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# A NEW KIND OF INSTABILITY IN A PARTIALLY IONIZED PLASMA

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We study the stability of a partially ionized plasma, taking into account ionization and recombination processes. Owing to these processes, specific ionization-recombination vibrations may occur in such a plasma. We show that there may then occur an instability connected with the parametric excitation of ion-acoustic oscillations while the largest build-up increment characterizes the ion-acoustic oscillations, whose doubled frequency is close to the frequency of the ionization-recombination oscillations.

## 1. INTRODUCTION

ONE usually assumes that the unperturbed state of the plasma is a stationary one when one studies the oscillations of a partially ionized plasma. The oscillations propagating in such a plasma turn out in that case to be weakly damped (in the case of a not very strong external electric field); the build-up of the oscillations leading to an instability of the plasma arises only in sufficiently strong external electrical fields.[1-3]

However, it has recently been shown by Roth[4] that the unperturbed state of a partially ionized plasma is in general not a stationary one. Due to ionization and recombination processes in such a plasma oscillations arise which are accompanied by a change in the total number of charged particles: ionization-recombination oscillations.[1)] By their nature these oscillations are analogous to the oscillations of the numerical population in the Volterra problem of two species of fish devouring one another (see[5]).

Oscillations propagating in a plasma with a charged particle number which changes periodically with time may turn out to be building up even when there is no external electrical field. Therefore, even when there is no external electrical field a partially ionized plasma may be unstable.

The present paper is devoted to a study of the stability of a partially ionized plasma, taking into account processes in which the total number of charged particles changes. We show that in such a plasma an instability is possible which is connected with the parametric excitation of ion-acoustic oscillations. The largest build-up increment then characterizes those ion-acoustic oscillations for which the doubled frequency lies close to the frequency of the ionization-recombination oscillations.

## 2. EQUATIONS FOR THE ION-ACOUSTIC OSCILLATIONS

We consider the oscillations of a partially ionized plasma in which the average energy of the random motion of the electrons appreciably exceeds the ion temperature. We can describe the electron component of such a plasma by the kinetic equation

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)f-\frac{e}{m}\frac{\partial\varphi}{\partial\mathbf{r}}\frac{\partial F}{\partial\mathbf{v}}+J\{f\}=0, \tag{2.1}$$

where f is the deviation of the electron distribution function from its unperturbed value F, $\varphi$ is the electrostatic potential, and J{f} is the linearized collision integral.

We shall be interested in the ion-acoustic oscillations the phase velocity of which is appreciably less than the average thermal velocity of the electrons $v_e$, $\omega/k \ll v_e$ ($\omega$ and k are the frequency and wave vector of the oscillations). In the case of not too large wavelengths ($kl \gg 1$, $l$ is the electron mean free path) we can write Eq. (2.1) in the form

$$\mathbf{v}\frac{\partial f}{\partial\mathbf{r}}-\frac{e}{m}\frac{\partial\varphi}{\partial\mathbf{r}}\frac{\partial F}{\partial\mathbf{v}}=0. \tag{2.2}$$

If we integrate this relation over the velocities we get the following expression for the deviation $\delta n_e$ of the electron density $n_e$ from its unperturbed value $n_{e0}$:

$$\delta n_e=-n_{e0}\frac{e\varphi}{T^*}, \tag{2.3}$$

where T* is an effective temperature

$$T^*=-\frac{1}{mn_{e0}}\int\frac{d^3v}{v_{\parallel}}\frac{\partial F}{\partial v_{\parallel}}, \tag{2.4}$$

while $v_{\parallel}$ is the component of the electron velocity in the direction of propagation of the wave.

In the case of a Maxwell distribution of the electrons T* = T, where T is the electron temperature. We note that if then the main contribution to the collision integral J{f} is given by the electron-electron collisions, Eq. (2.3) will be valid for any ratio of the wavelength of the oscillations and the electron mean free path (and not only when $kl \gg 1$). Indeed, when the mean free path of the electrons is small a Boltzmann distribution for the electrons, $n_e = n_{e0}\exp(-e\varphi/T)$, can be set up at every time and in every point in space. When $|n_e - n_{e0}| \ll n_{e0}$ we get from this Eq. (2.3).

If the partially ionized plasma is in an external constant and uniform electrical field $\mathbf{E}_0$, the unperturbed distribution function F has the form[6]

$$F=F_0+\frac{\mathbf{v}}{v}\mathbf{F}_1,\quad F_0=C\exp\left\{-\left(\frac{mv^2}{2T_e}\right)^2\right\}\quad \mathbf{F}_1=-\frac{e\mathbf{E}_0l}{mv}\frac{dF_0}{dv},$$
$$T_e=\sqrt{\frac{M_0}{3m}}\,eE_0l,\quad C=\frac{n_{e0}}{\pi\Gamma(3/4)}\left(\frac{2T_e}{m}\right)^{-3/2} \tag{2.5}$$

[1)] The review papers [8,9] are devoted to phenomena in partially ionized which are connected with ionization-recombination processes.

($M_0$ is the mass of the neutral particles). In that case we must in general still add in Eq. (2.1) a term $(e/m)E_0\partial f/\partial v$. One can, however, show that

$$\frac{e}{m}\mathbf{E}_0\frac{\partial f}{\partial \mathbf{v}} \leqslant \sqrt{\frac{m}{M_0}}J\{f\};$$

we shall therefore neglect this term. Substituting (2.5) into (2.4) we get for the effective temperature

$$T^* = \frac{2\Gamma(3/4)}{\Gamma(1/4)}T_e.$$

Bearing in mind that the phase velocity of the ion-acoustic oscillations is much larger than the ion thermal velocity we can describe the ionic component of the plasma by the hydrodynamic equations

$$\frac{\partial \delta n_i}{\partial t} + n_{i0}\,\mathrm{div}\,\mathbf{u}_i = 0, \qquad \frac{\partial \mathbf{u}_i}{\partial t} - \frac{ze}{M_i}\frac{\partial \varphi}{\partial \mathbf{r}} + \nu_i\mathbf{u}_i = 0, \tag{2.6}$$

where $n_{i_0}$ and $\delta n_i$ are the unperturbed value and the perturbation in the ion density ($n_{i_0} = n_{e_0}/z$), $\mathbf{u}_i$ their hydrodynamic velocity and $\nu_i^{-1}$ the average free flight time of the ions ($-ze$ and $M_i$ are the ion charge and mass).

If the unperturbed state of the plasma is spatially uniform we get, by using the Poisson equation and Eqs. (2.3) and (2.6), the following equation for the spatial Fourier component of the ion density $n_i$:

$$\frac{d}{dt}\left\{\frac{1}{n_{e0}}\frac{dn_i}{dt}\right\} + \frac{\nu_i}{n_{e0}}\frac{dn_i}{dt} + \omega^2 n_i = 0, \tag{2.7}$$

where $\omega^2 = k^2V_S^2/(1 + a^2k^2)$, $V_S = (zT^*/M_i)^{1/2}$ is the sound velocity and $a = (T^*/4\pi e^2 n_{e0})^{1/2}$ is the electron Debye radius.

This equation describes the ion-acoustic oscillations in a plasma the unperturbed state of which is uniform but not necessarily stationary. In the case of a stationary unperturbed state ($n_{e_0}$ = const) these oscillations are the usual ion-acoustic waves with frequency $\omega$ and damping decrement $\nu_i/2$ (here and henceforth we shall assume that $\nu_i \ll \omega$). In the case, however, of an unperturbed state when the unperturbed electron density $n_{e_0}$ is a function of the time the damping decrement of such oscillations may differ appreciably from $\nu_i/2$; in particular, it may become negative which corresponds to a build-up of the ionic sound leading to an instability of the unperturbed state of the plasma.

## 3. IONIZATION-RECOMBINATION OSCILLATIONS

The unperturbed state of a partially ionized plasma, in which processes are possible which change the total number of charged particles, is in general not a stationary one.[4]

Let there be injected $\Omega_1$ neutral particles per second in a plasma occupying a volume V. If the dimensions of the plasma are large compared with the Debye radius the total number of electrons $N_e$ and ions $N_i$ will be connected with one another through the relation $zN_i = N_e$. The ions may leave the volume V through the surface S of the plasma (for instance, an intersection perpendicular to the magnetic force lines); the loss of ions per unit time is clearly

$$\Omega_2 N_i = -N_i\frac{v_i S}{4V},$$

where $\Omega_2$ is a coefficient characterizing the ion loss and $v_i$ the average thermal velocity of the ions.

We shall assume that the main source of ions is the ionization of neutral particles when they collide with the electrons in the plasma. Introducing the cross-section $\sigma$ for this process we can write the changes in the numbers of neutral and charged particles per unit time caused by these processes in the form

$$\Omega_3 NN_e = -\Omega_4 NN_e = \frac{\langle\sigma v\rangle}{V}NN_e,$$

where $\Omega_{3,4}$ are coefficients characterizing the speeds of the ionization and recombination processes which are due to collisions between particles (N is the total number of neutral particles; v the relative velocity of the colliding particles; the brackets $\langle\ldots\rangle$ indicate an average over the velocities of the colliding particles). Neglecting other possible processes which can change the number of neutral and charged particles we get the following equations to determine N and $N_e$:

$$\frac{dN}{dt} = \Omega_1 - \Omega_3 NN_e, \quad \frac{dN_e}{dt} = N_e(\Omega_2 + \Omega_3 N). \tag{3.1}$$

These equations can be appreciably simplified in the case of a weakly ionized plasma, when $N_e \ll N$. Eliminating the quantity dN/dt from (3.1) we have

$$\frac{d^2N_e}{dt^2} - \Omega_3\Omega N_e + \Omega_3^2N_0N_e^2 + \Delta N\Omega_3N_e(\Omega_3N_e - 2\Omega_3N_0 - 2\Omega_2) - (\Delta N)^2\Omega_3^2N_e = 0, \tag{3.2}$$

where $N_0$ is the (time) average of the function N(t); $\Delta N(t) = N(t) - N_0$, and

$$\Omega = \Omega_1 + \Omega_3^{-1}(\Omega_2 + \Omega_3N_0)^2.$$

In the case of a weakly ionized plasma we shall show below that the amplitude of the oscillations in the number of neutral particles is small, $\Delta N \ll N_0$. We can thus neglect in (3.2) terms containing $\Delta N$. As a result we get

$$\frac{d^2N_e}{dt^2} - \Omega_3\Omega N_e + \Omega_3^2N_0N_e^2 = 0. \tag{3.3}$$

Introducing the notation

$$N_e = N_{e0}(1 + Q), \quad \eta = \Omega / \Omega_3 N_0 N_{e0}, \qquad \omega_0 = \Omega_3\sqrt{N_0N_{e0}} \equiv \frac{\langle\sigma v\rangle}{V}\sqrt{N_0N_{e0}}, \tag{3.4}$$

where $N_{e0}$ is the extremal value of the function $N_e(t)$ we can write Eq. (3.3) in the form

$$\omega_0^{-2}\frac{d^2Q}{dt^2} + Q^2 + (2-\eta)Q + (1-\eta) = 0. \tag{3.5}$$

We can express the solution of this equation in terms of Jacobi's elliptic functions. To do this we multiply Eq. (3.5) by dQ/dt and integrating once we get

$$\left(\omega_0^{-1}\frac{dQ}{dt}\right)^2 + \frac{2}{3}Q(Q-Q_1)(Q-Q_2) = 0, \tag{3.6}$$

where

$$Q_{1,2} = -3/4\{(2-\eta) \pm \sqrt{(\eta - 2/3)(\eta+2)}\}.$$

We consider first the case when $2/3 < \eta < 1$. Integrating Eq. (3.6) we have

$$Q = Q_2\,\mathrm{sn}^2\left(\sqrt{\frac{|Q_1|}{6}}\,\omega_0 t;\ \sqrt{\frac{Q_2}{Q_1}}\right), \tag{3.7}$$

where sn(x; k) is Jacobi's elliptic sine function. The frequency $\omega_r$ of the ionization-recombination collisions is in this case equal to

$$\omega_r = \frac{\omega_0}{2}\sqrt{\frac{|Q_1|}{6}}K^{-1}\left(\sqrt{\frac{Q_2}{Q_1}}\right), \quad (3.8)$$

where K(k) is the complete elliptic integral of the first kind; the quantity $N_{e0}/V$ is the maximum value of the electron density.

In the case $\eta > 1$ we can as before use for the function Q of Eq. (3.7), but, bearing in mind that in that case $Q_2 > 0 > Q_1$, it is convenient to write the equation for Q in the form

$$Q = \frac{Q_1 Q_2 \,\mathrm{sn}^2(x_1; k_1)}{Q_1 - Q_2 + Q_2\,\mathrm{sn}^2(x_1; k_1)},$$
$$x_1 = \frac{\omega_0 t}{2}\left[\left(\eta - \frac{2}{3}\right)(\eta + 2)\right]^{1/4}, \quad k_1 = \sqrt{\frac{Q_2}{Q_2 - Q_1}}. \quad (3.9)$$

The frequency of the oscillations is then equal to

$$\omega_r = \frac{\omega_0}{4}\left[\left(\eta - \frac{2}{3}\right)(\eta + 2)\right]^{1/4} K^{-1}\left(\sqrt{\frac{Q_2}{Q_2 - Q_1}}\right); \quad (3.10)$$

the quantity $N_{e0}/V$ is the minimum value of the electron density.

When $|\eta - 1| \ll 1$, we have $Q \ll 1$. In that case Eq. (3.3) describes harmonic oscillations with a small amplitude and frequency $\omega_0$ and we have for the unperturbed electron density $n_{e0}(t)$

$$n_{e0}(t) = \bar{n}_{e0}(1 + h\cos\omega_0 t), \quad (3.11)$$

where $\bar{n}_{e0}$ is the (time) average of the unperturbed electron density ($\bar{n}_{e0} = N_{e0}\eta/V$), $h = 1 - \eta$.

The case $\eta = 1$ corresponds to the stationary regime, $N_e(t) \equiv N_{e0}$.

To conclude this section we estimate the quantity $\Delta N$ which we neglected when going from Eq. (3.2) to Eq. (3.3). Bearing in mind that according to the first of Eqs. (3.1) $\Delta N \sim \omega_0^{-1} N_0 N_{e0} \Omega_3$, and using (3.4), we see that $\Delta N/N_0 \sim \sqrt{N_{e0}/N_0}$). Equation (3.3) therefore validly describes the plasma oscillations if its degree of ionization is small, $N_{e0} \ll N_0$.

## 4. INSTABILITY OF THE ION-ACOUSTIC OSCILLATIONS

We show now that when there are ionization-recombination collisions in the plasma the ion-acoustic waves may turn out to be building-up leading thereby to an instability of the plasma.

Substituting Eq. (3.11) into Eq. (2.7) and using the fact fact that $|h| \ll 1$, we have

$$\frac{d^2 n_i}{dt^2} + \alpha(t)\frac{dn_i}{dt} + \omega^2(t) n_i = 0, \quad \alpha(t) = h\omega_0 \sin\omega_0 t + \nu_i, \quad (4.1)$$
$$\omega^2(t) = \omega_s^2\left(1 + \frac{a_0^2 k^2}{1 + a_0^2 k^2} h\cos\omega_0 t\right),$$

where $\omega_s^2 = k^2 V_s^2/(1 + a_0^2 k^2)$, $a_0^2 = T^*/4\pi e^2 \bar{n}_{e0}$. Also making the substitution

$$n_i' = n_i \exp\left\{\frac{1}{2}\int \alpha(t)\,dt\right\}$$

we write Eq. (4.1) in the form

$$\frac{d^2 n_i'}{dt^2} + p n_i' = 0, \quad p = \omega^2 - \frac{1}{2}\frac{d\alpha}{dt} - \frac{1}{4}\alpha^2 \quad (4.2)$$

(we shall neglect the term in p proportional to $\alpha^2$ in what follows bearing in mind that $|h| \ll 1$).

Equation (4.2) is analogous to the well-known equation for the oscillations of a system with periodically changing parameters (see, for instance,[7]). We can look for its general solution in the form

$$n_i' = e^{\mu_1 t}\xi_1(t) + e^{\mu_2 t}\xi_2(t), \quad (4.3)$$

where $\xi_{1,2}$ are periodic functions with period $2\pi/\omega_0$, while $\mu_{1,2}$ are some, in general, complex numbers for which $\mu_1 + \mu_2 = 0$. If Re $\mu_j > \nu_i/2$ (j = 1, 2) the corresponding term in (4.3) will increase exponentially with time (so-called parametric resonance). It is well known that parametric resonance occurs for values of the frequencies $\omega_0$ which lie close to

$$2\omega_s/n, \quad \omega_0 = 2\omega_s/n + \varepsilon, \text{ where } n\varepsilon/2\omega_s \ll 1 \quad (n = 1, 2, \ldots).$$

The most intensive form of parametric resonance occurs when n = 1; when n increases the build-up of the oscillations and the width of the instability region change as $h^n$.

Putting $\omega_0 = 2\omega_s + \epsilon$ in Eq. (4.2) and assuming that $\xi_1 \sim \cos(\omega_s + \frac{1}{2}\epsilon)t$, $\xi_2 \sim \sin(\omega_s + \frac{1}{2}\epsilon)t$ we have for $\mu_j$

$$\mu_{1,2} = \pm\left\{h^2\omega_s^2\left[1 - \frac{a_0^2 k^2}{2(1 + a_0^2 k^2)}\right]^2 - \varepsilon^2\right\}^{1/2}. \quad (4.4)$$

The width of the instability region is thus determined by the inequality

$$\varepsilon^2 < h^2\omega_s^2\left[1 - \frac{a_0^2 k^2}{2(1 + a_0^2 k^2)}\right]^2 - \nu_i^2. \quad (4.5)$$

The build-up increment $\gamma$ of the ion-acoustic oscillations is then determined by the formula $\gamma = \frac{1}{2}(\mu_1 - \nu_i)$. We draw attention to the fact that resonance turns out to be possible not for any small h, but only when $|h| > h_c$, where

$$h_c = \nu_i/\omega_s\left[1 - \frac{a_0^2 k^2}{2(1 + a_0^2 k^2)}\right]. \quad (4.6)$$

One can show that for resonance close to the frequency $2\omega_s/n$, the threshold value $h_c$ is proportional to $\nu_i^{1/n}$, i.e., increases with increasing n.

In the long wavelength case, $a_0 k \ll 1$, Eqs. (4.5) and (4.6) become

$$\varepsilon^2 < (h\omega_s)^2 - \nu_i^2, \quad h_c = \frac{\nu_i}{\omega_s}. \quad (4.7)$$

In the case of short-wavelength oscillations, $a_0 k \gg 1$ (ionic Langmuir waves) we have

$$\varepsilon^2 < \left(\frac{h\omega_s}{2}\right)^2 - \nu_i^2, \quad h_c = \frac{2\nu_i}{\omega_s}. \quad (4.8)$$

We have obtained Eqs. (4.4)—(4.8) in the simplest particular case of small amplitude ionization-recombination collisions, $|1 - \eta| \ll 1$. One can show that in the general case of an arbitrary amplitude of such oscillations parametric resonance is also possible which leads to a build-up of ion-acoustic waves. Ion-acoustic oscillations with a frequency $\omega$ close to $\omega_s = n\omega_r/2$, with $\omega_r$ the frequency of the ionization-recombination oscillations determined by Eq. (3.8) or (3.10) will then be excited.

We note that apart from the problem about the excitation of ion-acoustic oscillations the problem may also arise of the parametric excitation of another kind of oscillations of a partially ionized plasma—low-frequency hydrodynamic waves—when a constant uniform external electrical field is present, which were considered in[2]. One can, however, show that in contrast to the case of the ion-acoustic oscillations the parametric excitation of low-frequency hydrodynamic oscillations by ionization-recombination collisions in the plasma is impossible. This is connected with the fact that the damping decrement $\gamma$ for hydrodynamic oscillations in an electrical field is not small compared with the frequency in the system of reference moving together with the electrical current, $|\omega - \mathbf{k}\cdot\mathbf{u}_e| < \gamma$, where $\mathbf{u}_e$ is the average velocity of the directed electron motion.

We discuss, in conclusion, the characteristic values of the parameters of the plasma for which one must observe the instability of the ion-acoustic oscillations discussed here. For instance, for a plasma with an electron temperature $T_e \sim 3 \times 10^4$ degree (with $T_0/T_e \sim 0.3$), a density $n_0 \sim 10^{15}$ cm$^{-3}$ and a ionization coefficient $n_{e0}/n_0 \sim 10\%$ the frequency of the excited oscillations is $\omega_s \sim 3 \times 10^4$ sec$^{-1}$. In order that oscillations with such a frequency could occur the length of the plasma discharge must be of the order of 20 cm in the case of an argon-potassium plasma or 10 cm in the case of a cesium plasma.

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