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THE ANOMALOUS SKIN EFFECT IN THIN CYLINDRICAL CONDUCTORS

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We construct a theory for the anomalous skin effect in thin cylindrical conductors the radius of curvature of the surface of which is much smaller than the electron mean free path, assuming a quadratic dispersion law for the electrons in the metal. We find a strong dependence of the surface impedance on the way the electrons are scattered from the surface. When the electrons are reflected specularly from the surface of the conductor an essentially new dependence of the surface impedance on the frequency, the radius of the cylinder and the mean free path of the electrons appears.

1. INTRODUCTION

IN connection with the wide use of pure metals for which the mean free path of the electrons is large compared with the dimensions of the sample it is of considerable interest to evaluate the surface impedance of thin cylindrical conductors.

In the present paper, assuming a quadratic dispersion law for the electrons in the metal, we find a general expression for the current density in a cylindrical conductor for an arbitrary value of the coefficient p of specular reflection of the electrons from the surface. Assuming that the electron mean free path is large compared with the radius we find an equation for the field in the cylinder. We consider in detail the limiting cases when the scattering of the electrons is basically through collisions with the surface (non-specular reflection) and when the electrons are scattered by impurities (specular reflection from the surface).

It turned out that in the first case the surface impedance of a cylindrical conductor depends in an essential way on the way electrons are scattered by the surface and is independent of the radius of the cylinder. The dependence of the surface impedance on the frequency of the external field is now exactly the same as for samples with a plane surface.

If the electrons are specularly reflected from the surface of the cylinder, a peculiar situation arises which is connected with the fact that even when the electron mean free path is large compared to the radius, the scattering of the electrons is by impurities. This leads to an essentially new dependence of the surface impedance on the frequency, the radius of the cylinder, and the mean free path.

In what follows we shall use the classical kinetic equation for the electrons in the metal to solve the problem. The largest error which is introduced in this way is connected with the neglect of the so-called surface levels. For the case where there is no magnetic field Prange[1] has considered these levels (see also[2]). To obtain an estimate of the region of applicability of the following formulae we shall first of all briefly consider the properties of surface levels.

2. QUANTUM SURFACE LEVELS

We consider the quantum states of electrons in a cylindrical sample with specularly reflecting walls corresponding to large centrifugal angular momenta. For the sake of simplicity we restrict ourselves to the free electron model with a quadratic dispersion law and we write down the solution of the Schrödinger equation

$$\nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0, \tag{2.1}$$

which satisfies the boundary condition

$$\psi|_{r=R} = 0, \tag{2.2}$$

in the form

$$\psi_{n,M,p_z}(r,\varphi,z) = C_{n,M,p_z}\exp\left\{\frac{ip_z z}{\hbar} + iM\varphi\right\}J_M\left(\frac{r}{R}\mu_n^{(M)}\right). \tag{2.3}$$

Here $J_M(x)$ is a Bessel function, $\mu_n^{(M)}$ the n-th root of the equation $J_M(x) = 0$, and m the effective mass of the electron. We find the following expression for the energy spectrum of the electrons

$$E_n = \frac{p_z^2}{2m} + \frac{\hbar^2}{2mR^2}[\mu_n^{(M)}]^2. \tag{2.4}$$

When the skin effect is present the electrons which spend a long time in the skin-layer give the main contribution to the conductivity. Those electrons correspond to a large classical centrifugal angular momentum

$$M \sim \frac{mv_0R}{\hbar} \gg 1, \tag{2.5}$$

where v_0 is the Fermi velocity. Using the well-known asymptotic expression for the Bessel function we can show that Eq. (2.4) for the electron energy spectrum for $M \gg 1$ and $n \sim 1$ has the form

$$E_n = \frac{p_z^2}{2m} + \frac{\hbar^2M^2}{2mR^2} + 2^{-1/3}\frac{\hbar^2M^{4/3}}{mR^2}a_n, \tag{2.6}$$

where a_n is the n-th root of the equation $\Phi(-x) = 0$ ($\Phi(x)$ is the Airy function[3]). One sees easily that the electron energy spectrum (2.6) has the same form as the electron energy spectrum in a plane conductor when there is a magnetic field present and that the radius of curvature of the cylinder plays the same role as the Larmor radius.

The electrons may make transitions from one surface state to another under the influence of a high-frequency field. We get for the transition frequency from level m to level n:

$$\omega_{mn} = \frac{1}{\hbar}(E_n - E_m) = 2^{-1/3}\frac{\hbar M^{4/3}}{mR^2}(a_n - a_m). \tag{2.7}$$

As a rule, the resonance character of the transitions between surface levels leads to an oscillating dependence of the surface impedance on the frequency ω of the external field. We can neglect the influence of the surface levels on the conductivity if the energy of the quanta of the high-frequency field is appreciably smaller than the energy of the first quantum level. This condition under which we can evaluate the surface impedance in the framework of the classical theory has the form

$$\omega R / v_0 \ll M^{1/3}. \tag{2.8}$$

Of course, all we have stated is valid only provided the reflection of the electrons from the surface is close to specular. As the scattering of the electrons by the surface becomes more diffuse, the surface levels will become more smeared out.

3. SOLUTION OF THE KINETIC EQUATION

Let there be a relatively long metallic cylinder of radius R with its axis along the z-axis in an axially symmetric high-frequency field whose electric vector E is along the axis of the cylinder and depends only on the distance r from the axis. The equation for the electric field has the form

$$\Delta^2 \mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} = \frac{4\pi i\omega}{c^2}\mathbf{j}. \tag{3.1}$$

To find still another equation which connects the current density j with the field we must solve the kinetic equation which we linearize in the field E and write in a cylindrical system of coordinates:[4]

$$\left(i\omega + \frac{1}{\tau}\right) f_1 + v_r\frac{\partial f_1}{\partial r} + \frac{v_\varphi^2}{r}\frac{\partial f_1}{\partial v_r} - \frac{v_r v_\varphi}{r}\frac{\partial f_1}{\partial v_\varphi} = e\frac{\partial f_0}{\partial \varepsilon} v_z E_z. \tag{3.2}$$

Here $f_0 = \{1 + \exp[(\epsilon - \epsilon_0)/T]\}^{-1}$ is the equilibrium electron Fermi distribution function, f_1 the correction to the equilibrium distribution function taking the influence of the external field into account, $\epsilon = \frac{1}{2}mv^2$, $\epsilon_0 = \frac{1}{2}mv_0^2$ is the Fermi energy, and τ the relaxation time. We shall assume the electron gas to be degenerate in Secs. 4 and 8 and we write the collision integral in the relaxation time approximation.

To solve the problem it is convenient to introduce instead of f_1 two functions f_+ and f_- which describe electrons moving away from and towards the center of the cylinder:

$$f_1 = \begin{cases} e\dfrac{\partial f_0}{\partial \varepsilon} f_+, & v_r > 0; \\ e\dfrac{\partial f_0}{\partial \varepsilon} f_-, & v_r < 0. \end{cases} \tag{3.3}$$

The distribution function must be continuous at $v_r = 0$:

$$f_+|_{v_r=0} = f_-|_{v_r=0}.$$

The boundary conditions for the functions $f_\pm$ at the cylinder surface have the form:

$$f_-|_{r=R-0} = pf_+|_{r=R-0}. \tag{3.4}$$

Here p is the coefficient of specular reflection of the electrons from the surface, introduced by Reuter and Sondheimer.[5] 1) The equations for the functions $f_\pm$ have the form

$$(i\omega + 1/\tau) f_\pm \pm |v_r| \frac{\partial f_\pm}{\partial r} \pm \frac{v_\varphi^2}{r}\frac{\partial f_\pm}{\partial |v_r|} \mp \frac{|v_r| v_\varphi}{r}\frac{\partial f_\pm}{\partial v_\varphi} = v_z E_z. \tag{3.5}$$

Introducing a spherical system of coordinates in velocity space

$$v_r = v\sin\theta\cos\varphi, \quad v_\varphi = v\sin\theta\sin\varphi, \quad v_z = v\cos\theta \tag{3.6}$$

and writing

$$\Phi(x, y) \equiv \frac{i\omega + 1/\tau}{v\sin\theta}[x^2 - y^2\sin^2\varphi]^{1/2}, \tag{3.7}$$

we find the solution of Eqs. (3.5) with the boundary conditions (3.4) in the form

$$f_+ = \frac{\operatorname{ctg}\theta\exp[-\Phi(r,r)]}{1 - p\exp[-2\Phi(R,r)]}\left\{\int_{r|\sin\varphi|}^{R} e^{-\Phi(r',r)} + \int_{r|\sin\varphi|}^{r} e^{\Phi(r',r)} \times pe^{-2\Phi(R,r)}\int_{r}^{R} e^{\Phi(r',r)}\right\}\left[1 - \left(\frac{r}{r'}\sin\varphi\right)^2\right]^{-1/2} E_z(r')\,dr'; \tag{3.8}$$

$$f_- = \frac{\operatorname{ctg}\theta\exp[\Phi(r,r)]}{1 - p\exp[-2\Phi(R,r)]}\left\{pe^{-2\Phi(R,r)}\left[\int_{r|\sin\varphi|}^{} e^{-\Phi(r',r)} + \int_{r|\sin\varphi|}^{R} e^{\Phi(r',r)}\right] \times \int_{r}^{R} e^{-\Phi(r',r)}\right\}\left[1 - \left(\frac{r}{r'}\sin\varphi\right)^2\right]^{-1/2} E_z(r')\,dr'. \tag{3.9}$$

For the current density in the conductor we get the following expression

$$j_z = -\frac{2e^2m^3}{h^3}\int_0^\infty \frac{\partial f_0}{\partial \varepsilon} v^3 dv \int_0^\pi \cos\theta\sin\theta\, d\theta \int_{-\pi/2}^{\pi/2} [f_+ + f_-]\, d\varphi, \tag{3.10}$$

which as $R \to \infty$ gives the well-known result obtained by Reuter and Sondheimer[5] in the theory of the anomalous skin-effect in a plane sample.

4. SURFACE IMPEDANCE OF A THIN CYLINDER WITH NON-SPECULAR WALLS

In the case of interest to us where the mean free path of the electrons is large compared to the cylinder radius we have the inequality

$$|\xi| \ll 1, \tag{4.1}$$

where $\xi = R(i\omega + 1/\tau)/v_0$. We shall in the present section also assume that the coefficient of specular reflection p of the electrons from the surface of the cylinder is not too close to unity so that

$$1 - p \gg |\xi|. \tag{4.2}$$

The opposite limiting case when $1 - p \ll |\xi|$ is considered in Secs. 5–7. In the region (4.1) when (4.2) holds Eqs. (3.8) and (3.9) can be simplified so that

$$f_+ + f_- = 2\frac{1+p}{1-p}\operatorname{ctg}\theta\int_{r|\sin\varphi|}^{R}\left[1 - \left(\frac{r}{r'}\sin\varphi\right)^2\right]^{-1/2} E_z(r')\,dr'. \tag{4.3}$$

It is clear from Eq. (4.3) that the electron distribution function under our assumptions is independent of the relaxation time and the choice of the form of the collision integral in the kinetic equation is thus immaterial.

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1) Generally speaking, one must assume the coefficient p to be a function of the angle of incidence (see, e.g., [6]). For the sake of simplicity we shall, however, assume p to be constant.

Using the well-known relation for elliptical integrals:[7]

$$(1+k)\mathbf{K}(k)=\mathbf{K}\left(\frac{2\sqrt{k}}{1+k}\right), \tag{4.4}$$

where $\mathbf{K}(k) = \int_0^{1/2\pi} [1 - k^2\sin^2\varphi]^{-1/2} d\varphi$ is the complete elliptical integral of the first kind, we find the following expression for the current density in a thin cylinder:

$$j_z(r)=\frac{4\pi e^2m^2v_0^2}{h^3}\frac{1+p}{1-p}\int_0^R E_z(r')\mathbf{K}\left(\frac{2\sqrt{rr'}}{r+r'}\right)\frac{r'\,dr'}{r+r'}. \tag{4.5}$$

Neglecting in the wave equation (3.1) the displacement current in comparison with the conduction current we are led to the following equation for the field in the cylinder:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dE_z(r)}{dr}\right]=\frac{2i}{\pi}\frac{1+p}{1-p}k_0{}^3\int_0^R E_z(r')\mathbf{K}\left(\frac{2\sqrt{rr'}}{r+r'}\right)\frac{r'dr'}{r+r'}, \tag{4.6}$$

where

$$k_0=(e^2m^2v_0{}^2\omega/c^2\hbar^3)^{1/3}, \tag{4.7}$$

$\delta_0 = k_0^{-1}$ is the penetration depth of the field into the metal in the theory of the anomalous skin effect.[5] Equation (4.6) describes the penetration of the field into the cylinder for an arbitrary ratio of the radius of curvature to the thickness of the skin-layer. Using the formula[7]

$$(r+r')^{-1}\mathbf{K}\left(\frac{2\sqrt{rr'}}{r+r'}\right)=\frac{\pi}{2}\int_0^\infty J_0(kr)J_0(kr')\,dk, \tag{4.8}$$

we can write Eq. (4.6) also in the form

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dE_z(r)}{dr}\right]=ik_0{}^3\frac{1+p}{1-p}\int_0^R E_z(r')r'dr'\int_0^\infty J_0(kr)J_0(kr')\,dk. \tag{4.9}$$

If the penetration depth of the field into the cylinder is appreciably larger than its radius, i.e., when $k_0R(1-p)^{-1/3} \ll 1$ it is convenient to solve Eq. (4.9) by the method of successive approximations. Putting in the first approximation $E_Z(r') = E_Z(R)$ we find

$$rE_z'(r)=ik_0{}^3\frac{1+p}{1-p}E_z(R)\int_0^r r_1dr_1\int_0^R r'dr'\int_0^\infty J_0(kr_1)J_0(kr')\,dk, \tag{4.10}$$

whence

$$E_z'(R)/E_z(R)=ik_0{}^3R^2\frac{1+p}{1-p}\int_0^\infty J_1{}^2(k)\frac{dk}{k^2}. \tag{4.11}$$

The integral in (4.11) is equal to $4/3\pi$ so that

$$E_z'(R)/E_z(R)=\frac{4i}{3\pi}k_0{}^3R^2\frac{1+p}{1-p}. \tag{4.12}$$

For the surface impedance[2)] in the region $k_0R(1-p)^{-1/3} \ll 1$ we find the expression

$$Z=\frac{4\pi i\omega}{c^2}\frac{E_z(R)}{E_z'(R)}=3\pi^2\frac{1-p}{1+p}\frac{\hbar^3}{(mev_0R)^2}. \tag{4.13}$$

In the opposite limiting case $k_0R(1-p)^{-1/3} \gg 1$ when the penetration depth of the field into the cylinder is small compared to the radius only a small region near the surface is important. Putting

$$r=R-\delta\rho,\quad r'=R-\delta\rho',\quad \delta=\delta_0\left(\frac{1-p}{1+p}\right)^{1/3}, \tag{4.14}$$
$$E_z(R-\delta\rho)=f(\rho),$$

and using the following asymptotic form of the elliptical integral:[7]

$$\mathbf{K}(k)=\ln\frac{4}{k'}+o(k'^2\ln k'),\quad k'=\sqrt{1-k^2}, \tag{4.15}$$

and extending the integration over ρ' to ∞ we get the equation

$$f''(\rho)=\frac{i}{\pi}\int_0^\infty f(\rho')\ln\frac{8R/\delta}{|\rho-\rho'|}d\rho'. \tag{4.16}$$

Continuing the function $f(\rho)$ to the region $\rho < 0$ such that it is an even function and changing in Eq. (4.16) to Fourier components,

$$F(k)=2\int_0^\infty f(\rho)\cos k\rho\,d\rho, \tag{4.17}$$

we get for $F(k)$ the equation

$$(k^2+i/k)F(k)-\frac{2i}{\pi^2}\int_0^\infty F(k')\frac{\ln k/k'}{k^2-k'^2}dk'=-2f'(+0). \tag{4.18}$$

An equation of the type (4.18) was obtained by Azbel' and Kaner[8] and solved by Hartmann and Luttinger.[9] Using the results of[9] we find the surface impedance of the cylinder in the form

$$Z=-\frac{4i\omega\delta}{c^2f'(+0)}\int_0^\infty F(k)\,dk=Z_0\left(\frac{1-p}{1+p}\right)^{1/3},\quad \delta\ll R, \tag{4.19}$$

where $Z_0 = 2\sqrt{3}\,\pi\omega e^{i\pi/3}/c^2k_0$ is the surface impedance of a plane metallic sample with a diffuse surface.

The surface impedance for the case of the strongly anomalous skin-effect in region (4.2) is thus independent of the radius of the cylinder and hence of the form of the cross section of the conductor. This should have been expected as the radius of the cylinder under condition (4.1) plays the role of a mean free path and for the strongly anomalous skin effect the surface impedance is independent of the mean free path. We note that the surface impedance of thin cylindrical conductors depends in an essential way on the way the electrons are scattered by the surface.

5. EQUATION FOR THE FIELD IN A THIN CYLINDER WITH A SPECULAR SURFACE

When the reflection of the electrons from the surface of the cylinder is specular, i.e., under the condition

$$1-p\ll|\xi|, \tag{5.1}$$

the scattering of the electrons is basically by impurities. We shall consider this case also on the basis of condition (4.1). In Secs. 5–7 we shall not assume that the electron gas is degenerate so that the final results will be valid for any temperature of the electron gas. Using conditions (4.1) and (5.1) the general expression (3.10) for the current density in the cylinder can be written in the form

$$j_z=-\frac{8(1+p)e^2m^3}{h^3}\int_0^\infty\frac{\partial f_0}{\partial\varepsilon}v^3\,dv\int_0^\pi\cos^2\theta\,d\theta\int_0^{\pi/2}[1-p+2p\Phi(R,r)]^{-1}d\varphi$$
$$\times\int_{r\sin\varphi}^R\left[1-\left(\frac{r}{r'}\sin\varphi\right)^2\right]^{-1/2}E_z(r')\,dr'. \tag{5.2}$$

Assuming, in accordance with condition (5.1) that

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2)The usual resistivity $\mathcal{R}$ of a conductor of length l and radius R is connected with the surface impedance Z through the formula $\mathcal{R} = lZ/2\pi R$.

p = 1 and also noting that for a Fermi distribution (see, e.g.,[10])

$$\int_0^\infty \frac{\partial f_0}{\partial \varepsilon} v^4 dv = -3N\pi^2\hbar^3/m^4,$$

where N is the electron concentration, we get the following expression for the current density

$$j_z = \frac{2}{\pi}\frac{e^2N}{m}\frac{1}{R(i\omega+1/\tau)}\int_0^{\pi/2}\left[1-\left(\frac{r}{R}\sin\varphi\right)^2\right]^{-1/2} d\varphi \quad (5.3)$$

$$\times\int_{r\sin\varphi}^{R}\left[1-\left(\frac{r}{r'}\sin\varphi\right)^2\right]^{-1/2} E_z(r')\,dr'.$$

Substituting Eq. (5.3) for the current density into the wave equation (3.1), neglecting the displacement current in comparison with the conduction current, and expressing the electron concentration in terms of the plasma frequency

$$N = m\omega_0^2/4\pi e^2, \quad (5.4)$$

we find after some transformations the following equation for the field in the conductor:

$$\frac{d}{dr}\left[r\frac{dE_z(r)}{dr}\right] = \frac{2i}{\pi}\frac{\omega\omega_0^2}{c^2(i\omega+1/\tau)}\int_0^R E_z(r')\mathcal{G}(r/R, r'/R)\,dr'. \quad (5.5)$$

The kernel of Eq. (5.5) can be expressed in terms of the complete elliptical integral of the first kind as follows:

$$\mathcal{G}(x,y) = \frac{xy}{x\sqrt{1-y^2}+y\sqrt{1-x^2}}\mathrm{K}\left[\frac{2(xy\sqrt{1-x^2}\sqrt{1-y^2})^{1/2}}{x\sqrt{1-y^2}+y\sqrt{1-x^2}}\right]. \quad (5.6)$$

Changing for the sake of convenience of the further discussion to dimensionless quantities:

$$x = r/R,\quad y = r'/R,\quad f(x) = E_z(Rx)/E_z(R),$$
$$\zeta = \left(\frac{\omega_0 R}{c}\right)^2 \Big/ (1+1/i\omega\tau), \quad (5.7)$$

we write Eq. (5.5) in the form

$$\frac{d}{dx}\left[x\frac{df(x)}{dx}\right] = \frac{2}{\pi}\zeta\int_0^1 f(y)\mathcal{G}(x,y)\,dy. \quad (5.8)$$

Using Eq. (4.8) we find easily the following integral representation for the kernel (5.6):

$$\mathcal{G}(x,y) = \frac{\pi}{2}\int_0^\infty J_0\left(k\frac{\sqrt{1-x^2}}{x}\right)J_0\left(k\frac{\sqrt{1-y^2}}{y}\right)dk. \quad (5.9)$$

Using the representation (5.9) and also the formulae[7]

$$\int_0^\infty J_0(kx)\frac{x\,dx}{(\alpha+x^2)^{3/2}} = \alpha^{-1/2}e^{-\sqrt{\alpha}k}, \quad (5.10)$$

$$\int_0^\infty J_0(kz)e^{-\sqrt{\alpha}k}\,dk = (\alpha+z^2)^{-1/2}, \quad (5.11)$$

we find the following identity

$$f'(1) = \zeta\int_0^1 f(y)\,y\,dy. \quad (5.12)$$

The integro-differential equation (5.8) is equivalent to the following Fredholm integral equation:

$$f(x) + \zeta\int_0^1 f(y)G(x,y)\,dy = 1, \quad (5.13)$$

whose kernel has the form

$$G(x,y) = \int_0^\infty J_0\left(k\frac{\sqrt{1-y^2}}{y}\right)dk\int_x^1\frac{dx'}{x'}\int_0^{x'}J_0\left(k\frac{\sqrt{1-x''^2}}{x''}\right)dx''. \quad (5.14)$$

The only parameter, ζ, which occurs in Eq. (5.13) is determined by the ratio of the penetration depth of the field into the cylinder to its radius. If the field freely penetrates into the cylinder, $\zeta \ll 1$. In the opposite case of a strong skin-effect, $\zeta \gg 1$.

6. SURFACE IMPEDANCE OF A CYLINDER WITH A SPECULAR SURFACE

To evaluate the surface impedance it is unnecessary to find the exact solution of (5.13), but it is sufficient to find only f′(1):

$$Z = \frac{4\pi i\omega}{c^2}\frac{E_z(R)}{E_z'(R)} = \frac{4\pi i\omega R}{c^2}[f'(1)]^{-1} \quad (6.1)$$

We shall write f′(1) as the ratio of determinants of infinite order. We introduce the notation

$$I_n \equiv \int_0^1 f(y)y^{2n-1}\,dy,\quad n = 1, 2, \ldots. \quad (6.2)$$

In accordance with the identity (5.12),

$$f'(1) = \zeta I_1. \quad (6.3)$$

Multiplying Eq. (5.13) by x^{2n-1} and integrating from zero to unity we get

$$I_n + \zeta\int_0^1 f(y)g_n(y)\,dy = 1/2n, \quad (6.4)$$

where

$$g_n(y) = \int_0^1 x^{2n-1}G(x,y)\,dx. \quad (6.5)$$

Substituting (5.14) into (6.5) we get after simple transformations

$$g_n(y) = \frac{1}{4n^2}\int_0^\infty J_0\left(k\frac{\sqrt{1-y^2}}{y}\right)dk\int_0^\infty J_0(ku)[1-1/(1+u^2)^n]\frac{u\,du}{(1+u^2)^{3/2}}. \quad (6.6)$$

Moreover applying the obvious identity

$$(\alpha+u^2)^{-n-3/2} = (-1)^n\frac{\Gamma(3/2)}{\Gamma(n+3/2)}\frac{\partial^n}{\partial\alpha^n}(\alpha+u^2)^{-3/2}, \quad (6.7)$$

using Eqs. (5.10) and (5.11) to integrate, and using Leibnitz's formula for the n-th derivative of a product we find $g_n(y)$ in the form

$$g_n(y) = \frac{1}{4n^2}\left\{y - \frac{\Gamma(n+1)}{2\sqrt{\pi}\,\Gamma(n+3/2)}\sum_{s=1}^{n+1}\frac{\Gamma(s-1/2)\Gamma(n-s+3/2)}{\Gamma(s)\Gamma(n-s+2)}y^{2s-1}\right\} \quad (6.8)$$

Here $\Gamma(x)$ is the gamma function. Substituting (6.8) into (6.4) we find the following infinite set of equations for the I_n:

$$\sum_{s=1}^{n+1}\{\delta_{ns}+a_{ns}\}I_s = 1/2n,\quad n = 1.2, \ldots, \quad (6.9)$$

where δ_{ns} is the Kronecker symbol, and

$$a_{ns} = \frac{\zeta}{4n^2}\left\{\delta_{1s} - \frac{\Gamma(n+1)\Gamma(s-1/2)\Gamma(n-s+3/2)}{2\sqrt{\pi}\,\Gamma(n+3/2)\Gamma(s)\Gamma(n-s+2)}\right\}. \quad (6.10)$$

We show that the set (6.9) is normal, i.e., that the series

$$\sum_{n,s=1}^{\infty}|a_{ns}| \quad (6.11)$$

converges. Indeed, when $s = 1$ the series over n is bounded:

$$\sum_{n=1}^{\infty}|a_{n1}| < \frac{\zeta}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\zeta}{4}\zeta(2). \tag{6.12}$$

Here $\zeta(x)$ is Riemann's zeta function. Using the obvious inequality

$$\Gamma(s-{}^1\!/_2)\Gamma(n-s+{}^3\!/_2) < \Gamma(s)\Gamma(n-s+2), \tag{6.13}$$

we can majorize the series (6.11) for $s > 1$ by the series

$$\frac{\zeta}{8\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{\Gamma(n)}{\Gamma(n+{}^3\!/_2)} \tag{6.14}$$

the convergence of which can easily be established, e.g., using Raabe's test.[11]

It is well known[12] that there exists for a normal infinite system a determinant Δ of infinite order which can be evaluated as the limit (as the number of rows and columns tends to infinity) of a sequence of determinants of finite order. There also exists a determinant Δ_s which is obtained from Δ by replacing the s-th column by the numbers $1/2n$. We can thus find the solution of the set (6.9) by Cramer's rule:

$$I_s = \Delta_s/\Delta, \quad s = 1, 2, \ldots. \tag{6.15}$$

We get for the surface impedance the following expression:

$$Z = \frac{4\pi i\omega R}{c^2\zeta}\frac{\Delta}{\Delta_1}. \tag{6.16}$$

If $\zeta \ll 1$, when the radius of the cylinder is small compared to the penetration depth of the field into the metal, only the diagonal elements are important in the determinants Δ and Δ_1 so that $\Delta = 1$, $\Delta_1 = {}^1\!/_2$ and the surface impedance is equal to

$$Z_0 = \frac{8\pi}{\omega_0^2 R}(i\omega + 1/\tau). \tag{6.17}$$

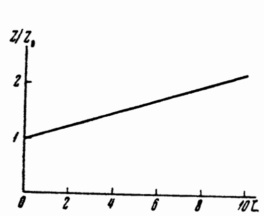
Owing to the fact that the a_{ns} tend rather fast to zero as n and s increase we can in the range $\zeta \sim 1$ limit ourselves to second order determinants. We give in the figure the ζ-dependence of the surface impedance for $\zeta \leq 10$ for the case when $\omega\tau \gg 1$.

7. STRONGLY ANOMALOUS SKIN-EFFECT

In the region of very large ζ it is inconvenient to use the general Eq. (6.16) as large n and s become important. We find an expression for the surface impedance of a cylinder with a specular surface which is asymptotic as

$$\zeta \gg 1 \tag{7.1}$$

We note that the ζ-dependence of the surface impedance when condition (7.1) holds can be obtained directly from the general expression (6.16). We shall, however,



start from Eq. (5.8) as in such an approach the physical content of the theory is revealed more clearly.

It is natural to assume that when condition (7.1) holds the same situation arises as in the case of the strongly anomalous skin effect. In that case the main contribution in Eq. (5.8) is given by a small region of the order of the penetration depth near the surface of the cylinder. Putting

$$x = 1-\delta\xi,\ y = 1-\delta\eta,\ f(1-\delta\xi) = g(\xi),\ \delta = \zeta^{-1/5} \ll 1, \tag{7.2}$$

we can reduce Eq. (5.8) to the dimensionless equation:

$$g''(\xi) = \frac{1}{\sqrt{2}}\int_0^{\infty} g(\eta)Q(\xi,\eta)\,d\eta, \tag{7.3}$$

where

$$Q(\xi,\eta) = \int_0^{\infty} J_0(k\sqrt{\xi})J_0(k\sqrt{\eta})\,dk. \tag{7.4}$$

We get thus for the surface impedance the following expression

$$Z = \frac{4\pi i\omega R\delta}{c^2|g'(0)|}$$
$$= \frac{4\pi}{c}|g'(0)|^{-1}\left[\left(\frac{\omega}{\omega_0}\right)^4\frac{\omega R}{c}\left(1+\frac{1}{\omega^2\tau^2}\right)\right]^{1/5}\exp\left[i\left(\frac{\pi}{2}-\frac{2}{5}\operatorname{arctg}\frac{1}{\omega\tau}\right)\right], \tag{7.5}$$

where $g'(0)$ is a constant determined by Eq. (7.3). In a well-known way we can estimate it as follows. We consider the functional

$$I\{g\} = \frac{1}{2}\left\{\int_0^{\infty}[g'(\xi)]^2 d\xi + \frac{1}{\sqrt{2}}\int_0^{\infty}d\xi\int_0^{\infty}d\eta\,Q(\xi,\eta)g(\xi)g(\eta)\right\}. \tag{7.6}$$

Varying the function g in such a way that at the boundaries of the domain of integration the variation δg vanishes and using the symmetry of the kernel (7.4) one can show that the function $g(\xi)$ satisfying Eq. (7.3) minimizes the functional (7.6). To estimate $g'(0)$ we can substitute $g(\xi) = \exp(-\lambda\xi)$ into (7.6) and find λ such that the function $I\{\exp(-\lambda\xi)\}$ is minimized. Proceeding in this way we find

$$|g'(0)| \approx \lambda = (9\pi/4)^{1/5} \approx 1.48. \tag{7.7}$$

It is clear from Eq. (7.5) that the dependence of the surface impedance on the various parameters for the case of specular reflection of the electrons from the surface under the conditions of the strongly anomalous skin-effect is essentially different from the case of a cylinder with a non-specular surface and for a plane sample. Under those conditions the surface impedance depends in an essential way on the relaxation time. The dependence of the surface impedance on the cylinder radius and hence on the shape of the cross-section of a conductor is relatively weak.

8. SURFACE IMPEDANCE OF A VERY THIN CYLINDRICAL CONDUCTOR

The expressions obtained above for the surface impedance of thin cylindrical conductors refer to two opposite limiting cases (4.2) and (5.1). It is of great interest to trace how the appropriate formulae change from one into the other when the parameter p is changed. In the present section we calculate for an arbitrary value of the parameter of specularity the surface impedance of a cylindrical conductor which is so thin that the penetration depth of the field into the metal is much larger than the radius of the cylinder. We assume in this section that the electron gas in the metal is degenerate.

The condition that the cylinder radius is small compared to the penetration depth of the field into the metal enables us when using Eq. (5.2) to evaluate the current density to put $E_z(r') = E_z(R)$. After evaluating the integrals we find

$$\frac{E_z'(R)}{E_z(R)} = \frac{4i}{3\pi} k_0^3 R^2 \frac{1+p}{1-p} f\left(\frac{1-p}{2p\xi}\right), \tag{8.1}$$

where k_0 is defined by Eq. (4.7) and

$$f(k) = 1 + \frac{k}{2} + \frac{2k^3}{5} + \frac{1}{5k^2}[k^2(k^2-4)K'(k) - (2k^4+3k^2-8)E'(k) - 8]. \tag{8.2}$$

The functions $K'(k)$ and $E'(k)$ occurring in Eq. (8.2) are the complete elliptical integrals of the first and second kind of the complementary modulus:

$$K'(k) = \begin{cases} K(\sqrt{1-k^2}), & |k|<1; \\ \frac{1}{k}K(\sqrt{1-k^{-2}}); & |k|>1; \end{cases} \qquad E'(k) = \begin{cases} E(\sqrt{1-k^2}), & |k|<1; \\ kE(\sqrt{1-k^{-2}}), & |k|>1. \end{cases} \tag{8.3}$$

We obtain for the surface impedance the following expression

$$Z = 3\pi^2 \frac{1-p}{1+p} \frac{\hbar^3}{(mev_0R)^2} \left[f\left(\frac{1-p}{2p\xi}\right)\right]^{-1}. \tag{8.4}$$

In the region (4.2) when the scattering of the electrons by the surface dominates over the scattering due to collisions with impurities, Eq. (8.4) gives

$$Z = 3\pi^2 \frac{1-p}{1+p} \frac{\hbar^3}{(mev_0R)^2}, \tag{8.5}$$

which is the same as Eq. (4.13). The surface impedance is then independent of the mean free path.

In the opposite limiting case of specular reflection under condition (5.1) the electron scattering is basically due to collisions with impurities. The surface impedance then depends in an essential way on the relaxation time:

$$Z = 12\pi^2 \frac{p}{1+p} \frac{\hbar^3\xi}{(mev_0R)^2}. \tag{8.6}$$

If we express the plasma frequency in terms of the Fermi velocity

$$\omega_0^2 = 4v_0^3 m^2 e^2 / 3\pi\hbar^3,$$

we see easily that for a degenerate electron gas Eq. (6.17) is the same as (8.6) for $p = 1$.

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