PROPERTIES OF LONG AND SHORT RANGE CORRELATIONS IN THE CRITICAL REGION

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A rule for calculating the asymptotics of many-point correlations in the critical region is proposed for cases when the distance between some points is much larger than that between others. A simple physical interpretation of the phenomenon is presented. Variation of the correlation functions induced by magnetic fields of different strengths is derived. It is shown that at the phase transition point the magnetic-moment correlator decreases with the distance at least as rapidly as predicted by the Ornstein-Zernike formula. As a function of temperature, the correlator has an extremum near the critical point. The problem of universality of critical correlations is considered. The results obtained are compared with the exact solution for the plane Ising model. A new method for summation of "parquet" diagrams is proposed in the Appendix.

1. INTRODUCTION

IN a system of many particles, at temperatures close to critical, the effective interaction between the particles becomes strong, and no small parameters are left in the problem. In this region, only the exact relations are valid, and this raises the question of which of these relations can be used.

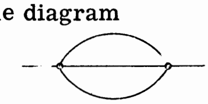
The present author has previously established[1] an analogy between the problem of phase transitions and relativistic quantum field theory. It was observed that a correspondence exists between the correlation functions of the system experiencing the phase transition, and the different propagation functions defined in field theory. The role of the quantum mass was played by the reciprocal correlation radius r_c. The ultraviolet divergences were cut off at momenta of the order of $r_0^{-1} \gg r_c^{-1}$ (where r_0 are the interatomic distances). Owing to the renormalizability of the interaction, the dependence on r_0 entered in the correlation functions in the form of an inessential factor, and therefore all the correlations were determined by a single characteristic radius r_c. To study the concrete properties of the different correlations, use was made of the S-matrix theory formalism, viz., the unitarity and analyticity relations, and also the consequences of the conservation laws (the Ward identities).

In this paper, using the same methods, we obtain a number of new results concerning the structure of the critical correlations. In Sec. 2 we explain the analytic properties of the correlation functions in the momentum representation, we carry out their analytic continuation (which will be needed later) to imaginary momenta, and obtain an inequality for the critical index. In Sec. 3 we investigate the condition for the compatibility of the unitarity relation and the Dyson equations. It is shown that the correlation function of the magnetic moments in the momentum representation $G(\mathbf{k}, \tau)$ has a minimum with respect to τ at a certain $\tau = \tau_0(\mathbf{k})$. This minimum can be observed experimentally in neutron scattering. In Sec. 4 we obtain the asymptotic forms of the many-point correlation functions in those cases when certain relative distances exceed others. A simple physical interpretation of such asymptotic forms is presented. In Sec. 5 we investigate the influence of a weak magnetic field on the correlations. In Secs. 6 and 7 we compare the results with the data known from an exact solution of the Ising model, and analyze the question whether the critical indices and the functions are universal.

2. UNITARITY OF THE GREEN'S FUNCTION AND INEQUALITY FOR THE CRITICAL INDEX

The method of analyzing phase transitions, proposed in[1], can be applied to any system in the critical region. In order to specify the terminology concretely, we shall consider a three-dimensional Ising model and use the ordinary diagram technique. We start with the analytic continuation of individual diagrams.

Let us consider the diagram

$\Sigma^{(1)}(q) =$  (2.1)

where the thin lines correspond to bare Green's functions $(\mathbf{k}^2 + \tau)^{-1}$ ($\tau \propto |T - T_c|/T_c$). We have

$$\Sigma^{(1)}(\mathbf{q}) = \frac{(-\lambda)^2}{(2\pi)^6} \int \frac{d^3k_1\, d^3k_2}{(\mathbf{k}_1^2+\tau)(\mathbf{k}_2^2+\tau)((\mathbf{q}-\mathbf{k}_1-\mathbf{k}_2)^2+\tau)}. \quad (2.1')$$

Here λ is the coupling constant, with $\lambda > 0$ corresponding to repulsion.

For an analytic continuation to imaginary q, i.e., $q^2 < 0$, it is necessary to go over to pseudo-Euclidean integrals. Let us consider this question in great detail. We compare expression (2.1′) with the following expression:

$$\frac{(-\lambda)^2}{(2\pi)^6} \int \frac{(-id^3k_1)(-id^3k_2)}{(\tau - k_1^2 - i0)(\tau - k_2^2 - i0)(\tau - (q - k_1 - k_2)^2 - i0)};$$

$$d^3k = dk_x dk_y dk_z, \quad k^2 = k_x^2 - k_y^2 - k_z^2.$$

if during the course of integration with respect to k_x we perform the standard rotation of the contour through 90°, which is permissible in the case of "spacelike" $q^2 = q_x^2 - q_y^2 - q_z^2 < 0$, we find that (2.2) coincides with (2.1′) only if $q^2 = -\mathbf{q}^2$. Since the integral (2.2) is an analytic function for all values of q^2, Eq. (2.2) gives the desired analytic continuation of (2.1′) to $q^2 < 0$. Analytically continued diagrams of the Euclidean technique give the usual diagrams of the relativistic quantum theory[2], where each line is set in correspondence with a propagator $i/(k^2 - \tau + i0)$, each bare vertex corresponds to $-i\lambda$, and all the momenta are pseudo-Euclidean. Repulsion in Euclidean space (i.e., the case when the initial diagrams are of alternating sign) corresponds to repulsion in the relativistic theory. All these follow from the fact that in integrals of the type (2.2) rotation of the contour denotes the substitutions $d^3k \to id^3k$, $k^2 \to -k^2$, and $\tau \to \tau - i0$.

Let us consider now the exact properties of the Green's function $G(q^2)$ obtained in the "relativistic" theory after the analytic continuation. The singularities of $G(q^2)$ are arranged in the following manner: when $q^2 < 0$, the function G is certainly regular and real, since $G(-q^2)$, in the sense of the analytic continuation, is the Fourier transform of the correlation function of the magnetic moments. Since the correlations fall off exponentially, or at least (at the very point of the phase transition) like the reciprocal power of the distance, their Fourier transforms have no singularities on the real axis. In the language of relativistic theory, the singularities at $q^2 < 0$ would correspond to a "ghost" state. Further, G has no complex singularities. The first singularity of G, as a rule, is a pole at $q^2 = m^2 = r_c^{-2}$. This pole corresponds to a single-particle state with mass m. In some cases $G(q^2)$ may have no poles, and this possibility is discussed in Sec, 7, but will not be considered now.

The singularity following the pole is a branch point. It is located at $q^2 = (2m)^2$ if one particle can virtually be transformed into two, and at $q^2 = (3m)^2$ if this transition is forbidden by the symmetry of the problem and only transformation of one particle into three is possible. The first case corresponds to the system below the transition point, or placed in an external magnetic field; the second case corresponds to a system above the critical point, in the symmetrical phase.

Further, there is an infinite set of branch points, connected with the transition of one particle into end particles and located at $q^2 = (nm)^2$. The jumps on the cuts drawn from these points are subject to the unitarity condition and are positive, owing to the hermiticity of the theory.

We introduce the renormalized Green's function in accordance with the formula

$$G(q^2) = Z(\tau) G_c(q^2).$$

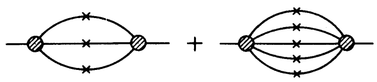
The renormalization constant Z is defined as the residue at the pole of $G(q^2)$ as $q^2 \to m^2$, thus,

$$G_c(q^2) \to \frac{1}{q^2 - m^2}, \quad q^2 \to m^2.$$

To calculate the imaginary part it is necessary to cut up the given diagram in all possible ways and to make in each section the substitution

$$(k^2 - m^2 + i0)^{-1} \to 2\pi\theta(k_0)\delta(k^2 - m^2).$$

the obtained unitarity condition for the renormalized Green's function[1)] is of the form[3,4]

$\operatorname{Im} G_c^{-1}(q^2) =$  $+ \cdots$ (2.3)

Here each line with a cross, carrying a momentum k, corresponds to the quantity $2\pi\theta(k_0)\delta(k^2 - m^2)$. The shaded blocks are set in correspondence with the renormalized vertex parts defined by the formula

$$\Gamma_n^c = (Z^{1/2})^n \Gamma_n$$

(where Γ_n is the sum of the Feynman diagrams with n ends, and the Green's functions corresponding to the ends are not included in Γ_n). For each of the Γ_n^c there is a separate unitarity condition. From all these equations it is possible to eliminate the mass m by introducing the dimensionless functions g and γ_n by means of the formulas

$$G_c(q^2) = m^{-2} g(q^2/m^2),$$
$$\Gamma_n^c(k_1 \ldots k_n) = m^{3-n/2}\gamma(k_1/m, \ldots, k_n/m). \qquad (2.4)$$

It is easy to verify that after substituting (2.4) in (2.3), the latter assume the form

$$\operatorname{Im} g^{-1}(\varkappa) = \sum \int \prod d^3\varkappa_i \, |\gamma_n(\varkappa_i)|^2 \delta\left(\varkappa - \sum \varkappa_i\right) \theta(\varkappa_{i0}) \delta(\varkappa_i^2 - 1) \qquad (2.5)$$

and the mass actually drops out of the equations.

The observed correlation functions are connected, as shown in[1], with non-renormalized quantities. We must require that in the limit as $\tau \to 0$ the dependence on τ, which enters in these quantities via the factors $Z(\tau)$ and $m(\tau)$, disappear. Since $m(\tau) \sim \tau^\beta$ and $Z \propto \tau^{\beta(2-\alpha)}$, where α and β are the critical indices, this requirement yields

$$g(\varkappa^2) \sim (\varkappa^2)^{-\alpha/2}, \quad \varkappa^2 \to \infty;$$
$$\gamma_n\left(\frac{\varkappa_1}{m}, \ldots, \frac{\varkappa_n}{m}\right) \sim \varkappa_1^{3-n\alpha} \varphi\left(\frac{\varkappa_2}{|\varkappa_1|}, \ldots, \frac{\varkappa_n}{|\varkappa_1|}\right), \quad \varkappa_i \to \infty. \qquad (2.6)$$

The asymptotic condition (2.6) makes it possible to obtain a limitation on the quantity α. For $g(\kappa)$ we can write, by virtue of (2.6) and (2.5), the dispersion relation

$$g(\varkappa^2) = \frac{1}{\varkappa^2 - 1} + \int_4^\infty \frac{\operatorname{Im} g(z)\, dz}{\varkappa^2 - z + i0}. \qquad (2.7)$$

By virtue of (2.5) we have $\operatorname{Im} g > 0$, therefore as $\kappa^2 \to \infty$ the integral (2.7) decreases no more rapidly than $1/\kappa^2$. Consequently $\alpha \leq 2$. This result is closely connected with the hermiticity of the Hamiltonian, which leads to the positiveness of Im g. The meaning of the inequality in the coordinate representation lies in the fact that the correlations at the transition point decrease more rapidly than in accordance with the Ornstein-Zernike formula (1/r). We note that a numerical calculation in the three dimensional Ising model yields a variation like $r^{-1.008}$ [2].

3. DEPENDENCE OF SPIN CORRELATIONS ON THE TEMPERATURE

The thermodynamic properties of the Ising model are determined by the dependence of the average energy E per spin on $\tau \propto |T - T_c|/T_c$. The function $E(\tau)$ is connected with the correlation function $G(\mathbf{k}, \tau)$ by the obvious equality

$$E(\tau) = \sum_{\mathbf{r}'} V_{\mathbf{r}\mathbf{r}'} \langle \sigma_{\mathbf{r}} \sigma_{\mathbf{r}'} \rangle = \int d^3k \, V_{\mathbf{k}} G(\mathbf{k}, \tau), \qquad (3.1)$$

where $\sigma_{\mathbf{r}}$ is the spin and $V_{\mathbf{r}\mathbf{r}'}$ is the spin interaction

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1)In [1] there was used a less convenient form for the unitarity condition, although containing the same information as (2.3).

potential. In the sum (3.1), the significant distances are of the order of the interaction radius r_0, and consequently the basic role in the momentum representation should be played by $|\mathbf{k}| \sim r_0^{-1}$. Therefore we cannot substitute in the integral a Green's function in the form

$$G(\mathbf{k}, \tau) = |\mathbf{k}|^{-\alpha} f(\mathbf{k}^2\tau^{-2\beta}). \tag{3.2}$$

This formula is valid only when $|\mathbf{k}| \ll r_0^{-1}$. In[1] we used the unitarity condition and showed that the singular part of $E(\tau)$ is proportional to $\tau^{3\beta-1}$. On the basis of formula (3.1) it is natural to assume that $G(\mathbf{k}, \tau)$ has for all values of $\mathbf{k}$ a singularity with respect to τ of the type $\tau^{3\beta-1}$. This result is already given in[1]. We shall show in Sec. 4 that this singularity actually arises and is a particular case of one general property of the correlation functions. We consider here another aspect of formula (3.1).

The specific heat C is determined by the integral

$$C \infty \frac{\partial E}{\partial \tau} \infty \int V_{\mathbf{k}} \frac{\partial G(\mathbf{k}, \tau)}{\partial \tau} d^3k. \tag{3.3}$$

In this integral there is a region of large momenta $|\mathbf{k}| \sim r_0^{-1}$, where (3.2) is not valid, but there is, in addition, a region of small momenta $|\mathbf{k}| \ll r_0^{-1}$. Let us examine the contribution made to C by this last region. We have

$$\begin{aligned} C \infty & \int\limits_{|\mathbf{k}| \ll r_0^{-1}} V_{\mathbf{k}} \frac{\partial G(\mathbf{k}, \tau)}{\partial \tau} d^3k + \int\limits_{|\mathbf{k}| \sim r_0^{-1}} V_{\mathbf{k}} \frac{\partial G(\mathbf{k}, \tau)}{\partial \tau} d^3k \\ &= V_0 \int\limits_{|\mathbf{k}| \ll r_0^{-1}} |\mathbf{k}|^{-\alpha} \frac{\partial}{\partial \tau} f(\mathbf{k}^2\tau^{-2\beta})\, d^3k + \int\limits_{|\mathbf{k}| \sim r_0^{-1}} \\ &= -2\beta V_0 \tau^{(3-\alpha)\beta-1} \int \frac{d^3\varkappa f'(\varkappa^2)}{|\varkappa|^{\alpha-2}} + \int\limits_{|\mathbf{k}| \sim r_0^{-1}} . \end{aligned} \tag{3.4}$$

In the last equality we have made, after differentiation with respect to τ, the change of variable $\kappa = \mathbf{k}\tau^{-\beta}$ and extended the integral with respect to κ to all of space. We had a right to do this, since, as will be shown now, the values $|\kappa| \sim 1$ predominate in the integral. To prove this, let us examine $f'(\kappa^2)$. Since the Green's function has at $\tau \to 0$ a singularity with respect to τ in the form $\tau^{3\beta-1}$, it follows that

$$f(\mathbf{k}^2\tau^{-2\beta}) \approx f(\infty) + \text{const}\cdot(\tau^{2\beta}\mathbf{k}^{-2})^{(3\beta-1)/2\beta},$$

and

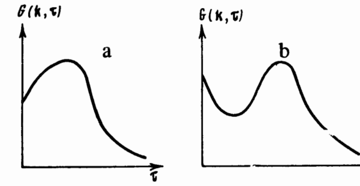
$$f'(\varkappa^2) \approx \text{const}\cdot(\varkappa^2)^{-(5\beta-1)/2\beta} \quad \text{as} \quad \varkappa^2 \to \infty. \tag{3.5}$$

At small κ^2, i.e., when $k^2 \ll \tau^{2\beta}$, the Green's function becomes a function of τ. We have

$$G(\mathbf{k}, \tau) \infty \tau^{-\alpha\beta}, \quad f(\varkappa^2) \approx \text{const}\cdot(\varkappa^2)^{\alpha/2}, \quad \varkappa \to 0.$$

Consequently, the integral with respect to κ behaves like $\int d^3\kappa/|\kappa|^{3+\alpha-1/\beta}$ at the upper limit and like $\int d^3\kappa$ at the lower limit. If $\alpha > 1/\beta$, as is the case for most systems, then the significant values of $|\kappa|$, as already stated, are of the order of unity. Under the same condition ($\alpha > 1/\beta$) the singularity turns out to be so strong, that a contradiction arises, namely, we know that the singularity of the specific heat is $\tau^{3\beta-2}$ and the singularity of the integral of (3.4) is $\tau^{(3-\alpha)\beta-1}$. Therefore the integral (3.4) should vanish[2]:

[2]The integral over the region $|\mathbf{k}| \sim r_0^{-1}$ does not cancel out the indicated singularity, since the singularity of $G(\mathbf{k},\tau)$ with respect to τ is $\tau^{3\beta-1}$.



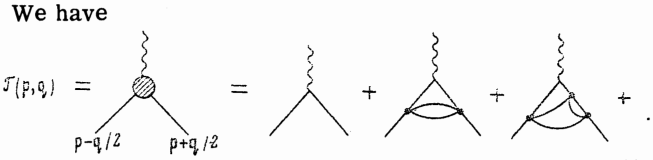
Plot of $G(\mathbf{k},\tau)$ at fixed k in the following cases: a – $\partial G/\partial\tau|_{\tau=0} > 0$, b – $\partial G/\partial\tau|_{\tau=0} < 0$.

$$\int \frac{d^3\varkappa}{|\varkappa|^{\alpha-2}} f'(\varkappa^2) = 0. \tag{3.6}$$

It follows from this equality that the derivative $\partial G(\mathbf{k}, \tau)/\partial\tau$ should reverse sign for all values of τ, and consequently there exists a value $|\mathbf{k}| = k_0(\tau) = x_0\tau^{\beta}$ such that $\partial G/\partial\tau = 0$. Depending on the sign of $\partial G/\partial\tau$ at $\tau = 0$, two types of dependence of $G(\mathbf{k}, \tau)$ on τ can be realized, as shown in the figure. We have taken into account the fact that at large τ we get $G \infty \tau^{-\alpha\beta}$. Since G is the scattering cross section integrated over the energy[5], these relations can be observed experimentally, but the existing experiments on critical scattering[2] are still too inaccurate.

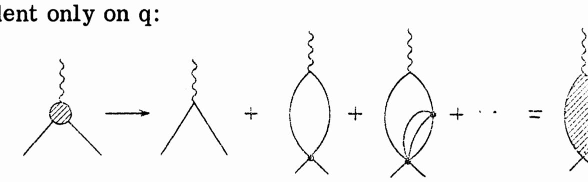
4. ASYMPTOTIC FORM OF MANY-POINT GREEN'S FUNCTION AND RULE OF COALESCENCE OF CORRELATIONS

In those cases when some momenta of the many-point diagrams exceed others, it is possible to obtain rules for the calculation of their asymptotic forms. We start with the vertex part of the energy density $\mathcal{T}(p, q)$, where $\mathcal{T}(p, q)G(p + q/2)G(p - q/2)$ is the Fourier transform of the correlation function $\langle \epsilon_r \sigma_{r_1} \sigma_{r_2} \rangle$ (where ϵ_r is the energy density and σ_r is the magnetic moment). We have



(4.1)

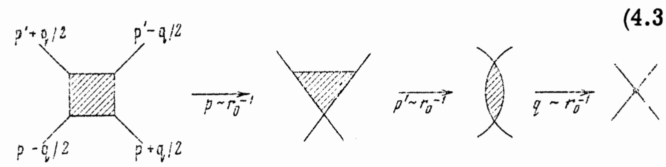
The lines in these diagrams correspond to exact Green's functions. Let us assume that $p \sim r_0^{-1} \gg q$ ($p \equiv |\mathbf{p}|$, $q \equiv |\mathbf{q}|$). All the lines that carry momenta p compress in this case to a point, and the vertex becomes dependent only on q:



(4.2)

(with $p \sim r_0^{-1}$). In perfect analogy, the four-point diagram experiences, following a successive increase of the external momenta, the evolution:

(4.3)



Such a transition will be called coalescence of correlations.

It seems natural that the polygons that enter in formulas of this type are universal. For example, the two-angle diagrams in (4.3) and (4.2) should be equal to the energy correlation function (4.1). Indeed, (4.2) is proportional to the correlation $\langle \epsilon_{\mathbf{r}} \sigma_{\mathbf{r}_1} \sigma_{\mathbf{r}_2} \rangle$ in the case when

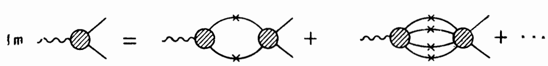
$$|\mathbf{r} - \mathbf{r}_{1,2}| \gg r_0 \sim |\mathbf{r}_1 - \mathbf{r}_2|. \tag{4.4}$$

The energy correlation function is given by

$$\langle \varepsilon_{\mathbf{r}} \varepsilon_{\mathbf{r}_1} \rangle = \sum_{\mathbf{r}_2} V_{\mathbf{r}_1 \mathbf{r}_2} \langle \varepsilon_{\mathbf{r}} \sigma_{\mathbf{r}_1} \sigma_{\mathbf{r}_2} \rangle \tag{4.5}$$

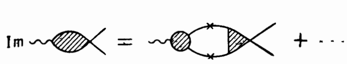
($V_{\mathbf{r}_1 \mathbf{r}_2}$ is the spin interaction with radius r_0). We are therefore dealing in both cases with correlations of two pairs of spins, and the spins making up the pair are close to each other, while the distance between pairs is large and equal to $|\mathbf{r} - \mathbf{r}_{1,2}|$. Obviously the dependence of the correlation on $|\mathbf{r} - \mathbf{r}_{1,2}|$ does not change if the spins of one pair are shifted relative to each other slightly (by a distance much smaller than $|\mathbf{r} - \mathbf{r}_{1,2}|$), and therefore when $|\mathbf{r} - \mathbf{r}_{1,2}| \gg |\mathbf{r}_1 - \mathbf{r}_2|$ the correlation $\langle \epsilon_{\mathbf{r}} \sigma_{\mathbf{r}_1} \sigma_{\mathbf{r}_2} \rangle$ decreases with increasing $|\mathbf{r} - \mathbf{r}_{1,2}|$ exactly as the correlation $\langle \epsilon_{\mathbf{r}} \epsilon_{\mathbf{r}_1} \rangle$ does with increasing $|\mathbf{r} - \mathbf{r}_1|$. It follows from this reasoning that it is not necessary to impose the condition $|\mathbf{r}_1 - \mathbf{r}_2| \lesssim r_0$ in order for the statement made above to be correct. Nonetheless, we begin the formal proof for the case when this condition is satisfied, and then eliminate this condition.

Let us examine the unitarity condition for $\mathscr{T}$:



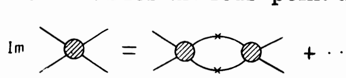
(4.6)

assuming that $p \sim r_0^{-1}$ and using the rule for coalescence of the correlations, we obtain



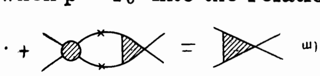
(4.7)

The unitarity condition for the four-point diagram



(4.8)

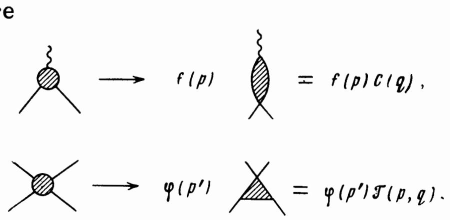
is transformed when $p \sim r_0^{-1}$ into the relation



(4.9)

Comparison of (4.9) and (4.6) shows that these quantities satisfy the same equations and therefore coincide.

We can now lift the limitation $p \gtrsim r_0^{-1}$, leaving only the conditions $p \gg \max(q, r_c^{-1})$. The only difference in the formulation of the rule for the coalescence of the correlations in this case is as follows. Since the essential momenta in the Feynman diagrams are of the order of $\max(q, r_c^{-1})$, we can, as before, contract the lines carrying the momentum p into a point. But when $p \ll r_0^{-1}$ one cannot neglect the dependence of these points on p. Therefore



(4.10)

where $p \gg \max(q, r_c^{-1})$, $p' \gg \max(q, p, r_c^{-1})$, and $\mathscr{T}(\mathbf{p}, \mathbf{q})$ and $C(\mathbf{q})$ are the same as above. The unknown functions can be found from the requirement that all the amplitudes have correct dimensionalities. Then (4.10) will yield the asymptotic forms of the dimensionless correlation functions.

We begin with $\mathscr{T}(\mathbf{p}, \mathbf{q})$. Its dimensionality is defined by the Ward identity

$$\mathscr{T}(\mathbf{p}, 0) = \frac{\partial G^{-1}(\mathbf{p})}{\partial \tau} \sim \frac{p^{\alpha}}{\tau} \sim p^{\alpha - 1/\beta}. \tag{4.11}$$

Consequently

$$\mathscr{T}(\mathbf{p}, \mathbf{q}) = p^{\alpha - 1/\beta}\, t\left(\frac{\mathbf{p}}{\tau^{\beta}}, \frac{\mathbf{q}}{\tau^{\beta}}\right). \tag{4.12}$$

From (4.10) and (4.12), with allowance for the fact that

$$C(\mathbf{q}) = q^{(3\beta-2)/\beta} c(\mathbf{q}\tau^{-\beta}),$$

it follows that

$$f(p) \sim \mathscr{T}/C \sim \mathrm{const}/p^{3-\alpha-1/\beta}. \tag{4.13}$$

We see therefore that the function $\mathscr{T}(\mathbf{p}, \mathbf{q})$ defined in (4.12) has the following form when p is large:

$$\mathscr{T}(\mathbf{p}, \mathbf{q}) \approx q^{(3\beta-2)/\beta} p^{\alpha+1/\beta-3} c(\mathbf{q}\tau^{-\beta}), \quad p \gg q, \tau^{\beta}. \tag{4.14}$$

This corresponds to the formula

$$t(\mathbf{x}, \mathbf{y}) \approx \mathrm{const}\cdot(|\mathbf{y}|/|\mathbf{x}|)^{(3\beta-2)/\beta} c(\mathbf{y}), \quad |\mathbf{x}| \gg |\mathbf{y}|. \tag{4.15}$$

When q = 0 we obtain the already known results

$$\mathscr{T}(\mathbf{p}, 0) \approx \mathrm{const}\cdot\tau^{3\beta-2} p^{\alpha+1/\beta-3}. \tag{4.15$'$}$$

The asymptotic form of $\mathscr{T}(\mathbf{p}, \mathbf{q})$, in the case when the momentum of one of the solid lines is $p \ll q$ (we have replaced $\mathbf{p} - \mathbf{q}/2$ by $\mathbf{p}$), can also be determined. It is simplest to use physical considerations. In the coordinate representation the given values of the momenta represent the configuration when there is a group of three closely located spins and one spin far from them. From the foregoing reasoning it is clear that the correlations do not change if the group of three spins is replaced by a single spin. (This can be proved formally with the aid of unitarity.) Consequently

$$\mathscr{T}(\mathbf{p}, \mathbf{p}+\mathbf{q}) G(\mathbf{p}) G(\mathbf{p}+\mathbf{q}) \approx h(\mathbf{q}) G(\mathbf{p}), \quad q \gg p$$

or

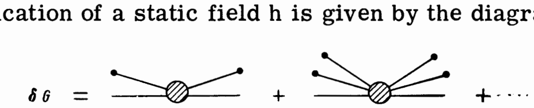
$$\mathscr{T}(\mathbf{p}, \mathbf{p}+\mathbf{q}) \approx \mathrm{const}\cdot q^{\alpha-1/\beta}, \quad q \gg p. \tag{4.16}$$

(Dimensionality considerations have been used in the last equality.)

It is possible to investigate analogously the asymptotic forms of all other correlations. These asymptotic forms are determined by two known critical indices and do not require introduction of new constants.

5. WEAK MAGNETIC FIELD

In this section we use the previously derived rule of correlation coalescence to determine the change of the correlation in a system under the influence of a magnetic field. The change of the Green's function following application of a static field h is given by the diagrams



(5.1)

The shaded blocks correspond to the amplitudes Γ_n introduced above. Thus,

$$\partial^2 G^{-1} / \partial h^2 = \Gamma_4(\mathbf{p}, \mathbf{p}, 0) G^2(0) + \ldots \quad (5.2)$$

This identity sets the dimensionality of the magnetic field:

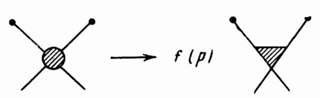
$$h \sim (\Gamma_4 G^3)^{-1/2} \sim p^{(3+\alpha)/2}. \quad (5.3)$$

Formula (5.3) is well known[2]. It only tells us the dimensionless combinations of h and p on which G depends. Our problem consists of finding the explicit dependence of G on h for small values of h.

When $h \ll \tau^{\beta(3+\alpha)/2}$ and for arbitrary values of p, the solution is trivial, and it follows from (5.2) that the expansion of $G^{-1}(p, h)$ begins with terms proportional to h^2. Further, in the region $h \gg [\max(p, \tau^\beta)]^{(3+\alpha)/2}$ an important role is played in the diagrams for Γ_4 only by virtual momenta of order $h^{2/(3+\alpha)}$, therefore Γ_4 depends only on h. In this case we obtain the known[2] result:

$$G(p, h) \infty h^{-2\alpha/(3+\alpha)}. \quad (5.4)$$

We now examine the region $p \gg h^{2/(3+\alpha)} \gtrsim \tau^\beta$. In spite of the relative weakness of the magnetic field, the dependence on this field does not drop out of Γ_4, since momenta of order of τ^β and not of order of p are of importance in the diagrams. All the lines carrying a momentum p must be contracted into a point that depends on p. The obtained contracted diagram depends only on h. We must find this dependence. Taking into account the correlation coalescence rule, this can be readily done. We have



(5.5)

i.e., $\Gamma_4 = f(p)\mathcal{T}(h, \tau)G^2(h, \tau)$ (where $\mathcal{T}$ is the energy-density vertex at zero values of the external momenta). From dimensionality considerations (see formulas (4.12) and (3.2)) we get

$$\mathcal{T} = \tau^{\alpha\beta-1} t(h\tau^{-\beta(3+\alpha)/2}), \quad G = \tau^{-\alpha\beta} g(h\tau^{-\beta(3+\alpha)/2}). \quad (5.6)$$

When $h \gg \tau^{\beta(3+\alpha)/2}$ the dependence on τ drops out of (5.6), and it turns out that $\Gamma_4 \approx f(p) h^{-2(1+\alpha\beta)/\beta(3+\alpha)}$. Therefore

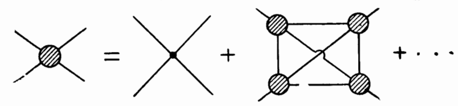
$$G^{-1}(p, h) \approx G^{-1}(p, 0)\left[1 + \text{const}\cdot\left(\frac{h^{2/(3+\alpha)}}{p}\right)^{3-1/\beta}\right]. \quad (5.7)$$

This formula contains no new critical indices. A result similar to (5.7) but for a plane Ising model is given in Sec. 7.

6. NON-UNIVERSALITY OF THE CRITICAL BEHAVIOR

The magnitude of the critical indices in the form of the dimensionless functions are determined both by the type of symmetry, which is violated in the phase-transition process, and also by the concrete structure of the interaction. Since at a given symmetry the characteristics of the interaction (the bare coupling constants) do not enter in the unitarity conditions, their influence on the Green's function can appear either via the subtraction constants in the dispersion relations, or in the form of a non-uniqueness of the solution of the unitarity and analyticity equations.

However, owing to the asymptotic condition (2.6), the dispersion relations must apparently be written without subtractions. In diagram language, this corresponds to the fact that in the expansion of the vertex part Γ_4



(6.1)

it is possible, in the case of small p, to discard the first term if $\Gamma_4 \gg 1$ when $p \ll r_0^{-1}$. If $\Gamma_4 \ll 1$, then this first term is cancelled out by the contribution of the momenta of order r_0^{-1} in the right side of (6.1), and can again be neglected.

We arrive at the conclusion that the system of equations for the correlation functions knows nothing concerning the bare interaction. If its solution were unique, then this would imply that the critical indices and functions do not depend on the bare constants. However, computer calculations have revealed that in Heisenberg ferromagnets with different values of the spin s, the critical indices are different[2]. This means that the solution of the system is not unique and the correct branch is determined by the conditions for continuity in the region of momenta of order of r_0^{-1}.

These conditions, of course, contain s. From the theoretical point of view, how can the non-uniqueness appear? There are two known types of ambiguities in S-matrix theory. First, obviously, it is possible to specify an arbitrary mass spectrum in the system; namely, there can exist not only one type of particle with mass m, but an arbitrary set with different masses. This set is determined by the bare interactions. As indicated in an analogous situation by Gribov and Migdal[6], this type of ambiguity leads apparently to a jumplike dependence on the coupling constants (i.e., in our case on s), since the process of formation of the bound states is also jumplike. However, in our problem there can appear another ambiguity, which can cause a smooth dependence on s. This is an ambiguity of the CDD type[7].

The unitarity condition for the Green's function G determines the imaginary part of G^{-1}. The dispersion relation for G^{-1} is of the form

$$G^{-1}(q^2) = \frac{q^2 - m^2}{\pi} \int_{9m^2}^{\infty} \frac{\operatorname{Im} G^{-1}(z)\, dz}{(z - q^2 - i0)(z - m^2)} + \frac{q^2 - m^2}{\pi} \sum \frac{R_n}{(z_n - m^2)(z_n - q^2)} + c(q^2 - m^2) \quad (6.2)$$

(where $c \geq 0$ and $R_n \geq 0$). Using the asymptotic condition (2.6), we find that $c = 0$ when $\alpha < 2$ (the integral in (6.2) increases more slowly than q^2 as $q^2 \to \infty$). However, R_n and z_n can be arbitrary, and this creates an ambiguity that is limited only after continuity is established in the region of momenta of the order of r_0^{-1}. Thus, we verify that the dependence of the critical indices on the bare parameters does not contradict our approach.

7. COMPARISON OF THE RESULTS WITH THE EXACT SOLUTION OF THE PLANE ISING MODEL

In this section we verify our results with the aid of the plane Ising model. Unfortunately, even in the plane Ising model it is impossible to calculate to conclusion many of the important quantities. Thus, we are unable to verify a formula of the type (5.7), which in the Ising model, where $\alpha = 7/4$, $\beta = 1$, and the number of dimensions is equal to 2, takes the form

$$G^{-1}(p, h) \approx p^{7/4}(1 + \text{const}\cdot h^{8/15}/p), \quad p \gg h^{8/15} \gg \tau. \qquad (7.1)$$

Formulas of the type (3.4) and (4.15), as already noted in[1], have been confirmed. In[9] it is shown that in the Ising model

$$G^{-1}(p, \tau) \approx p^{7/4}\left[1 + \text{const}\cdot\frac{\tau}{p}\ln\frac{p}{\tau}\right], \quad p \gg \tau, \qquad (7.2)$$

which agrees with formula (4.15).

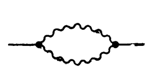
To verify the correlation-coalescence rule it is necessary to calculate the function $\langle \epsilon_r \sigma_{r_1} \sigma_{r_2} \rangle$ and other many-point correlators. Such a calculation is possible in the Ising model, but it has not yet been carried out, so that a verification is still impossible.

At the same time, the results calculated in this model yield interesting and somewhat unexpected information. We refer to the asymptotic forms of the correlation functions when $r \gg r_c$. It was shown in[1] that these asymptotic forms are determined by the first singularities in q^2 in the momentum representation. The use of the unitarity condition has made it possible to obtain the pre-exponential factor in the asymptotic form when $r \gg r_c$.

Above the transition point, according to Kadanoff[10], the spin correlations in the Ising model have an asymptotic form $r^{-1/2}\exp(-r/r_c)$. This means, according to[1], that the first singularity of the Green's function $G(q^2)$ is a pole at $q^2 = m^2 = r_c^{-2}$, as was indeed assumed in Sec. 2 of this paper. However, below the transition, the calculation[10] has led to an asymptotic form $r^2\exp(-2r/r_c)$. According to[1], this indicates that the function G has no pole, but only a two-particle branch point. This raises the question whether this fact contradicts our initial premises concerning the existence of a single-particle state. We propose a hypothesis that lifts this apparent contradiction. The absence of a pole in $G(q^2)$ suggests that some selection rules forbid the single-particle state in the Lehmann expansion. The simplest possibility consists in the fact that a single-particle state with a nonzero spin or with negative parity is produced. Since $G(q^2)$ is a correlation of two (spatially) scalar quantities, a transition of the type

(7.3)

(where the wavy line is the line of the particle existing the theory, and the solid line corresponds to the magnetic moment σ_r) is forbidden by the momentum (or parity) conservation law. At the same time, a transition of the type


(7.4)

is possible, and this indeed produces the two-particle singularity.

Depending on whether it is the angular momentum or the parity which forbids the pole of the Green's function, the term following (7.4) will be of the three- or four-particle type, since the angular momentum of three particles with nonzero spin can be equal to zero, while the parity of three identical particles with negative internal parities and with zero total angular momentum must be negative. Calculations[10] show that the term following $\exp(-2r/r_c)$ is asymptotically $\exp(-4r/r_c)$. This means apparently that the second of the possibilities considered by us is realized. We note, finally, that factorization of the coefficients in the asymptotic form with $r \gg r_c$, noted by Johnson[11], is valid regardless of whether the pole or the branch point is the leading singularity.

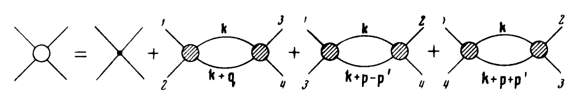
The correlation coalescence rule for the four-dimensional Ising model is verified in the appendix.

Thus, all the relations existing in the Ising model do not contradict our approach.

APPENDIX

Inasmuch as the correlation coalescence rule cannot be regarded as rigorously proven, it is advantageous to verify in some exactly solved problem. Let us consider the four-dimensional Ising model. This model imitates the critical behavior of uniaxial ferroelectrics[12]. In this model, the fluctuations interact logarithmically, and the problem reduces to a summation of ''parquet'' diagrams[12]. Let us calculate the four-point diagram (4.3) at momenta $p' \gg p \gg q$. The summation of ''parquet'' diagrams by the methods of[13] is in this case too cumbersome, and we therefore propose a modification of these methods.

The logarithmic integrals are due to two-particle cross sections[13]. Separating in each diagram that two-particle cross section in which the integration momentum is the smallest, we represent the scattering amplitude in the form[13]



(A.1)

where $p_{1,2} = q/2 \pm p$ and $p_{3,4} = q/2 \pm p'$. The shaded blocks are the scattering amplitudes in which all the virtual momenta are $\kappa \gg k$, where k is the integration momentum in (A.1) ($\kappa \equiv |\boldsymbol{\kappa}|$, $k \equiv |\mathbf{k}|$). The last two terms produce a logarithmic integral only because of the region $k \gg p' \gg p \gg q$. Let

$$\xi = \ln\frac{\Lambda}{p'}, \quad \zeta = \ln\frac{\Lambda}{p}, \quad \eta = \ln\frac{\Lambda}{q}, \quad x = \ln\frac{\Lambda}{k}.$$

Then the contribution of the last two terms in $\Gamma(\xi \le \zeta \le \eta)$ is given by

$$2\int_0^{\xi} dx\,\Gamma^2(x, x, x). \qquad (A.2)$$

We now consider the second term in (A.1). The logarithmic integrals occur in it as the result of three regions:

$$1)\ k \gg p' \gg p \gg q, \quad 2)\ p' \gg k \gg p \gg q, \quad 3)\ p' \gg p \gg k \gg q. \qquad (A.3)$$

Recognizing that $\kappa_i \gg k$, we can easily write the contributions of all three regions:

$$3)\ \int_{\zeta}^{\eta} dx\,\Gamma(\xi, x, x)\,\Gamma(\zeta, x, x). \qquad (A.4)$$

adding (A.2) and (A.4), we find an equation for

$$\Gamma(\xi, \zeta, \eta) = -\gamma + 3\int_0^{\eta} \Gamma^2(x, x, x)\,dx + \qquad (A.5)$$

$$+\int_{\xi}^{\zeta}\Gamma(\xi,x,x)\Gamma(x,x,x)\,dx+\int_{\zeta}^{\eta}\Gamma(\xi,x,x)\Gamma(\zeta,x,x)\,dx.$$

When $\xi = \zeta = \eta$ we obtain the equation of[13], which was already used in[12] to calculate $\Gamma(\xi, \xi, \xi)$. The solution of (A.5) is obtained immediately:

$$\Gamma=-\frac{2\gamma}{(1+3\gamma\xi)^{2/3}(1+3\gamma\zeta)^{1/3}}+\frac{\gamma(1+3\gamma\eta)^{1/3}}{(1+3\gamma\xi)^{2/3}(1+3\gamma\zeta)^{2/3}}. \quad (A.6)$$

For comparison with the correlation coalescence rule, we note that[12] when $p \sim q$ $\mathscr{T}(p, q)$ takes the form

$$\mathscr{T} \infty [1+3\gamma\ln(\Lambda/p)]^{-1/3}=(1+3\gamma\zeta)^{-1/3}.$$

Putting $p \sim q$ and $\eta = \zeta$ in (A.6), we get $\Gamma(\xi, \zeta, \zeta) = \varphi(\xi)\mathscr{T}(\zeta, \zeta)$. Similarly, $\mathscr{T}(\zeta, \eta) = f(\xi)C(\eta)$ when $\eta \gg \xi \gg 1/\gamma$, where $C(\eta) \approx \gamma^{-1}(3\gamma\eta)^{1/3}$ is the specific heat. Thus, in this model the correlation-coalescence rule is already satisfied. We note that the method proposed in this appendix for summing the "parquet" is suitable and convenient for all practical problems.

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