CONCERNING THE ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON

I. M. TERNOV, V. G. BAGROV, V. A. BORDOVITSYN and O. F. DOROFEEV

Moscow State University

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The anomalous magnetic moment of an electron moving in a stationary and uniform magnetic field is investigated in the first order of the expansion in terms of the fine-structure constant. It is shown that the moment is a complex function of the field strength and of the particle energy.

S is well known, a theoretical explanation of the nature of the anomalously magnetic moment of the electron, and a calculation of its magnitude, were first presented by Schwinger[1]. By considering part of the energy of the vacuum interaction of the electron, which appears in the presence of an electric field, Schwinger has shown that the terms linear in the magnetic-field intensity vector lead in the nonrelativistic approximation to a change of the g-factor of the electron.

The calculations, which were first made by perturbation theory with accuracy to terms of order $\alpha = e^2/\hbar c$[1], were subsequently continued[2-4] up to α^3. It then turned out that an electron in an external constant magnetic field behaves as if it had a static magnetic moment equal to

$$\mu = \mu_B\left(1 + \frac{\alpha}{2\pi} - 0{,}328\frac{\alpha^2}{\pi^2} + 0{,}13\frac{\alpha^3}{\pi^3}\right) \quad (1)$$

($\mu_B = -\mu_0 = -e\hbar/2mc$ is the Bohr magneton), where e is the positive elementary charge and m is the rest mass of the electron.

In connection with the great possibilities afforded by the highly developed precision experimental technique of measuring the magnetic moment of the electron[5], it is of interest to investigate in greater detail the energy of the vacuum interaction.

It should be noted first that the magnitude of the magnetic moment of the electron is in general a rather complicated function dependent on the intensity of the external magnetic field. The first to point out this circumstance was Gupta[6], who considered in first order of perturbation theory in α not only the linear term but also the higher terms of the expansion of the magnetic moment in the characteristic parameter a^{-1}, where $a = H_0/2H$, $H_0 = m^2c^3/e\hbar = 4.41 \times 10^{13}$ Oe.[1)] This circumstance may be of interest from the point of view of measuring the magnetic moment of the electron, since the correction discarded in (1), which depend on the field intensity in first order in α, can be noticeable against the background of the higher powers of the perturbation-theory expansion in the square of the charge.

It must be further emphasized that in all the foregoing papers they considered the nonrelativistic approximation of the problem. This is particularly pronounced in[6], where the electron in the external magnetic field is taken in the "ground" state, in which the energy of the orbital motion and the energy of the interaction of the magnetic moment with the external field cancel each other ($E = mc^2$) (see also[8]).

In this connection we can expect, in the case of a relativistic electron, the magnitude of its magnetic moment to be not only a function of the field intensity, but also dependent on the electron energy.

Finally, we note that in all the indicated investigations they considered only the case of a "weak" magnetic field, when the field intensity H is limited to the range $0 < H \ll H_0$. In spite of the fact that the critical value H_0 greatly exceeds in magnitude all the hitherto attained magnetic fields[2)], an investigation of the behavior of the magnetic moment of the electron under the conditions of a "strong" field may nevertheless be of interest from the point of view of the general theory.

In this paper we wish to investigate the vacuum magnetic moment of an electron in first order of perturbation theory in terms of α without the aforementioned limitations. We shall follow the method proposed by Luttinger[8] and developed for nonrelativistic particles in[10].

1. RADIATIVE CORRECTIONS TO DIRAC'S EQUATION

Dirac's equation for an electron moving in an external magnetic field, with allowance for the interaction of the electron with the electromagnetic vacuum, can be written in the form[11]

$$(\gamma_\mu P_\mu - m)\Psi(x_\mu) = \int M(x_\mu, y_\mu)\Psi(y_\mu)\,d^4y,$$

where $\gamma^\mu = -i\rho_3\alpha_\mu$, $\alpha_\mu = (\alpha, iI)$ are Dirac matrices, $P_\mu = i\partial/\partial x_\mu + e(A_\mu + A_\mu^{vac})$ is the kinetic momentum ($c = \hbar = 1$), and the mass operator in first order of the expansion with respect to the fine-structure constant[11,12]

$$M(x, y) = ie^2 \sum_{k,m} \gamma^k S^c(x, y)\gamma^m D_{km}{}^c(y - x)$$

contains the Greenian of the Dirac equation with allowance for the external magnetic field $S^c(x, y)$, and also the Greenian for the photons $D^c_{km}(y - x)$.

Let us consider a constant and homogeneous external

[1)]The constant H_0 is numerically equal to the intensity of the electric field on the boundary of the so-called vacuum radius of the electron [7].

$$r_{vac} = \sqrt{\frac{e^2}{mc^2}\frac{\hbar}{mc}}.$$

[2)]The question on the possible occurrence of superstrong magnetic fields is discussed in connection with the problem of collapsing protostars [9].

magnetic field directed along the z axis of a Cartesian coordinate system. Then the stationary Dirac equation with allowance for the radiative corrections can be readily reduced to the form (see[13,10])

$$(E-\mathcal{H})\Psi(\mathbf{r})=\hat{R}\Psi(\mathbf{r})=\int\mathcal{K}(\mathbf{r},\mathbf{r}')\Psi(\mathbf{r}')\,d^3\mathbf{r}', \tag{2}$$

where $E = c\hbar K$ is the energy of the electron and the Hamiltonian operator $\mathcal{H}$ contains only the external field

$$\mathcal{H}=c(\alpha\mathbf{P})+\rho_3 mc^2.$$

We have discarded here terms containing A^{vac} and not pertaining to the anomalous moment of the electron (see[11,13]), and have gone over to the usual representation of the Dirac matrices $\boldsymbol{\alpha}$ and to the usual system of units.

The kernel in the right side of a Dirac equation

$$\mathcal{K}(\mathbf{r},\mathbf{r}')=\frac{e^2}{4\pi^2}\sum_{n',\varepsilon}\int\frac{d^3\varkappa}{\varkappa}\frac{e^{i\varkappa\mathbf{r}}\alpha_\mu\Psi_{n'}(\mathbf{r})\Psi_{n'}^{+}(\mathbf{r}')\alpha_\mu e^{-i\varkappa\mathbf{r}}}{K-\varepsilon(K_{n'}+\varkappa)}$$

presupposes summation over all the intermediate states, including the electronic state ($\epsilon = 1$) and the positronic state ($\epsilon = -1$). Regarding the right side of (2) as a perturbation, we note that the radiative corrections to the energy, and also the transition probability, are completely determined by the matrix elements of the operator $\hat{R}$, which characterizes the effective energy of the vacuum interaction.

Calculation of the matrix elements of the operator $\hat{R}$ can be carried out with the aid of the exact wave functions of an electron moving in a constant and uniform magnetic field (see, for example,[13,14]). We confine ourselves henceforth to the motion of an electron in the plane of the orbit of revolution. Then the energy of the unperturbed state depends only on the principal quantum number n = 0, 1, 2, ...:

$$E=c\hbar K=c\hbar\sqrt{k_0^2+2\gamma n},\quad k_0=mc/\hbar,\quad \gamma=eH/c\hbar.$$

The radiative correction to the energy is determined by the diagonal matrix element of the operator $\hat{R}$:

$$W_{\zeta\zeta'}=\int\Psi_{n\zeta'}^{+}(\mathbf{r})\hat{R}\Psi_{n\zeta}(\mathbf{r})\,d^3\mathbf{r}$$
$$=\int\Psi_{n\zeta'}^{+}(\mathbf{r})\mathcal{K}(\mathbf{r},\mathbf{r}')\Psi_{n\zeta}(\mathbf{r}')\,d^3\mathbf{r}\,d^3\mathbf{r}', \tag{3}$$

with the indices ζ, $\zeta' = \pm 1$ characterizing the dependence of the energy of the vacuum interaction of the electron on the initial (ζ) and final (ζ') orientations of the spin. As a result of simple calculations, the details of which can be found in a number of earlier papers (see, for example,[14]), we obtain for the radiative correction to the energy (3) the expression

$$W_{\zeta\zeta'}=W_{\zeta\zeta'}^{(1)}+W_{\zeta\zeta'}^{(2)}=\frac{e^2}{4\pi}\sum_{n',\varepsilon}\int_0^\infty\int_0^\pi\frac{\varkappa d\varkappa\sin\theta d\theta}{K-\varepsilon(K'+\varkappa)}[F_1(\varkappa,\theta)+F_2(\varkappa,\theta)], \tag{4}$$

in which the functions F_1 and F_2 are expressed in terms of the well known Laguerre functions

$$I_{n,n'}(y)=\sqrt{\frac{n'!}{n!}}e^{-y/2}y^{(n-n')/2}L_{n'}^{n-n'}(y)$$

of argument $y = \varkappa^2\sin^2\theta/2\gamma$. Here

$$F_1(\varkappa,\theta)=(AA'^{+}+BB'^{+})\left\{\left(1-\varepsilon\frac{k_0^2}{KK'}\right)[I_{n,n'-1}^2(y)+I_{n-1,n'}^2(y)]\right.$$
$$\left.-\varepsilon\frac{k_0^2}{KK'}[I_{n-1,n'-1}^2(y)+I_{n,n'}^2(y)]-2\varepsilon\frac{2\gamma\sqrt{nn'}}{KK'}I_{n,n'}(y)I_{n-1,n'-1}(y)\right\}, \tag{5}$$

$$F_2(\varkappa,\theta)=(BB'^{+}-AA'^{+})\left[\frac{k_0}{K}\left(1-\varepsilon\frac{K}{K'}\right)\frac{n'-n}{y}-\varepsilon\frac{K}{K'}\right]$$
$$\times[I_{nn'}^2(y)-I_{n-1,n'-1}^2(y)]. \tag{6}$$

The spin coefficients A and B which enter in the expression for the wave functions $\Psi_{n\zeta}$ satisfy the orthonormality requirement

$$AA'^{+}+BB'^{+}=\delta_{\zeta\zeta'},$$

and their concrete form is determined by a suitable choice of spin integral of motion (see[14,15]). It is seen from (4)—(6) that we should relate the quantity $W_{\zeta\zeta'}^{(2)}$ directly with the presence of the anomalous magnetic moment of the electron. Indeed, just this quantity, which is proportional to the factor $(AA'^{+} - BB'^{+})$, either depends explicitly on the spin orientation (spin polarization along the field), or gives rise to transitions with a change of this orientation (longitudinal polarization), depending on the choice of the spin polarization. Obviously, $W_{\zeta\zeta'}^{(1)} \sim \delta_{\zeta\zeta'}$ does not possess these properties.

We note further that part of the vacuum operator R, connected with the anomalous magnetic moment, is usually replaced in the nonrelativistic approximation by the operator $\hat{R}' = \mu\rho_3(\boldsymbol{\sigma}\cdot\mathbf{H})$ (see[11,13]), where the constant quantity $\mu = -\alpha\mu_0/2\pi$ is interpreted as the anomalous moment of the electron. As already noted by Pauli[16], such a generalization of the Dirac equation is covariant, and it can therefore be assumed that in the general case the operator substitution $\hat{R}\to\hat{R}'$ remains valid, but μ can now turn out to be dependent on the intensity of the magnetic field and on the energy of the electron. Comparing the matrix elements of the operator $\hat{R}'$

$$W_{\zeta\zeta'}'=\int\Psi_{\zeta}^{+}(\mathbf{r})\hat{R}'\Psi_{\zeta}(\mathbf{r})\,d^3\mathbf{r}=\mu(AA'^{+}-BB'^{+})H$$

with the energy of the vacuum interaction $W_{\zeta\zeta'}^{(2)}$ (see formulas (4)—(6)), we can arrive at the conclusion that, indeed, the operator substitution $\hat{R}\to\hat{R}'$ is possible if μ is chosen in the form

$$\mu=\frac{W_{\zeta\zeta'}^{(2)}}{H(AA'^{+}-BB'^{+})}=-\frac{\alpha}{2\pi}\mu_0 f(n,a).$$

Summing in (4) over the sign factor $\epsilon = \pm 1$, we obtain for the function f(n, a) (see also[10]):

$$f(n,a)=-8a\sum_{n'}\int_0^\infty\int_0^{\pi/2}\frac{xdx\sin\theta d\theta}{(\sqrt{\xi+x^2\cos^2\theta}+x)^2-1}$$
$$\times\left[1+\frac{\xi-1+x^2\sin^2\theta}{x\sin^2\theta\sqrt{\xi+x^2\cos^2\theta}}\right][I_{n,n'}^2(z)-I_{n-1,n'-1}^2(z)], \tag{7}$$

with $a = k_0^2/2\gamma = H_0/2H$, $\xi = (n' + a)/(n + a)$, and $z = (n + a)x^2\sin^2\theta$. It is important to emphasize that (7) contains no divergences and is finite in the entire range of variation of the energy and of the external field.

The case of the "ground" state of the electron (n = 0) is in a certain sense special. In this state, the electron spin can be oriented only strictly opposite to the direction of the magnetic field (A = 0, B = 1), and therefore the vacuum interaction cannot be subdivided into parts by the method indicated above. In this case the entire energy (4) is connected with the magnetic moment of the electron (see[8]) and for the function f(0, a) we obtain

$$f(0,a) = -8a\sum_{n'=0}^{\infty}\int_0^{\infty}\int_0^{\pi/2}\frac{x\,dx\sin\theta\,d\theta}{(\sqrt{\xi}+x^2\cos^2\theta+x)^2-1}$$

$$\times\left[\frac{x}{\sqrt{\xi'+x^2\cos^2\theta}}I^2_{0,n'-1}(z')+\left(2+\frac{2x^2\sin^2\theta+\xi'-1}{x\sin^2\theta\sqrt{\xi'+x^2\cos^2\theta}}\right)I^2_{0,n'}(z')\right], \quad (8)$$

with

$$\xi' = \xi_{n=0} = 1 + n'/a, \quad z' = z_{n=0} = ax^2\sin^2\theta.$$

This expression contains a divergence which is not connected with the magnetic field and which can be eliminated by mass renormalization.

We now consider particular cases of the formulas for the anomalous moment of the electron.

2. ANOMALOUS MAGNETIC MOMENT OF AN ELECTRON IN A WEAK MAGNETIC FIELD

The case of a weak magnetic field ($H \ll H_0$, $a \to \infty$), as applied to the "ground" state of the electron ($n = 0$), was investigated in[8], where it was shown that $f(0, \alpha) \to 1$ as $a \to \infty$. The "excited" states of the electron ($n \neq 0$) were investigated in[10], and it was likewise concluded that $f(n, a) \to 1$ as $a \to \infty$, regardless of the electron energy (the quantum number n). Let us return to this problem in order to introduce some refinements.

In accordance with[10], it is convenient to replace the sum over the quantum number n in (7) and (8) by an integral, by introducing a δ-function:

$$\sum_{n'} f_{n'}(n') = \sum_{n'}\int_{-\infty}^{\infty} f_{n'}(k)\,\delta(k-n')\,dk,$$

since the sums with the Laguerre functions can be readily evaluated:

$$\sum_{n'=0}^{\infty} I^2_{n,n'}(x)\,e^{-in'\varphi} = e^{-i(n\varphi+x\sin\varphi)}I_{n,n}\left(4x\sin^2\frac{\varphi}{2}\right).$$

With the aid of these relations we obtain

$$f(n,a) = \frac{4a}{\pi}\int_0^{\infty}x\,dx\int_0^{\pi/2}\sin\theta\,d\theta\int_{-\infty}^{\infty}d\eta\int_{-\infty}^{\infty}d\lambda\frac{e^{i\lambda\eta}F}{(\sqrt{1+\eta+x^2}+x)^2-1},$$

where the function F is given by

a) $n \neq 0$; $\quad F = \dfrac{yL^1_{n-1}(y)}{n}e^{-\Phi}\left[1+\dfrac{\eta+2x^2\sin^2\theta}{x\sin^2\theta\sqrt{1+\eta+x^2}}\right],$

$$y = 4(n+a)x^2\sin^2\theta\sin^2(\lambda/2(n+a)),$$

$$\Phi = (n+a)x^2\sin^2\theta\left(1-\cos\frac{\lambda}{n+a}+i\sin\frac{\lambda}{n+a}-i\frac{\lambda}{n+a}\right). \quad (9)$$

b) $n = 0$ (regularized expression):

$$F = \left(2+\frac{4+3x^2\sin^2\theta}{x\sin^2\theta\sqrt{1+\eta+x^2}}\right)(1-e^{-\Phi_0})$$

$$+\frac{x}{\sqrt{1+\eta+x^2}}\left(1-\exp\left\{-\Phi_0-i\frac{\lambda}{a}\right\}\right). \quad (10)$$

Here Φ_0 is determined by the same expression (10) as Φ, except that we must put in it $n = 0$. In the case of a weak field it can be assumed that $a \gg 1$, and consequently $n + a \gg 1$, so that it is possible to expand all the expressions in terms of the quantity $(n + a)^{-1}$. As a result of expansion and integration we get

$$f(n,a) = 1-\frac{7}{3a^2}\left(\ln a-\frac{576\ln 2-83}{420}\right),$$

$$f(0,a) = 1-\frac{1}{a}\left(\frac{4}{3}\ln a-\frac{13}{18}\right). \quad (11)$$

The last formula was first obtained in[6], but our result contains the numerical coefficient 13/18 in lieu of 47/90. It follows from these formulas that the nonlinear corrections to the magnitude of the anomalous magnetic moment are largest in the "ground" state of the electron $n = 0$. In the excited state ($n \neq 0$) the field corrections have a quadratic-logarithmic character and do not depend on the energy of the electron (such a dependence can appear only in the next higher terms of the expansion, $\sim 1/a^3$).

This conclusion, however, calls for a refinement. Formula (11) is valid if $n \ll a$. In order to determine the limiting value of the energy at which these results are valid, we investigate the second extreme case, $n \gg a$. After expanding (9) in terms of the quantity $(n + a)^{-1}$ and performing the necessary integration, we obtain the expression (see also[10])

$$f(n,a) = -\frac{a}{\pi(n+a)^n}\int_0^{\infty}\lambda^2 d\lambda\int_{-\infty}^{\infty}d\eta\int_0^{\pi/2}\sin^3\theta\,d\theta\cos\lambda\eta\,e^{-q/2}L^1_{n-1}(q)\,\Phi_1,$$

$$\Phi_1 = \frac{1}{1+\eta}\left(\frac{2}{\sin^2\theta}-\frac{3}{2}\right)+\frac{3}{2}\ln(1+n), \quad q = \frac{\lambda^2\sin^2\theta}{4(n+a)},$$

which can be reduced, by means of relatively simple transformations, to the form

$$f(n,a) = \frac{1}{n}\left\{(n+a)\int_0^{\infty}\sin\left[\frac{n+a}{a}x-2n\,\mathrm{arctg}\frac{x}{2a}\right]dx-a\right\}.$$

In a weak field $a \to \infty$ ($a \gg 1$), and therefore

$$f(n,a) = \frac{1}{n}\left\{(n+a)\int_0^{\infty}\sin\left(x+\frac{n}{12a^3}x^3\right)dx-a\right\},$$

from which it follows that

$$f(n,a) = \begin{cases} 1, & \text{if} \quad n \ll a^3 \\ \dfrac{a\Gamma(1/3)}{(18n)^{1/3}} = \dfrac{3^{1/6}}{6}\Gamma(1/3)\left(\dfrac{E}{mc^2}\dfrac{H}{H_0}\right)^{-2/3}, & \text{if} \quad n \gg a^3 \end{cases}.$$

Thus, the anomalous magnetic moment of the electron is actually independent of its energy only if $n \ll a^3$.³⁾ When $n \gg a^3$, the anomalous magnetic moment decreases sharply with increasing electron energy. In particular, a numerical calculation of f in the critical case $n = 12a^3$ leads to the result $f \approx 0.53$; that is to say, in the critical energy region $E \approx E_{cr} = mc^2H_0/H$ the anomalous magnetic moment of the electron is equal approximately to half its nonrelativistic value.

Thus, the vacuum magnetic moment of the electron behaves in a rather complicated manner even in a weak magnetic field.

3. ANOMALOUS MAGNETIC FIELD OF AN ELECTRON IN A STRONG MAGNETIC FIELD

An even greater deviation from the known results is obtained in the case of a strong magnetic field $H \gg H_0$

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3) We note that an analogous criterion appears as the limit of applicability of the classical theory of synchrotron radiation [14].

$$E < E_{cr} = mc^2H_0/H.$$

$(a \to 0)$.[4)] By making the change of variable $x \to x/\sin^2\theta$ and by means of an exact integration, the expression (7) can be represented in the form

$$f(n,a) = \frac{a \ln a}{n+a} - 8a \int_0^\infty \sum_{n'=0}^{\infty} \frac{x^2(1+\xi)-(1-\xi)^2}{x\sqrt{s}} \cdot$$
$$\cdot \{I^2_{n,n'}[(n+a)x^2] - I^2_{n-1,n'-1}[(n+a)x^2]\}\, \Phi(x,\xi)\, dx, \qquad (12)$$

where

$$s = |[(x-\sqrt{\xi})^2 - 1][(x+\sqrt{\xi})^2 - 1]|,$$

and the discontinuous function Φ is equal to

$$\Phi(x,\xi) = \begin{cases} \operatorname{arctg} \dfrac{\sqrt{s}}{(x+\sqrt{\xi})^2 - 1} & \text{for } |\sqrt{\xi} - 1| \leqslant x \leqslant |\sqrt{\xi} + 1| \\ \dfrac{1}{2} \ln \left| \dfrac{x^2 + \xi - 1 + \sqrt{\xi}}{2x\sqrt{\xi}} \right| & \text{for all other } x. \end{cases}$$

In the asymptotic case when $a \to 0$, all that remains of the entire sum (12) is the single term with $n' = 0$. The situation is somewhat more complicated in the calculation of the "ground" state $n = 0$. However, after regularizing the divergent expression by means of mass renormalization, we can verify that in this case the main contribution is made by the term with $n' = 0$. Ultimately we get

$$f(n,a) = \frac{2a \ln a}{n},$$
$$f(0,a) = -2a(\ln^2 a + 2C \ln a),$$

where C is Euler's constant. Consequently, in this case

$$f(n,a) < 0.$$

Thus, in first order of the expansion in the fine-structure constant, the anomalous magnetic moment of an electron moving in a constant and uniform magnetic field is a complicated function of the field intensity and of the particle energy. Depending on the field intensity, the anomalous magnetic moment can even reverse sign, and has a tendency to decrease with increasing electron energy.

4) As noted by Heisenberg and Euler [17], there are serious grounds for assuming that quantum electrodunamics remains valid in such superstrong magnetic fields (unlike in electric fields); moreover, the use of perturbation theory is valid.

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