears when taking the logarithm of (8) since \( \exp (2\pi i n) = 1 \). Let us determine the "critical" value \( \alpha \) for which the bound level passes beyond the boundary of the continuous spectrum \( \epsilon < -1 \). Since \( \psi (z) = \ln z - \gamma z + \ldots \) for \( z \to \infty \), we find
\[ \ln R - \frac{2}{2 \gamma - 1} - 2 \psi (1) = \frac{\pi}{\gamma} \tag{10'}, \]
and (for the ground state)
\[ \psi = 2 \pi n + \ldots, \quad \alpha = \frac{1}{2} + n \eta \| + \ldots \tag{11} \]

The expansion parameter in (11) is \( \eta \) and it is for this reason that we had to impose the condition \( |\ln R| > 1 \). We note that the main term in the asymptotic expression (11) contains only the cut-off radius \( R \), whereas the specific form of the cut-off function \( \psi (r / R) \) affects only the subsequent terms (via the parameter \( \gamma \)). From (10) and (10') we find an equation for the energy of the level:
\[ \epsilon = \frac{1}{2} - \frac{\pi}{\gamma}, \tag{12} \]
where
\[ \epsilon = - \frac{1}{\alpha} - \frac{1}{2} \exp \left( \frac{\pi}{\gamma} \right) \tag{13} \]

The function \( \epsilon (\lambda) \) is real in the interval \( 2k^2 > \lambda > -1 \), which corresponds to the existence of a bound state for \( \gamma > \alpha > \alpha \). For \( \epsilon < -1 \) it acquires an imaginary part
\[ \ln 1 - \epsilon = \frac{1}{2} \left( 1 - \frac{\pi}{2} \right) \tag{13'} \]
which is exponentially small near \( \epsilon = -1 \). For \( \epsilon < -1 \) we have the asymptotic expansion
\[ \epsilon = \frac{1}{3 \alpha} (1 + e) + \frac{1}{2} \exp \left( \frac{\pi}{\gamma} \right) \tag{14} \]
where \( \delta (x) = 1 \) for \( x > 0 \), \( \delta (x) = 0 \) for \( x < 0 \); the value of \( \epsilon (\lambda) \) is taken on the lower rim of the cut:
\[ \ln \epsilon < 0, \quad \alpha = \pm k \tag{15} \]
From (12) and (14) we find the energy of the quasi-stationary level for \( \alpha > \alpha \):
\[ \epsilon < -1 \] corresponds to the nonrelativistic problem to states of the continuous spectrum \( E > 0 \), for which the penetrability of the Coulomb barrier (cf. Fig. 1) is equal to \( (k = \sqrt{2E} = \sqrt{\epsilon^2 - 1}) \):

\[
D \omega \exp \left\{ -2 \frac{n}{r} \left[ \ln r - 2|\epsilon|a - \alpha - \gamma \right] \right\} = \exp \left\{ -2\pi \left[ \frac{a|\epsilon|}{r} - \frac{1}{4\pi} \right] \right\}.
\]

(17)

For \( \epsilon = -1 \) the penetrability is \( D = 0 \); therefore the state with \( \epsilon = -1 \) is localized. This can also be seen from the explicit formula for \( \chi(r) \):

\[
\chi(r) = r^n K_n(r \gamma) \propto r^\epsilon \exp(-r\gamma),
\]

(18)

where \( K_n(z) \) is the MacDonald function. This state occurs for \( \alpha = \alpha_{cr} \). With further increase of \( \alpha \) it goes over into the quasistationary state (15) whose wave function goes over into the divergent wave (15') for \( r \to \infty \). From (15) we obtain the probability for pair production for \( \alpha = \alpha_{cr} \):

\[
\omega = -2i\pi \exp \left\{ -i\frac{\pi}{3} \left[ \frac{\epsilon}{\gamma} v_{cr} - \frac{y_{cr}}{v_{cr}} \right] \right\},
\]

(19)

For spin \( S = \frac{1}{2} \) the calculations are analogous. We only quote the final results. The equation for the energy has the form (22) as before, where now \( \nu = 2\sqrt{\epsilon^2 - 1} \), and the function \( f_{\nu}(\epsilon) \) becomes

\[
f_{\nu}(\epsilon) = \frac{1}{2} \ln \left[ \frac{\nu + \epsilon + \frac{1}{\nu} + \gamma}{\nu - \epsilon + \frac{1}{\nu} + \gamma} \right].
\]

(20)

For \( \epsilon < -1 \) the function \( f_{\nu}(\epsilon) \) acquires an imaginary part; however, this has a different sign from (13):

\[
\text{Im} f_{\nu}(\epsilon) = \frac{1}{4} \left( 1 - \text{cth} \left( \frac{\pi}{\nu} \right) \right). \tag{20'}
\]

This was to be expected, since we are not working in a second-quantized theory: as is known\(^2\) the vacuum polarization matrix elements with an odd number of closed loops must be taken with different signs for bosons and fermions. In the formalism of second quantization (or in the equivalent Feynman method of transition amplitudes) the required sign appears automatically: cf. in this connection\(^6\). For \( \alpha > \alpha_{cr} \) the energy of the quasistationary state is equal to

\[
\epsilon = -\left( 1 + \frac{x^2}{2} \right) - i\pi \text{cosec} \, x, \quad x = \frac{1}{2a} \left( \frac{\nu - \nu_{cr}}{\nu_{cr}} \right). \tag{21}
\]

From this we obtain for the probability of positron production in the Coulomb field for \( \alpha > \alpha_{cr} \),

\[
\omega_0 = \frac{6\pi}{5} \exp \left\{ -\frac{3\pi}{5} \left[ \frac{\nu}{\nu_{cr}} \right] \right\}, \quad v = 2y_{cr}^2 - 1. \tag{22}
\]

With regard to formulas (19) and (22) we make the following remarks.

1. The probabilities \( \omega_0 \) and \( \omega_{1/2} \) are here measured in the units \( mc^2/\hbar \).

2. The pair production is a threshold effect and occurs only for \( \alpha > \alpha_{cr} \).

3. For \( \alpha > \alpha_{cr} \) the probability \( \omega \) vanishes exponentially owing to the presence of a Coulomb barrier for positrons. The static field produces pairs only in that region where \( |V(r)| > 2mc^2 \), i.e., near the nucleus. In order that the positron escape to infinity

it must penetrate through the Coulomb barrier (Fig. 1), whose penetrability is exponentially small for \( \epsilon \to -1 \).

4. Formulas (19) and (22) have been obtained in the framework of the single-particle approximation. For their validity it is necessary, therefore, that the effect of pair production is small (otherwise an appreciable rearrangement of the vacuum would set in). This determines the region of applicability of the expressions obtained: it is necessary that the exponent be large compared to unity. In our case \( \nu_{cr} < 1 \), and already for \( \nu_{cr} \approx \nu_{cr}^2 \) the effect of pair production has the probability \( \sim 1 \), i.e., the dependence of \( \omega \) on \( \alpha \) is very sharp.

5. The analytic properties of the function \( f_{\nu}(\epsilon) \) are also of interest. Let, for example, \( s = \frac{1}{2} \). Going over to the variable \( t = y_{cr}^2 + 1 \) (here \( \epsilon = -1 \) corresponds to \( t = 0 \)), we have from (20)

\[
f_{\nu} = -\frac{1}{2\pi} \left\{ \ln t + \psi \left( \frac{1}{t} \right) + \frac{1}{2} \left[ t - t^2 - 1 + t + t^2 - \ln(t + t^2) \right] \right\}. \tag{23'}
\]

and

\[
f_{\nu}(t) = \sum_{n=0}^\infty \frac{B_{2n}}{(2\pi)^{2n}} \left( 1 - \frac{\Gamma(n - 1)}{\Gamma(n + 1)} \right) \tag{23''}
\]

(here \( B_2 \) is a Bernoulli number).

Since \( t^2 = (2z - z^2)/(1 - z)^2 \), where \( z = 1 + \epsilon \), and \( t = t(z) \) is a function which is analytic inside the circle \( |z| < 1 \), we can rewrite (23) in the form of a series in powers of \( z = 1 + \epsilon \):

\[
f_{\nu}(\epsilon) = \frac{5}{6\pi} \left[ z + \frac{33}{106} z^2 + \frac{346}{525} z^3 + \ldots \right]. \tag{23'''}
\]

As is known, for large \( n \)

\[
B_{n+2} \approx \left( -\frac{1}{2\pi} \right) \frac{(2n)!}{2^{2n-1} n!}. \tag{24}
\]

It follows from (23') that

\[
|\omega_{1/2}| \sim \frac{1}{\nu_{cr}} \left( \frac{\pi}{\nu_{cr}} \right)^{2n} \tag{25}
\]

(here \( \epsilon = 2.718 \ldots \) and therefore the series (23'') does not converge, except asymptotically. This is reflected by the fact that the point \( \epsilon = -1 \) is a singular point of the function \( f_{\nu}(\epsilon) \). We note, however, that this singularity is not connected with an infinity of the function itself or its derivative\(^5\) but with the appearance of the imaginary part (20') on the cut \( -\alpha < \epsilon \).

\(^5\) It is easy to see that \( \frac{d f_{\nu}(\epsilon)}{d \epsilon} = 0 \) and that all derivatives \( \frac{d^n f_{\nu}(\epsilon)}{d \epsilon^n} \) exist and are finite in the point \( \epsilon = -1 \).
The first term of the series \( V_1(r) \) corresponds to the graph of Fig. 2 a; this is the so-called Uehling potential. Usually only this term is taken into account in estimates of the role of vacuum polarization.\(^{[19,20]}\) Calculations show\(^{[6]}\) that numerically \( |V_\alpha(r)| \ll |V_1(r)| \), so that even for \( \alpha \approx 0.7 \) the corrections to the level shifts in \( \mu \) mesic nuclei due to \( V_\alpha(r) \) are negligibly small. For \( \alpha \to 1 \) one must, strictly speaking, sum over the entire infinite set of graphs (Fig. 2) since the point \( \alpha = 1 \) is a singularity of the series (26) in the case of a point charge. With account of the finite dimensions of the nucleus this singularity moves to the point \( \alpha = \alpha_{CR} > 1 \). What is the character of this singularity? In analogy to \( I_2(\epsilon) \) one should suppose that \( \alpha_{CR} \) is an essential singularity of the series (26) connected with the appearance of an imaginary part for \( \alpha > \alpha_{CR} \) (which corresponds physically to pair production). The series (26) itself remains convergent at \( \alpha = \alpha_{CR} \) and therefore the polarization potential \( V_\alpha(r) \) gives, even for \( \alpha = \alpha_{CR} \), in only a small correction of order \( \alpha_0 = \frac{\gamma}{137} \) to the "bare" Coulomb potential \( V_0(r) = -\frac{1}{r} \) [this would evidently not be the case if the singularity of the series (26) were, say, a simple pole]. Thus, in calculating the wave functions of the electron one can neglect the distortion of the Coulomb potential owing to the polarization of the vacuum.

7. Gershtein and Zel'dovich\(^{[1]}\) have proposed that a delocalization of the vacuum polarization occurs for \( \alpha \to \alpha_{CR} \), i.e., that the polarization charge extends over arbitrarily large distances from the nucleus. The principal argument in favor of this was that the wave function of the bound state \( \chi_0(\mathbf{r}) \approx e^{-\alpha r} \) and therefore the electron cloud would seem to become delocalized for \( \epsilon = -1 \). Our analysis shows that this is not so: as seen from (18), \( \chi_0(\mathbf{r}) \approx \exp(-\sqrt{8\alpha r}) \) for \( r \to \infty \) and \( \epsilon = -1 \). The reason for this striking difference in the behavior of the wave function of the electron for \( \epsilon = \pm 1 \) is the dependence of the effective potential \( V_\alpha \) on the sign of \( \epsilon \).

3. Threshold Behavior of the Probability \( w \)

After having studied the qualitative nature of the phenomenon for \( \alpha > \alpha_{CR} \) we now turn to a determination of the formulas which are valid in the region of realistic values of the nuclear radius \( R \approx 10^{-12} \) cm. We obtain the threshold behavior of the probability \( w \) for the production of positrons for \( \alpha \to \alpha_{CR} \) without assuming that \( \alpha_{CR} \to 1 \). Up to a factor we have

\[
 w = \exp\left(-\frac{2\alpha a_0 |\epsilon|}{k}\right), \quad k = \frac{\sqrt{2}}{\alpha_{CR}}.
\]

Here we can set \( \alpha = \alpha_{CR} \), \( |\epsilon| = 1 \), and the momentum of the outgoing positron \( k \) must be expressed through \( \delta \alpha = \alpha - \alpha_{CR} \). As the coupling constant is increased from \( \alpha_{CR} \) to \( \alpha = \alpha_{CR} + \delta \alpha \), the energy of the level is lowered by \( \delta \varepsilon \); \( \Re \varepsilon = -1 + \delta \varepsilon \). Setting \( \beta = -\left(\epsilon/\alpha\right)a_{CR} \) we have

\[
 w \sim \exp\left(-\sqrt{\frac{a}{a - a_{CR}}}ight), \quad a = 2 \sqrt{\frac{\alpha_{CR}^2}{\beta^2}}.
\]

The coefficient \( \beta \) determines the width of the threshold region in which the pair production is still a small effect. The value of \( \beta \) depends on \( R \) and can be calculated for different cut-off models. To this end we note that the level shift \( \delta \varepsilon \) can be found by perturbation theory (since the state of the electron for \( \epsilon = -1 \) remains localized). Together with (3) this gives

\[
 \beta = \frac{1}{R} \int \left(G^2 + F^2\right) \left(\frac{\varepsilon}{R} \right) dr + \int \left(G^2 + F^2\right) \frac{dr}{r}, \quad \nu = 2\sqrt{\alpha \alpha_{CR}} - 1
\]

where \( G = gG(r) \), \( F = r f(r) \), and \( g \) and \( f \) are radial functions for the upper and lower components of the Dirac bi-spinor normalized by

\[
 \int_G (G^2 + F^2) dr = \int (G^2 + F^2) dr = 1.
\]

In our case \( \epsilon = -1 \) and therefore for \( R > R \)

\[
 G = \kappa_{K} (\tilde{gS} \tilde{r}), \quad F = \frac{1}{\alpha} (G^2 - G),
\]

where \( \nu = 2\sqrt{\alpha \alpha_{CR}} - 1 \) and \( G(r) \) satisfies the equation

\[
 G'' - \frac{1}{r} G' + \left(\nu^2 - 1\right) \frac{2\alpha}{r} G = 0 \quad (r > R).
\]

For the calculation of the normalization constant \( \kappa \) one can use (30) up to \( r \to 0 \) since all integrals converge and the region \( r < R \) makes a small contribution to \( R \ll 1 \) to the normalization integral. As a result

\[
 c = \left[\frac{12a_{CR}^2}{3 + 2a_{CR}^2} \pi\right]^{1/2}.
\]

Expressing \( F \) in (29) through \( G \) according to (30) we have
Integrating the first term in (31) by parts and using (30'), we find after some calculation

\[ \int \frac{G'(r)dr}{r} = \frac{1}{a} \int \left( G'' - \frac{2G'G + G'}{r} \right) dr. \]  

(31)

In the internal region \( r < R \) we go over to the variable \( x = r/R \) and take into account that \( G \) and \( F \) satisfy the equations (for \( R \ll 1 \))

\[ G' = \frac{1}{x} G + a(x) F, \quad F' = -a(x) G - \frac{1}{x} F, \]  

(32)

then, for \( 0 < x < 1 \),

\[ G' - \frac{F'}{x} = \left[ a(x) + \frac{1}{x} \right] G = 0 \]  

(32')

[the dash in (32) and (33) denotes the derivative with respect to \( x \)]. Therefore

\[ \frac{1}{R^2} \int \frac{F(r)dr}{r} = \frac{1}{a} \int \frac{dz}{z} \left( G'' - \frac{2}{x} G' + \frac{1}{x^2} G \right). \]  

(33)

With the help of (32') we easily see that

\[ \frac{1}{r} \int \left[ G' - \frac{1}{x} G \right] dr = \frac{1}{r} \int \left( G'' - \frac{2}{x} G' + \frac{1}{x^2} G \right) dr = a'G. \]  

(33')

Since \( f(1) = 1 \) we find

\[ \frac{1}{R^2} \int \frac{G''(r)dr}{r} = \frac{1}{a} \int \frac{dz}{z} \left( G'' - \frac{2}{x} G' + \frac{1}{x^2} G \right). \]  

(33'')

Combining (31') and (33') we have

\[ \frac{\beta}{2} = \frac{1}{R^2} \int \frac{G''(r)dr}{r} + \frac{1}{a} \int \frac{dz}{z} \left( G'' - \frac{2}{x} G' + \frac{1}{x^2} G \right) dr. \]  

(34)

Thus the function \( F \) is completely eliminated from the expression for the coefficient \( \beta \). This has been possible because the Schrödinger equation simplifies significantly for \( \epsilon = -1 \). Using now expression (30) for \( G(x) \), we have finally

\[ a_\alpha = \frac{\alpha (3 + 2\alpha')}{24\alpha' \Lambda(\alpha)}, \]  

(35)

where everywhere \( \alpha = \alpha_\text{cr} \) and

\[ \Lambda(\alpha) = \frac{\hbar \alpha' n v}{2\pi} \left( 1 + \rho R' \right) - \frac{1}{16 a^2}. \]  

(36)

Here

\[ v = 2\alpha', \quad z = \gamma \alpha aR, \]  

\[ \rho = \frac{1}{2R} \int \frac{G(r)}{G(R)} \left( \frac{r}{R} \right)^2 \frac{dr}{r} = \rho_\alpha. \]  

(36')

Thus, for a rectangular cut-off \( f(\alpha) = 1 \)

\[ \rho = \frac{1}{4 \sin^2 \alpha} \left( 1 - \sin 2\alpha \right). \]  

(37)

For spin \( s = 0 \) the calculation is analogous to the one just presented (some details are given in Appendix B), and the final formula for the coefficient \( a_\alpha \) has the form

\[ a_\alpha = \frac{\alpha a}{12 \Lambda(\alpha)}, \]  

(38)

where \( \Lambda(\alpha) \) agrees formally with (36), differing only in the value of the parameters \( v = \sqrt{4\alpha^2 - 1} \) and \( \rho \):

\[ \rho = \rho_\alpha = \frac{1}{2R} \int \frac{G(r)}{G(R)} \left( \frac{r}{R} \right)^2 \frac{dr}{r} \]  

(38')

[we note that for \( f(x) = 1 \) the value of \( \rho_\alpha \) agrees with (37); in general \( \rho_\alpha = \rho v/2 \).]

For \( R \ll 1 \) that integral term dominates in (36) which represents the contribution from the outer region \( r > R \) in (29). We note that

\[ \int K_r(\nu) \frac{dx}{x} = \frac{\pi}{4 \nu^2} e^{\nu} \left( 1 + \frac{\nu}{2} \right) + \ldots \]  

(39)

as \( z \to \infty \),

\[ \int K_r(\nu) \frac{dx}{x} = \frac{\pi}{2 \nu \sin \frac{1}{2}} \ln \frac{1}{z} \]  

(39')

For \( \alpha_\text{cr} = 1.25 \left( Z_\text{cr} = 170 \right) \) the ratio of the term \( \rho K_r(\nu) \) over the integral term in formula (39) amounts to about \( 1/\nu \). Therefore the coefficient \( a_\alpha \) in (28) depends weakly on the specific form of the cut-off function.

In the limit \( R \to 0 \) we find, using the asymptotic form (39') and \( \nu_\text{cr} = 2 \pi \ln (1/R)^{1/3} \), that (35) and (38) agree with (19) and (22) of the previous section. In the region \( R \sim 10^{-12} \text{ cm} \) the asymptotic formulas (19) and (22) are inaccurate and a numerical calculation of the function \( \Lambda(\alpha_\text{cr}) \) is required. The results of such a calculation are shown in Fig. 3 for the simplest case of a rectangular cut-off \( f(\alpha) = 1 \).

If we extrapolate the dependence \( R = r_0 A^{1/3} \) to the region \( Z > 137 \) assuming (as for heavy nuclei) \( r_0 = 1.1 \text{ F} \) and \( A = 2.5 \text{ Z} \), then \( Z_\text{cr} = 170 \), where this value of \( Z_\text{cr} \) is weakly sensitive to the specific form of the nuclear charge distribution.\(^{[1]}\) From Fig. 3 we find: \( \alpha_\text{cr} = 3.75 \), so that for \( Z = Z_\text{cr} + 1 \) the probability for the production of positrons \( w \sim 10^{-10} \text{ mc}^{-1} \) is increased by three orders of magnitude as \( Z \to Z_\text{cr} \). One can see from the fact that the quantity \( w \) increases by three orders of magnitude as \( Z \to Z_\text{cr} + 2 \).

4. Pair Production in a Short-Range Potential

The case of a short-range potential differs significantly from the case just considered, since the Coulomb "tail" for \( r \to \infty \) is not present in \( U(r) \). For simplicity we take the potential in the form of a square well, assuming a vector type interaction:

\[ V(r) = -\frac{1}{\rho_\text{cr}} (r - r) \]  

(40)

(the depth \( \rho \) of the well is measured in the units \( mc^2 \)). The motion of the level's in such a potential has been considered in\(^{[3]} \). For spin \( s = 0 \) and \( l = 0 \) the equation for the level energy \( \epsilon \) coincides in form with the nonrelativistic equation;\(^{[3]} \)

\[ k \tan k r = -\lambda, \]  

(41)

differing only in the values of the parameters \( k \) and \( \lambda \).

\[ \text{FIG. 3. Dependence of the coefficient } a_\alpha \text{ in (28) on } \alpha_\text{cr}. \text{ Curves 1 and 2 refer to the cases with spin } s = 0 \text{ and } s = 1/2. \]
\[ k' = (\varepsilon + v)^2 - 1, \quad \lambda = \gamma 1 - e^2. \]

As the level appears, \( \lambda = 6 \). Denoting the corresponding depth of the well by \( w \) (for \( \varepsilon = \pm 1 \)) we have
\[ \nu_n^2 = \left( 1 + \frac{n^2}{r_0^2} \right), \quad n = 1, 2, \ldots. \]

The character of the dependence \( \varepsilon = \varepsilon(v) \) is most easily studied in the limiting case of a narrow well \( r_0 \ll 1 \). Then we obtain from (41) for \( n = 1 \) (ground level)
\[ v_1 = \frac{v}{2} + \frac{\gamma v^2 - e}{n}. \]

From Fig. 4 we note the following somewhat unexpected result: the curve \( \varepsilon = \varepsilon(v) \) folds back.\(^8\) This can be interpreted in the following way: for \( v = v_1 \) a bound state for antiparticles appears in the well (lower branch of the curve). With further increase of \( v \) the levels for particles and antiparticles approach each other, touch for \( v = v_{cr} \), and go off into the complex \( \varepsilon \) plane. The imaginary part of \( \varepsilon \) for \( v > v_{cr} \) describes pair creation, where at threshold
\[ v_{cr} \sim \gamma v^2. \]

It can be shown that in a narrow well of arbitrary form the function \( \varepsilon = \varepsilon(v) \) for a scalar particle has qualitatively the same form as in Fig. 4 (cf. Appendix C). Thus, the results obtained do not depend on the sharp cut-off of \( V(r) \) at \( r = r_0 \).

Let us now turn to the case of spin \( s = \frac{1}{2} \). The matching condition at \( r = r_0 \) for the Dirac equation is the continuity of the ratio \( F/G \), which gives (for \( k < 1 \), i.e., for \( j = \frac{1}{2} \))
\[ k \cos k r_o = -\left[ \lambda + v \frac{1 + \lambda r_o}{(1 + e^{-2})} \right], \]

where \( k \) and \( \lambda \) have the same values as above. The values of \( \gamma = \gamma^{(n)}_n \) for which a level with \( \varepsilon = 1 \) appears in the well are found from the equation
\[ k \cos k r_o = -\frac{v}{2} \quad (k = \gamma \nu (v + 2)), \]

and \( v_{cr} \) corresponding to the energy \( \varepsilon = 1 \) is obtained from (45) in explicit form:
\[ v_{cr}^2 = 1 + \frac{\gamma (1 + e^{-2})}{n} \quad (n = 1, 2, \ldots). \]

For a narrow well \( r_0 \ll 1 \) we have approximately from (45)
\[ v = n / r_0 = (1 + 2e) \quad (n = 1), \]

i.e., the curve \( \varepsilon = \varepsilon(v) \) falls off monotonically without a turning point. This difference between the spins \( s = 0 \) and \( s = \frac{1}{2} \) can be interpreted in the following way (cf. also Appendix C). The wave function for a scalar particle becomes, as in the nonrelativistic case, non-normalizable for \( \lambda = 0 \): \( \psi_0(r) \to c \) for \( r \to \infty \) (\( c \neq 0 \)).

Calculating the level energy for a slightly deeper well, we find in first order of perturbation theory
\[ \delta E = \int \frac{dV(r)}{r} \psi_0^*(r) dr / \int \frac{dV(r)}{r} dr = 6, \]

since the integral in the numerator converges [owing to \( dV(r) \)], while the denominator diverges. Hence \( \delta E \sim (\Delta V)^2 \), so that curve 1 of Fig. 4 has a horizontal tangent at \( \varepsilon = \pm 1 \). Furthermore, it follows from

\[ (16) \text{ and } (40) \text{ that the pairs of points } (\varepsilon, v) \text{ and } (-\varepsilon, v + 2e) \text{ correspond to the same effective potential } U(r) \text{ and to the same wave function. If the values } (\varepsilon, v) \text{ satisfy } (41), \text{ then } (\varepsilon, v + 2e) \text{ do as well; this leads immediately to the form of Fig. 4 for the qualitative behavior of curve } \varepsilon = \varepsilon(v) \text{ for spin } s = 0. \]

On the other hand, for spin \( s = \frac{1}{2} \) the state with \( \varepsilon = -1 \) is normalizable in an arbitrary short-range potential.\(^\star\) Indeed, for \( \varepsilon = -1 \) and \( V(r) = 0 \)
\[ F = \frac{1}{1 + e} \left( G - \frac{1}{r} \right) G. \]

For \( \varepsilon = -1 \) we have \( G(r) \to c \) for \( r \to \infty \), and \( F(r) \) would tend to infinity if \( c = 0 \). Therefore
\[ |G(r)| < C r^{-1}, \quad F(r) = C r^{-1} \quad (r \to \infty). \]

which guarantees the convergence of the normalization integral (29').

The normalizability of the states with \( \varepsilon = -1 \) in a short-range potential can also be explained in another way. Outside the range of the potential (\( r \gg r_o \))
\[ G = C k_s(\lambda r), \quad F = -C \int \frac{1 - e}{1 + e} k_s(\lambda r) \]

where \( I \) and \( l' \) are the angular orbital momenta for the upper and lower components of the bispinoor, and
\[ k_s(z) = \int_{\pi}^{\pi} k_s(\lambda r) = e^{-r} \sum_{l=0}^{l_0} \frac{(l + 1)!}{(l - n)! n! (2z)^l} \]

Since the wave function remains finite for \( \varepsilon = -1 \), it follows from this that the constant \( C \) in (52) must tend to zero in a definite way for \( \varepsilon \to -1 \):
\[ C < k^s_0, \quad L = \text{max}(l, l' + 1). \]

Therefore we have for the states with \( \varepsilon < 0 \) (\( j = \frac{1}{2}, l' = l + 1 \))
\[ G(r) = 0, \quad F(r) = cr^{-l} \quad (r \gg r_o), \]

i.e., for \( \varepsilon = -1 \) only the lower component of the bispinoor corresponding to the largest orbital angular momentum \( l' \) survives in the asymptotic expression for \( r \to \infty \). Since \( l' \geq 1 \) the normalization integral (29') remains convergent for \( \varepsilon = -1 \).

The situation is somewhat different for the states with \( \varepsilon > 0 \) (\( j = \frac{1}{2}, l' = l - 1 \)), for which we obtain instead of (54) \( (\varepsilon = 1), \quad r \gg r_o \)
\[ G(r) = c r^{-l}, \quad F(r) = -2 c r^{-l}. \]

Here both angular momenta \( l \) and \( l' \) are present in the asymptotic expression; therefore the state of the type \( F/r^l \) is not normalized for \( \varepsilon = -1 \). The other states \( (j > l' \) remain normalized.

\[ \text{Coulomb Field with } Z > 137 \]

\[ FIG. 4. \text{ Energy of the ground state } \varepsilon \text{ as a function of the well depth for spin } s = 0 \text{ (curve 1) and } s = 1/2 \text{ (curve 2) for a narrow well. The abscissa is marked by the quantity } (v - \pi/2r_0) \text{ for } s = 0 \text{ and by } (v - \pi/2r_0) \text{ for } s = 1/2. \]

\[ \text{In the nonrelativistic case } \chi(\lambda r) \to \lambda^2 r^2 \text{ as } r \to \infty \text{ as soon as a level with orbital angular momentum } l \text{ appears, i.e., } \chi(\lambda r) \text{ is normalized for } l > 1. \text{ Correspondingly, for the Dirac equation the states with } j > l' \text{ are normalized for } \varepsilon = -1 \text{ as well as for } \varepsilon = +1. \]
For the lowest level $1S_{1/2}$ we have $\kappa = -1$ and (54) agrees with (51). Owing to the normalizability of the wave function for this state the level shift $\Delta \varepsilon$ differs from zero already in first order perturbation theory \cite{[49]}, i.e., $\Delta \varepsilon \sim 6 \varepsilon$ and $\varepsilon = \sqrt{\varepsilon^2 - 1} = \sqrt{2} \Delta V$. The behavior of $u_{\text{eff}}(r)$ for $r \gg r_0$ is determined by the centrifugal barrier, whose penetrability increases with the momentum $k$ according to a power law:

$$D \propto k^{2-\kappa}$$

for $k \to 0$, and therefore $\omega \sim (V - V_{\text{cr}})k^{2-\kappa}/2$ (these formulas refer to states with $\kappa < 0$).

In the concrete case of a square well the behavior of the probabilities for pair production $\omega$ can easily be found directly from (45). According to what has been said, the critical value of the potential is obtained from (47) for $n = 1$:

$$\varepsilon_{\text{cr}} = 1 + \sqrt{1 + (\pi/r_0)^2}.$$  \hspace{1cm} (56)

For $\omega > \varepsilon_{\text{cr}}$ the root of Eq. (45) becomes complex:

$$\omega = \varepsilon - i \sqrt{\varepsilon^2 - \varepsilon_{\text{cr}}^2}.$$  \hspace{1cm} (57)

where $(q = r_0/z)$

$$A = \frac{q}{2} \left( \frac{1}{1 + q} \right)^{1/2} \left( 1 + \frac{q}{2} \right) \left( \frac{1}{1 + q} \right)^{1/2} = \frac{\rho_{n=2}^2}{\pi/\rho_0^2} \text{ for } r < 1,$$  \hspace{1cm} (58)

It is seen from this that the probability $\omega$ does not vanish at threshold according to an exponential law in the case of short-range potential.

5. Discussion of Results

1. As the charge $Z$ goes through the critical value $Z_{\text{cr}}(R)$, spontaneous creation of positrons in the nuclear Coulomb field sets in. This process has the following characteristics. In the Coulomb field of a nucleus $Z$ there is an unoccupied electron level for an electron lying in the lower continuum $\varepsilon = -m^2 c^2$. An electron of the Dirac sea makes a transition to this level, remaining localized near the nucleus for times of the order of the nuclear volume. The values of $\alpha_{\text{cr}}$ for the states $2P_{1/2}$ and $2S_{1/2}$ are calculated on the basis of model I. 3. The characteristic feature of the problem under consideration is the existence of a Coulomb barrier for the electron whose energy is close to the boundary of the lower continuum. The wave function $\chi_{\varepsilon}(r)$ for energy $\varepsilon$ has an asymptotic form of the type (5) for $r \to \infty$. The Coulomb interaction at large distances determines the coefficient $r a^{2s-1}/\varepsilon$ of the exponential in (5), which depends on the sign of $\varepsilon$. For $\omega > 1$ the maximum of $\chi_{\varepsilon}(r)$ goes off to large distances and the system becomes more and more dilute—delocalization sets in. This behavior of the states at the boundary of the continuous spectrum is well known from nonrelativistic quantum mechanics.\footnote{For example, for a hydrogen atom in a state with principal quantum number $n$ we have $X_{n\ell}(r) \sim e^{\xi r}/r^n$ for $r \to \infty$.}

For $\omega = -1$ a completely different picture obtains: the coefficient $r a^{2s-1}/\varepsilon$ decreases more rapidly than any finite power of $r$ and the electron remains localized near the nucleus. Accordingly, the behavior of the wave functions for $\omega = \pm 1$ is different:

$$\chi_{\varepsilon}(r) \sim \begin{cases} \sin(\sqrt{8a\rho} + b) & \text{for } \omega = +1 \\ \exp(-\sqrt{8a\rho}) & \text{for } \omega = -1 \end{cases}$$  \hspace{1cm} (59)

[the exponential decrease of $\chi_{\varepsilon}(r)$ in the second case is nothing but the damping due to the Coulomb barrier].

By continuity, it is clear that the functions $\chi_{\varepsilon}(r)$ and $\chi_{-\varepsilon}(r)$ must differ strongly also in the continuous spectrum ($|\varepsilon| > 1$), especially for $|\varepsilon| \to 1$. In the quasiclassical approximation

$$\chi_{\varepsilon}(r) \propto \exp(\sqrt{2a} r), \quad p(r) = \sqrt{\varepsilon^2 + 2ae^{-\sqrt{2a} r}}.$$  \hspace{1cm} (60)

For $\varepsilon > 1$ the quantity $p^2(r) > 0$ and $\chi_{\varepsilon}(r)$ oscillates for all. If $\varepsilon < -1$, we have a turning point $r_2 = 2a |\varepsilon|/k^2$, which is located far from the nucleus for small $k = \sqrt{\varepsilon^2 - 1}$. In the classically forbidden region $r < r_2$ the wave function $\chi_{\varepsilon}(r)$ is close to (18) and the oscillations described by the asymptotic form (5') set in only for $r > r_2$ (these oscillations describe the positron wave going out to infinity). The probability for pair production $\omega \sim \exp(\sim 2 \text{Im } S)$, where $S$ is the classical action calculated along the trajectory beneath the barrier. The action $S$ is independent of the spin, so that the probabilities $\omega$ behave similarly for $s = 0$ and $s = \frac{1}{2}$. For $\varepsilon = -1$ the exit point $r_2$ moves out to infinity and the probability $\omega$ vanishes exponentially at threshold. Therefore the rearrangement

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$R_1/F_1$ & $R_2/F_2$ & $R_3/F_3$ \\
\hline
1 & 2 & 3 \\
\hline
8 & 1.248 & 1.204 & 1.35 & 1.72 \\
10 & 1.271 & 1.243 & 1.38 & 1.78 \\
12 & 1.291 & 1.260 & 1.41 & 1.83 \\
\hline
\end{tabular}
\end{table}
of the vacuum for \( Z > Z_{\text{cr}} \) sets in continuously, not
suddenly.

4. After the emission of the two positrons an atom
with charge \( Z - 2 \) and a filled K shell remains. When
an external positron is scattered on this atom, a
narrow resonance will be observed;\(^ {13} \) the positron
penetrates through the Coulomb barrier and annihilates
with a K electron, as a result of which a nucleus with
an unoccupied 1S\(_2\) level in the lower continuum re­
 mains; then a pair is created spontaneously. This pro­
cess is analogous to the excitation of an atom by a pho­
ton with subsequent spontaneous emission of a photon of
the same frequency (within the limits of the line
width). The \( \epsilon \) section for the scattering of the
positrons must be described by a Breit-Wigner formula
with a width equal to the probability for pair produc­
tion \( \epsilon \).

5. For potentials with a finite range there also
exists a critical value of the well depth \( V = V_{\text{cr}} \); when
it is passed, spontaneous pair production sets in. How­
ever, in this case there is no Coulomb barrier for the
positron, so that the probability \( \epsilon \) increases at
the threshold \( V = V_{\text{cr}} \) much more rapidly than in the pre­
ceding case. Moreover, there is an appreciable spin
dependence in the threshold behavior of \( \epsilon \) as well as in
some qualitative features of the dependence
\( \epsilon = \epsilon (V) \). This is explained by the fact that here all
processes take place at small distances (assuming
\( r_0 < \hbar /mc \), where \( r_0 \) is the radius of the potential
well). Notwithstanding the absence of a Coulomb bar­
rrier, the states of the discrete spectrum remain
normalized for \( \epsilon = -1 \) (owing to the centrifugal bar­
rrier).

6. The problem considered is, at first glance, a
problem of strong coupling, since the perturbation
parameter \( \alpha = 2e^2 /hc \) > 1. Why is it possible to find
an exact (in the known sense) solution for it? The ex­
planation for this is that the interaction of the electron
with a strong external field can be taken into account
completely if the exact Coulomb wave functions are
used, and the interaction of the electrons (real or
virtual) between themselves can be neglected, since
the corresponding coupling constant is \( \alpha_0 = e^2 /hc = 1/137 \).

7. At present it is not known if nuclei with \( Z > 137 \)
exist in nature, so that the effects considered by us
must so far be brought into connection with thought
experiments.\(^ {13} \) However, the required situation can also
be realized in another (apparently more realistic)
way:\(^ {11} \) in collisions of two bare nuclei with charges
\( Z_1 \) and \( Z_2 \) such that \( Z_1, Z_2 < Z_{\text{cr}} (R) \), but \( Z_1 + Z_2 > Z_{\text{cr}} (R) \). At the instant where these nuclei approach
each other down to the distance \( R \), the electric field
for the electron is similar to the field of a nucleus with
charge \( Z_1 + Z_2 \), and spontaneous pair production
should also set in this case. For \( Z_1, Z_2 < Z_{\text{cr}} \) each
of the nuclei can be regarded as point-like, so that this
problem reduces to the relativistic two-center prob­
lem. Unfortunately, the variables of this problem are
no longer separable in elliptic coordinates \( \xi = (r_1 + r_2)/R, \eta = (r_1 - r_2)/R, \phi \) (in contrast to the nonrelativistic case). This is due to the term \( V^2 \) in
the effective potential (16). The non-separability of
the variables leads to additional difficulties of nume­
rical nature, but new qualitative features of the two­
center problem do not, to all appearances, arise.

The author expresses his deep gratitude to S. S.
Gershtein, Ya. B. Zel'dovich, and Ya. A. Smorodinski for
interest in this work, a discussion of the results and a
number of useful remarks. I should also like to thank B. L. Ioffe, I. Yu. Kobzarev, E. M. Lifshitz,
A. B. Migdal, L. B. Okun', A. M. Perelomov, L. P.
Pitaevskii, V. I. Ritus, V. V. Sudakov, and M. V.
Teren'tev for discussions of this work and T.
Rozhdestvenskii for help with the numerical calcula­tions.

\textbf{APPENDIX A}

In this Appendix we show how the limit \( \epsilon \rightarrow -1 \) in
the wave function (4) is taken, when the argument of
the Whittaker function \( x = 2\epsilon \rightarrow 0 \), its index
\( p = \alpha \lambda - t \rightarrow -\infty \), but their product remains finite. We start from the integral representation
\begin{equation}
W_{\alpha}(x) = \frac{e^{-\epsilon^2 x^2/4}}{\Gamma (k + \mu + 1/2)} \int e^{\epsilon^2 x^2 + -\epsilon x} (1 + t)^{-k/2} dt. \tag{A.1}
\end{equation}

The integral in this formula is written as
\begin{equation}
I = \frac{e^{-\epsilon^2 t^2/4}}{(1 + t)^{-k/2}} dt. \tag{A.2}
\end{equation}
when \( \psi (t) = xt + \kappa \ln (1 + t^{-1}) \). For \( x \rightarrow 0 \) and
\( k = -p \rightarrow \infty \) the function \( \psi (t) \) has a minimum for very
large \( t \) of order \( \sqrt{k/\kappa} \), so that 1 is conveniently writ­
ten in the form
\begin{equation}
I = \int e^{-(\eta+1)t^2/2} \psi (t) dt. \tag{A.3}
\end{equation}
\begin{equation}
\psi (t) = (1 + t^{-1})^{-\kappa/2} \exp (-k[\alpha (1 + t^{-1}) - \kappa])
\end{equation}
\begin{equation}
= 1 + \frac{\mu - \kappa}{t} + \frac{k}{2t^2} + \ldots \quad (\text{as } t \rightarrow \infty). \tag{A.4}
\end{equation}
Let us now use
\begin{equation}
\int e^{-(\eta+1)t^2/2} dt = 2 \left( \frac{\kappa}{\sqrt{2}} \right) K_n (2 \sqrt{k/\kappa}). \tag{A.5}
\end{equation}
where \( K_n (z) \) is the MacDonald function. Writing
\( z = 2 \sqrt{kx} = \sqrt{kx} r \), we find
\begin{equation}
W_{\alpha}(x) = Ce^{t-k/2} \left\{ K_\alpha (z) + \frac{1}{2} K_{\alpha-1} (z) + \frac{2\mu - 1}{z} K_{\alpha-1} (z) \right\} x + \ldots \}, \tag{A.6}
\end{equation}
where \( C = k^{\mu/2} / \Gamma (\kappa + \mu + 1/2) (k \rightarrow \infty) (k!)^{1/2} \). The co­
efficient of the first power of \( x \) in the expansion (A.6) is equal to
\begin{equation}
\left\{ K_{\alpha-1} (z) - K_{\alpha} (z) \right\} \frac{2\mu - 1}{z} K_{\alpha-1} (z) = 0 \tag{A.7}
\end{equation}
owing to the recurrence relations between the \( K \) func­
tions. As a result
\begin{equation}
\lim_{x \rightarrow 0, \alpha \rightarrow \infty} W_{\alpha}(x) = Cz K_{\alpha} (1 + O (x^2)). \tag{A.8}
\end{equation}
where \( \nu = 2g = \sqrt{4a^2 - 1} \), \( z = \sqrt{8a} \cdot R \). The limit for spin \( s \neq \frac{1}{2} \) is taken in analogous fashion. The function \( K_\nu(z) \) entering in (A.9) is much simpler than the Whittaker function; it is defined by the integral

\[
K_\nu(z) = \int e^{-zr} \cos rt \, dt,
\]

where the numerator of (B.2) contains the integral \((r-2)\), and \( a \neq 0 \) for \( r \neq 0 \). Then, for \( a = 0 \), we have

\[
\beta = -\frac{\partial}{\partial \nu} \left( \frac{1}{\nu} \right) \frac{4}{R^2} \int \chi'(r) f(R) \, dr + \int \chi'(r) \frac{dr}{r^2} - \frac{3}{a^2}.
\]

Using (B.3) and (B.4), we have

\[
\beta = 2a \left( \frac{\partial}{\partial \nu} \right) \int K_\nu(z) \frac{dz}{z} + \rho K_\nu(z) - \frac{1}{16a^2}.
\]

where \( z = \sqrt{8a} \cdot R \), and \( \rho \) is given by (38'). Substituting this in (28), we arrive at the desired expression (38) for the coefficient \( a_0 \).

**APPENDIX C**

The dependence of the energy of the ground state on \( v \), \( \epsilon = \epsilon(v) \), has been obtained for the special case of a rectangular well (cf. Sec. 4). Let us now show that also for an arbitrary short-range potential the curve \( \epsilon = \epsilon(v) \) has qualitatively the same form. Consider a narrow well of arbitrary form:

\[
V(r) = -v(R/r), \quad R \ll 1,
\]

where \( f(x) \) tends rapidly to zero for \( x \gg 1 \). For spin \( s = 0 \), introducing the variable \( x = r/R \) in the Klein-Gordon equation, and expanding all quantities in \( r \), we have

\[
\chi = \chi_0 + R \chi_1 + \ldots, \quad v = vR = a_0 + R a_1 + \ldots,
\]

where (for \( l = 0 \))

\[
\chi'' + a_1 f(x) \chi_0 = 0,
\]

(3.3)

\[
\chi'' + a_1 f(x) \chi_1 = -2a_0 (x + a_0) f(x).
\]

(3.4)

The boundary conditions for \( \chi(x) \) are

\[
\chi_0(0) = 0, \quad \chi(\infty) = 1.
\]

We note that \( a_0 \) is of order unity; it can be found from (C.3) and (C.5). Multiplying both sides of (C.4) by \( \chi_0 \) and using (C.3), we find

\[
[x, \chi_0 - \chi_0 x_0]\chi_0 + 2a_0 \left( f(x) + f(x) \right) = 0.
\]

(3.6)

For \( r \gg R \) the quantity \( \chi(r) \sim e^{-\lambda r} \), so that there exists a region \( R < r < 1 \) in which this exponential can be expanded in a series in terms of \( R \); thus the boundary condition for \( \chi_1 \) can be found:

\[
\chi_1(0) = 0, \quad \chi_1(\infty) = 1.
\]

Using (C.5) and (C.7) we find from (C.6) an equation for the energy of the level:

\[
v = \frac{a}{R} = \frac{a_0}{R} + a \frac{Y_1 - e - a_0}{a},
\]

(3.8)

where

\[
a_0 = \frac{1}{f(x)} \int f(x) \chi_0(x) \, dx, \quad a_1 = \frac{1}{f(x)} \int f(x) \chi_1(x) \, dx.
\]

(3.9)

Thus for \( R \ll 1 \) the concrete form of the potential affects only the values of the parameters \( a_0, a_1 \), and \( a_2 \). For an attractive potential \( f(x) > 0 \) and \( a_1, a_2 \) are positive. It follows from this that for \( s = 0 \) the curve \( \epsilon = \epsilon(v) \) always has a turn-over point of the form shown in Fig. 4. The critical value of the potential corresponds to the vertex of this curve and is equal to

\[
v_c = \frac{a_0}{R} + a_0 \frac{Y_1 - e - a_0}{a}.
\]

(3.10)

An analogous procedure applied to the Dirac equation leads, instead of (C.8), to

\[
v = \frac{b}{R} + b \frac{h_e}{R} - \frac{h_k}{R},
\]

(3.11)

(for the states with \( \kappa = -1, j = \frac{1}{2} \)). Here

\[
b_k = \int \left( G_{s}^\dagger + P_x^\dagger \right) dx \int \left( f(x) G_{s} + F_x^\dagger \right) dx,
\]

(3.12)

and \( G_0(x) \), \( F_0(x) \), and \( \beta_0 \) are solutions of the equations of zeroth order in \( R \):

\[
G' = \frac{1}{x} G + \beta f(x) F, \quad F' = -\beta f(x) G - \frac{1}{x} F.
\]

(3.13)
with the boundary conditions
\[
G(0) = F(0) = 0, \\
G(z) \approx 0, \quad F(z) \approx c/z \quad \text{for} \quad z \gg 1,
\]  
(C.14)
normalized according to (29). Here the dependence \( c = c(v) \) is linear and there is no turn-over; the critical value of the potential corresponds to \( c = -1 \) and is equal to
\[
\nu_c = \frac{\beta_+}{R} + (b_+ + b_-).
\]  
(C.15)

For a square well \( f(x) = \theta(1 - x) \) and we easily find
\[
\beta_+ = 2a = \pi, \quad a_1 = 2/\pi, \quad a_2 = \frac{1}{2}, \quad b_+ = 2.
\]
Here (C.8) and (C.11) go over into (43) and (48), respectively. Thus the picture obtained for the special example of the potential (40) is quite general.

---


Translated by R. Lipperheide