EFFECT OF CONCENTRATION NON-UNIFORMITY ON THE PROPERTIES OF SUPERCONDUCTORS WITH PARAMAGNETIC IMPURITIES

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The possibility of calculating the critical temperature of a superconducting alloy with a nonuniform distribution $c(r)$ of paramagnetic impurities is investigated by means of a generalization of the Abrikosov-Gor’kov theory (the characteristic inhomogeneity dimension $\delta_0$ and coherence length $\xi_0$ are assumed to be large compared to the mean distance between the impurities). By analogy with the problem of determining the spectrum of disordered systems, the concepts of a “renormalized” critical temperature $T_C$ (the same as introduced in the Abrikosov-Gor’kov theory) and of a “true” critical temperature $T_C^*$ ($T^*_C > T_C$) due to fluctuations, are introduced. When $T > T_C$ the superconductivity picture is that of “islets” of the superconducting phase separated by normal regions. The generalized Ginzburg-Landau equation is used to study localized states when $T > T_C$ (superconducting “nuclei”). It is shown that when $c(r)$ is not constant and the concentration is small the equation contains a “potential energy” term that is a linear functional of the impurity concentration.

1. INTRODUCTION

The study of the properties of superconducting alloys, i.e., superconductors with admixtures of other elements, is one of the urgent problems of modern theory of superconductivity. The theory of superconducting alloys (homogeneous solid solutions) containing both nonmagnetic and paramagnetic impurities was constructed in the well-known papers of Abrikosov and Gor’kov[1,2]. The most important result of this theory is the explanation of the dependence of the critical magnetic field $H_c$ on the concentration of the nonmagnetic impurities (in particular, the transition to superconductivity of the second kind, occurring at a definite concentration) and the critical temperature $T_c$ on the concentration of the paramagnetic impurities (accompanied by a vanishing in the superconductivity at a certain critical concentration).

In the Abrikosov-Gor’kov theory, the impurities are assumed to be distributed over the volume of the superconductor uniformly with a certain average concentration $\xi$. As noted in[3], the results of Abrikosov and Gor’kov can be obtained on the basis of a simplified equivalent Hamiltonian of the interaction between the electrons and the impurities; this Hamiltonian does not contain summation over the random positions of the impurities (and its spin), and takes the form

$$H_i = \sum_{\rho \neq \rho'} \delta_{\rho \rho'} \Gamma_{\alpha \beta} \rho \rho'$$

where $a_{\rho \alpha}$—operators of creation of an electron in the state $\rho$, $\alpha$ ($p$—momentum, $\sigma$—spin index), and $\Gamma_{\alpha \beta}$—a certain interaction matrix:

$$\Gamma_{\alpha \beta} = \gamma_\alpha \delta_{\alpha \beta} + \varepsilon_{\alpha \beta}, \quad \delta_{\alpha \beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varepsilon_{\alpha \beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

($\gamma_\alpha$ and $\varepsilon_{\alpha \beta}$—constants expressed in terms of the probabilities of the non-exchange and exchange scattering by the impurities $\tau_\alpha^n$ and $\tau_\beta^n$). This means that within the framework of the model used in[1,2] the real distribution of the impurities is replaced by a simplified uniform (in the mean) distribution, i.e., effects of concentration fluctuations are disregarded. At the same time it is obvious that for certain problems allowance for the inhomogeneity of the impurities is of decisive significance. For example, such problems include the problem of calculating the critical currents in the mixed state of superconductors of the second kind when $J \perp H$ (see[4]). Clusters of impurities, produced by fluctuations of their concentration, can play the role of blocking centers for the Abrikosov vortex filaments, leading to a finite value of the critical current at the concentration.

The purpose of the present paper is to study the influence of the inhomogeneities of the concentration on the properties of superconductors containing paramagnetic impurities, in the absence of an external magnetic field[1]. We consider here the case when the distribution of the impurities in a crystal can be characterized by their average concentration as a function of a point, $c(r)$. This can be done provided the characteristic distance over which the function $c(r)$ varies, and also the distance over which the average superconducting parameters change ($\xi_0 \sim \delta_0 / T_C$), are large compared with their “statistical” dimensions characterizing the microinhomogeneities of the concentration, i.e., actually compared with the average distances between the impurities[5]. At the concentrations of interest, on the order of $1\%$ (see[2]), the latter quantity amounts to several interatomic distances, i.e., it is small compared to the critical distance between the impurities.

NOTES

1) The influence of the fluctuations of the impurity concentration on the magnetic properties of superconductors, and particularly the problem of the critical currents in the mixed state, are not considered by us in this paper. These questions will be dealt with in a separate article.

2) The opposite limiting case, when the distance between the impurities is large compared with $\xi_0$, and therefore the individual impurities can be regarded as isolated defects, were investigated by Suhl et al. [5] and by Fetter [7] (in the case of nonmagnetic impurities).
pared with the correlation length $\xi_0 \sim 10^{-4}$ cm. We shall assume it to be small compared with the dimensions of the macroscopic inhomogeneities of the concentration $\delta_0$ (the characteristic distances over which a change of the function $c(\mathbf{r})$ takes place), but we shall make no assumption concerning any relation between the parameters $\xi_0$ and $\delta_0$.

Thus, the term "fluctuation" will mean in the present paper inhomogeneities of the average impurity concentration. Typical examples of systems to which the developed theory applies are, for example, the widely used inhomogeneous alloys produced in the decay of a solid solution quenched from high temperatures, or alloys produced by condensation from a gas mixture on a low-temperature substrate (in particular, "alloys" of mutually insoluble components can be obtained in this manner). As will be shown below, the critical temperature of such systems is essentially determined by the character of the distribution of the impurity concentration over the volume of the sample. Moreover, the very concept of critical temperature calls for a refinement.

Inasmuch as the critical temperature is a function of the concentration, which decreases with increasing $c$ (see [1]), the sections of the alloy in which the concentration exceeds the average value will have decreased values of $T_C$, and the sections with the lower concentrations will have higher $T_C$. It is therefore clear that the true critical temperature will be higher than the value corresponding to the average concentration. As was noted in an earlier paper [3], consideration of similar questions leads to a problem analogous to that of calculating the spectrum of disordered systems [8,9] (see also [10] etc.). In the latter case, as shown by I. Lifshitz [8,9], it is possible to introduce in the calculation of the state-density function $\rho(E)$ a "renormalized" end point $E_g$ of the spectrum and a true end point $E'_g$ of the spectrum, which can lie much higher than $E_g$ even at low concentrations. This situation is qualitatively illustrated in Fig. 1a, where the solid curve shows the density of states of a regular crystal ($E_g$—end point of the band), and the dashed line represents the function of the spectral density of the impurity-containing disordered system.

Figure 1b shows the qualitative dependence of the averaged ordering parameter $\Delta$ of the superconductor on the temperature $T$. The solid curve pertains to the pure superconductor, and the dashed line to a superconducting alloy containing paramagnetic impurities. Without taking the impurity-concentration fluctuations into account (if these impurities have a distribution that is homogeneous in the mean), $\Delta$ vanishes at a certain "renormalized" critical temperature $T'_C$ which differs from the transition temperature $T_{C0}$ of the pure superconductor. This is precisely the quantity introduced in the theory of Abrikosov and Gor'kov [2]. In fact, owing to the fluctuations, the superconductivity is conserved up to a certain temperature $T'_C$, exceeding $T_C$ ($T'_C > T_C$), which plays the role of the "true" critical temperature (in principle $T'_C$ may also coincide with $T_{C0}$). When $T > T'_C$, the picture of the superconductivity will consist of "islands" of the superconducting phase, separated by regions of the normal phase [4]. Such a superconductivity cannot be revealed by the vanishing of the resistance, which will be finite, but $\rho(T)$ can be observed, for example, in experiments on the tunnel effect [5].

In the present paper (in Sec. 2) we obtain a generalization of the Abrikosov-Gor'kov equations [2] to include the case of an arbitrary inhomogeneous distribution of the concentration of the paramagnetic impurities $c(\mathbf{r})$ (satisfying the properties indicated above). These equations are solved in Sec. 3 by perturbation theory with the aid of an expansion in powers of the concentration. We obtain for the "renormalized" critical temperature $T'_C$, an expression that coincides with the corresponding expression in the Abrikosov-Gor'kov theory. Finally, in Sec. 4 we investigate the question of calculating the "true" critical temperature $T_C$ caused by the localized states near regions with decreased concentration of the paramagnetic impurities. $T_C$ turns out to be here higher than the critical temperature defined in the homogeneous model [2]. To investigate the localized states, we used the modified Ginzburg-Landau equation, which contains a "potential energy" term that is a linear functional of the impurity concentration.

2. CHOICE OF MODEL. ABRIKOsov-Gor'kov EQUATIONS IN THE INHOMOGENEOUS CASE

Proceeding to solve our problem, let us discuss first the question of the maximum simplification of the calculation scheme, retaining at the same time the main physical premises which we shall assume to coincide with those on which the Abrikosov-Gor'kov theory [4] is based.

The Hamiltonian of the interaction between the electrons and the impurities is

$$
H_i = \sum_i \int \mathrm{d} r \delta \psi_+^{*}(\mathbf{r})[i\hbar \delta_{\alpha\beta} V_\alpha(\mathbf{r} - \mathbf{r}_i) + (\delta_{\alpha\beta} V_\alpha(\mathbf{r} - \mathbf{r}_i))] \psi_\beta(\mathbf{r}),
$$

(2.1)

where $V_R$ and $V_S$ are the non-exchange and exchange parts of the interaction potential, $\mathbf{r}_i$ are the coordinates of the impurities, $S_{ij}$ is the spin of the impurity located at the point $\mathbf{r}_i$, $\sigma_{\alpha\beta}$ are the spin matrices of the electron:

$$
\sigma^\alpha = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma^\beta = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma^z = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
$$

(2.2)

and $I_j$ are quantities whose meaning will be explained later.

Following [3], we assume $\delta$-function interactions between the electrons and the impurities, and we put accordingly

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Similiar considerations can be developed also for the magnetic properties of superconductors with nonmagnetic impurities, but we shall confine ourselves henceforth to superconducting alloys with paramagnetic impurities.
The possibility of such a substitution is connected with the fact that although the real interaction has a finite radius of action (on the order of the lattice constant a), nonetheless its magnitude is small compared with the characteristic distances over which the gap in the superconductor varies, i.e., the correlation length $\xi_0$.

The final formulas therefore do not contain detailed features of the interaction potential, but only the scattering amplitudes expressed in terms of the free-path length $\tau_0$.

The Hamiltonian (2.1), with allowance for (2.3), takes the form

$$
H_1 = \sum_i \psi_i^*(R_i)[i\hbar \partial_\tau \psi_i + (S_{\text{imp}}) \psi_i] \psi_i(R_i).
$$

Using 2.4 it is possible to obtain in first order expressions of the form

$$
\frac{1}{\tau_0} = 2nS(S + 1) \langle 0 | \psi \psi | 0 \rangle,
$$

where $S$ - spin of the impurity, $\langle \text{average concentration of the impurities, and } N(0) = mp_0/2\pi^2$ - density of the electron states on the Fermi surface.

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$$

Further simplification consists in the following. Using perturbation theory with respect to the Hamiltonian (2.5), we obtain in first order expressions of the form

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bution of a diagram shown in Fig. 2c turns out to be 
\[ (G_{00}^{2} - \text{Green's function of zeroth approximation}) \]
\[ (V^2 - S(S + 1) V^2) \]
\[ \delta \text{d}c(R) (s - R) G_{00}^0 (s - R) G_{00}^0 (R - r) G_{00}^0 (R - r), \]
where
\[ c(R) = \sum_i \delta (R - R_i) \]
is the average concentration of the impurities at the point \( R \).

In approximations of order higher than the second, it is necessary to discard all the diagrams with intersecting dashed lines. This follows from the fact that in expressions analogous to (2.12) we should assume that \( c(R) \) is a slowly varying function (at distances on the order of \( p_0 \sim a \), as a result of which the estimate of these diagrams is carried out in the same manner as when \( c = \text{const} \) \cite{1,4,11}, and shows that their contribution is small compared with the contribution of diagrams with non-intersecting lines. The calculation of \( K_w(s, r) \) then reduces to summation of diagrams of the ‘‘ladder’’ approximation \cite{11}.

A graphic equation for \( K_w(s, r) \) is shown in Fig. 3, where the shaded square corresponds to the exact function \( K_w \). Writing this equation analytically, we get
\[ K_w(s, r) = G_w(s, r) G_w(s, R) K_w(s, R). \]
(2.14)

The Green’s function \( G_w(s, r) \) itself is obtained from an equation summing the ‘‘ladder’’ diagrams shown in Fig. 4 (the thin lines correspond to the unperturbed Green’s function, and the heavy ones to Green’s functions in the presence of impurities). In accordance with Fig. 4, we have
\[ G_w(s, r) = G_0(s, r) + [V^2 - S(S + 1) V^2] \delta \text{d}c(R) G_0(s, R) G_w(R, R) G_w(R, r). \]
(2.15)

In the case of constant concentration (\( c(R) = \text{const} \)), Eqs. (2.14) and (2.15) go over into the corresponding equations of Abrikosov and Gor’kov \cite{2,3}. In this case the averaged Green’s functions \( G_w(s, r) \) and \( K_w(s, r) \) depend only on the difference of the arguments \( s - r \), making it possible to obtain them in explicit form (in the Fourier representation):
\[ G_w(p) = (i\omega \eta_0 - \xi_p)^{-1}, \quad \xi_p = \frac{\hbar^2}{2m} - \mu, \quad \eta_0 = 1 + \frac{1}{2\pi |\omega|}, \]
(2.16)
\[ K_w(p) \big|_{\omega = -\frac{\pi N(0)}{|\omega| + 1/\tau_s}}, \]
(2.17)
where
\[ 1/\tau_s = 2\pi N(0) [V^2 + S(S + 1) V^2], \]
(2.18)
and \( \tau_s \) is determined by formula (2.4). Since in this case \( \Delta = \text{const} \), substitution of (2.17) in (2.9) leads to
\[ S_w^0(R_1 - R_2) = \int \text{d}\rho G_w^0(R_1 - \rho) G_w^0(R_2 - \rho), \]
(2.19)

the well known equation for the critical temperature \cite{8}:
\[ \ln \frac{T_c}{T_s} = \psi \left( \frac{1}{2} + \frac{1}{2\pi \tau_s T_s} \right) - \psi \left( \frac{1}{2} \right), \quad \psi(z) = -\frac{d}{dz} \ln \Gamma(z). \]
(2.19)

In the case considered in the present paper, that of variable concentration, the situation turns out to be much more complicated, and Eqs. (2.9), (2.14), and (2.15) cannot be solved in general form. We consider approximate methods for their solution in the following sections.

In order not to interrupt the subsequent exposition, we introduce here certain functions which will be useful in what follows. We define besides \( G_w^0(R_1 - R_2) \) also the functions \( S_w^0(R_1 - R_2) \) and \( L_w^0(R_1 - R_2) \) in accordance with the formulas
\[ S_w^0(R_1 - R_2) = \int \text{d}\rho G_w^0(R_1 - \rho) G_w^0(\rho - R_2). \]
(2.20)

Using the zeroth-approximation Green’s function \( \tilde{G}_w(p) \) (formula (2.16) at \( \tau = \infty \)), we obtain the values of \( S_w^0(R_1 - R_2) \) and \( L_w^0(R_1 - R_2) \) in the coinciding points:
\[ S_w^0(0) = \frac{\pi N(0)}{2\omega}, \quad L_w^0(0) = -\frac{\pi N(0)}{2\omega}, \]
(2.21)

the values of the same functions in the case when the difference of the arguments is large compared with the interatomic distance (\( R = |R_1 - R_2| \gg p_0^2 \)) are determined by the asymptotic expressions
\[ G_w^0(R) \approx -\frac{m}{2\pi R} \exp \left( ip_0 R \text{sign} \omega \right) \exp \left( -\frac{|\omega| R}{\nu_t} \right), \]
(2.22)
\[ S_w^0(R) \approx \frac{m}{2\pi R} \sin p_0 R \exp \left( -\frac{|\omega| R}{\nu_t} \right), \]
(2.23)
\[ L_w^0(R) \approx -\frac{m}{2\pi R} \left[ \frac{\text{sign} \omega}{\nu^2} \sin p_0 R + \frac{R}{\nu^2} \exp \left( ip_0 R \text{sign} \omega \right) \right. \]
\[ \times \exp \left( -\frac{|\omega| R}{\nu_t} \right). \]
(2.24)

The derivation of (2.23) can be found in the book of Abrikosov, Gor’kov and Dzyaloshinskii \cite{11}. The remaining two expressions are obtained in analogy with (2.23).

3. CALCULATION OF THE RENORMALIZED CRITICAL TEMPERATURE (CASE OF LOW CONCENTRATION)

It is evident that the eigenvalue of Eq. (2.9) with kernel \( K_w(s, r) \) defined by (2.14) and (2.15) can be represented at small concentrations by a series in the form
\[ T_c = T_s + \delta T + \delta^2 T + \ldots \]
(3.1)
\[ \delta T = \frac{1}{2} \sum R (R) c (R) dR, \quad \delta T = \int R (R) c (R) dR, \quad dR, \quad \ldots, \ (3.2) \]

and from considerations of homogeneity of space it follows that the function \( f (R) \) should be a constant independent of \( R \), \( \epsilon (R_1, R_2) \) should be a function of the difference \( R_1 - R_2, \) etc. The expansion (3.1) is obtained if it is assumed that \( \Delta^* (r) \) can be represented in the form of a series in powers of the concentration

\[ \Delta^* (r) = \Delta_0 + \Delta_1 (r) + \Delta_2 (r) + \ldots \quad (3.3) \]

where \( \Delta_1 (r) \rightarrow 0 \) as \( c \rightarrow 0 \). This means that we seek the upper bound of the unlocalized solutions for the ordering parameter, i.e., the quantity called "renormalized" critical temperature \( T_C \).

Writing the kernel of (2.6) in the form of a series in powers of the concentration:

\[ K_0 (s, r) = K_0^c (s - r) + K_0 (s, r) + \ldots \quad (3.4) \]

and substituting (3.3) and (3.4) in (2.9), we get the successive approximations

\[ \Delta_0 - |\lambda| T_C \sum_a \int ds K_0^c (s - r) \Delta_0 = 0, \quad (3.5) \]

\[ \Delta_1 (r) - |\lambda| T_C \sum_a \int ds K_0^c (s - r) \Delta_1 (s) \]

\[ = -N(0)|\lambda| \frac{\delta T}{T_C} \Delta_0 + |\lambda| T_C \int a \int ds K_0^c (s, r) \Delta_0 \quad (3.6) \]

etc. We took account here of the fact that in the sums over the frequencies \( T \) is \( T_C + \delta T + \ldots \). The first equation of (3.5) defines the critical temperature of the pure superconductor \( T_{C0} \). Rewriting \( K_0^c \) on the basis of (2.23) in the form

\[ K_0^c (s, r) = G_{c, 0}^c (R) G_{c, 0}^c (R) = \left( \frac{m}{2 \pi N} \right)^{\frac{1}{2}} \exp \left( -\frac{2 |\omega| |R|}{v_0} \right), \quad (3.7) \]

cancelling \( \Delta_0 \) out of (3.5), integrating over space, and summing over the frequencies, we obtain with allowance for the fact that the summation should be terminated at the Debye frequency \( \omega_D \) the well known expression

\[ T_0 = \frac{2 \gamma}{\pi} v_0 \exp \left( -\frac{1}{N(0)|\lambda|} \right), \quad \ln \gamma = C = 0.577. \]

To find the correction to the critical temperature \( \delta T = T_C - T_{C0} \), we turn to Eq. (3.8). The right side of this equation should be orthogonal to the solution of the corresponding homogeneous equation, i.e., \( \Delta_0 \). This leads to a relation

\[ N(0)|\lambda| \frac{\delta T}{T_C} = \sum \int K_0^c (s, r) ds dr. \quad (3.8) \]

The quantity \( K_0^c (s, r) \) is obtained with the aid of Eqs. (2.14) and (2.15). It is represented in the form of a sum of the three diagrams shown in Fig. 2, and accordingly we have

\[ K_0^c (s, r) = S(S + 1) V^2 \int R (R) c (R). \]

\[ \cdot G_{c, 0}^c (R - s) G_{c, 0}^c (R - r) G_{c, 0}^c (R - r) \]

\[ + G_{c, 0}^c (R - s) G_{c, 0}^c (R - r) G_{c, 0}^c (R - r) \]

\[ - G_{c, 0}^c (R - s) G_{c, 0}^c (R - r) G_{c, 0}^c (R - r). \quad (3.9) \]

We note that the non-exchange part of the scattering has dropped out of this expression (compare with [12]). Recognizing that we are interested in the sum \( T_0^2 K_0^c \), we can replace \( \omega \) by \( -\omega \) in one of the first two terms of (3.9), after which the corresponding contributions coincide. Integrating with respect to \( s \) and \( r \), we obtain, with allowance for (2.20) and (2.21)

\[ \delta T = S(S + 1) V^2 \int R (R) T_{C0}^2 \sum \left[ 2 G_{c, 0}^c (R) G_{c, 0}^c (R) - (S, \lambda (0)) \right] \quad (3.10) \]

Finally, using (2.22) and summing over the frequencies, we get

\[ \delta T = -\frac{\pi^2}{2} S(S + 1) V^2 N(0) \int c (R) dr = -\frac{\pi^2}{2} S(S + 1) N(0) V^2. \quad (3.11) \]

The obtained expression coincides with the result of Abrikosov and Gor'k'ov [3]. Introducing the spin free path time \( \tau_s \) in accordance with formula (2.4) we represent (3.11) in the form

\[ \delta T = -\pi/4 \tau_s. \quad (3.12) \]

Thus, the "renormalized" critical temperature introduced by us coincides (in first order in the concentration) with the critical temperature of Abrikosov and Gor'k'ov. In this sense, the foregoing analysis yields nothing new compared with the "homogeneous" model [2], if we are interested in the "average" transition temperature \( T_C \). However, as will be shown in the next section, for the "true" critical temperature \( T_{C1} \), significant corrections appear even in first order in the concentration, and are connected with possible existence of localized solutions for \( \Delta^* (r) \), which cannot be represented in the form \( \Delta_0 + \Delta_1 (r) \) with small \( \Delta_1 \) (even if \( c \rightarrow 0 \)). The quantity \( T_{C1} \) is the limit separating the regions of existence of nonlocalized and localized solutions for the ordering parameter \( \Delta^* (r) \).

It is not difficult to calculate the next higher terms of expansions such as (3.1) and (3.2). It must be remembered, however, that \( T_{C1} \) is an effective quantity determining the limit of the existence of the nonlocalized solutions for \( \Delta^* \) of the type (3.3). In the next higher approximations in the concentration there appears not only a shift but also a smearing of this boundary, i.e., an expansion of the type (3.3) becomes incorrect, strictly speaking. A similar situation, as shown by I. Lifshitz [8,9], occurs also for spectra of disordered systems: in the higher approximation, the "renormalized" boundary of the spectrum, which separates the region of the existence of nonlocalized states, can, generally speaking, not be determined exactly. Only the concept of the "true" boundary of the spectrum has a rigorously strict meaning, and in our case this holds for the analogous concept of "true" critical temperature.

4. THE GINZBURG-LANDAU EQUATION AND LOCAL STATES

The purpose of this section is to investigate localized solutions for the ordering parameter (2.9) near the "true" critical temperature \( T_{C1} \). We confine ourselves to the case of first order in the concentration, i.e., we put \( c (r) \rightarrow 0 \). The critical temperature differs little in this case from \( T_{C0} \)—the critical temperature of a true superconductor. Because of this, it becomes possible to reduce the integral equation (2.9) to a differential equation of the Ginzburg-Landau type [13,14]. The localized states which occur near the minima of the
function \( c(r) \) will have a larger radius \( \sim \sqrt{T_{c0} - T} \), which becomes much larger than the BCS parameter \( \xi_0 \) when \( T \to T_{c0} \). In this case the kernel of (2.9) is a rapidly varying function compared with the ordering parameter \( \Delta^* (s) \), so that we can write the expansion

\[
\Delta'(s) = \Delta'(r) + \frac{\partial \Delta'}{\partial r} (r_i - r) + \frac{1}{2} \frac{\partial^2 \Delta'}{\partial r^2} (r_i - r) (s_i - r) + \ldots \quad (4.1)
\]

The rest of the derivation essentially duplicates the derivation of the Ginzburg-Landau equation from the microtheory presented by Gor'kov \cite{14}. Writing \( K_0 (s, r) \) in the form

\[
K_0 (s, r) = K_0' (s - r) + K_0'' (s, r),
\]

where \( K_0' (s, r) = G_0^2 (s - r) K_0^2 (s, r) \), and \( K_0'' (s, r) \) is determined by formula (3.9), and substituting (4.1), (4.2), and (2.9), we obtain

\[
\Delta'(r) \left[ 1 - \frac{\partial}{\partial r} \int \frac{dR K_0' (R)}{\sqrt{c_0^2 + \Delta'(r)^2}} \right] - \frac{1}{2} \frac{\partial^2 \Delta'}{\partial r^2} (s_i - r) (s_i - r) + \ldots \quad (4.11)
\]

The asymptotic form of \( Q(R) \) is given by (4.3). As a net result we obtain the following equation for \( \Delta'(r) \):

\[
\Delta' (r) \left[ 1 - \frac{\partial}{\partial r} \int \frac{dR K_0' (R)}{\sqrt{c_0^2 + \Delta'(r)^2}} \right] - \frac{1}{2} \frac{\partial^2 \Delta'}{\partial r^2} (s_i - r) (s_i - r) + \ldots \quad (4.12)
\]

Supplementing (4.8) with a term proportional to \( \Delta^* \), we obtain the Ginzburg-Landau equation \cite{13,14}, generalized to include the case of an inhomogeneous impurity concentration \( c(r) \). It is clear here that in the case of small concentrations the cubic term can be taken in the same form as when \( c = 0 \).

Let us consider some particular cases of (4.8).

If the impurity concentration changes slowly from point to point, so that the characteristic distances over which this change takes place are large compared with the "radius" of the nucleus \( Q(r) \), i.e., with the parameter \( \xi_0 \), we can take \( c(r) \) in (4.11) outside the integral sign. Integrating then the expression (4.9), we arrive at the relation

\[
V(r) = \frac{\pi^2}{2 n R_{c0}} N(0) (s + 1) V_0 c'(r),
\]

which shows that in the case of slow variation of \( c(r) \) the role of the "potential" in (4.8) is played directly (accurate to a constant factor) by the impurity concentration \( c(r) \). We note that at constant \( c(r) \) Eq. (4.8) again leads to formula (3.11) for the decrease of the critical temperature in first order in the concentration \( \xi_0 \) of the impurity.

In the case when the impurity concentration depends only on one coordinate \( x \), the potential \( V \) in (4.8) is likewise one-dimensional, and in this case it can be represented in the form

\[
V(x) = \frac{\pi^2}{2 n R_{c0}} N(0) (s + 1) V_0 c'(x),
\]

with a kernel \( q(x) \) normalized in accordance with the condition

\[
\int q(x) dx = 1.
\]

As can be readily shown on the basis of (4.9) and (4.11), the function \( q(x - x') \) is given by

\[
q(x - x') = \frac{1}{\xi_0} E \left[ (2s + 1) \frac{2 n R_{c0}}{v_0} |x - x'| \right],
\]

where \( E \) is the integral exponential function. The characteristic radius \( q(x - x') \) is \( |x - x'| \sim v_0 / R_{c0} \sim \xi_0 \).

The local states near the minima of the function \( c^*(x) \), corresponding to \( T_{c0} \), result from the one-dimensional nature of the problem at arbitrarily small depth of the "potential well" \( c^*(x) \). Writing \( c^*(x) \) near the minimum in the form

\[
c'(x) = c_0 - \frac{1}{2} k x^2 \quad (4.19)
\]

where \( k \) is conveniently represented in the form \( \xi_0 - 

6Actually the region of applicability of Eq. (4.8) is not limited to various concentrations. Thus, for example, according to \[15\], at concentrations up to 0.8 \( \xi_0 \), the deviation from the linear dependence of \( T_{c0} \) on the concentration \( \xi \) is 10% (for a homogeneous distribution of the impurity).
effective width of the well, \( c_{CR} \) —critical concentration introduced in the Abrikosov-Gor'kov theory \(^{1)} \)

\[
k = \frac{c_{CR}}{\delta^2}, \quad c_{CR} = \frac{T_a}{4\hbar^2 N(0) S(S+1) V^2}. \tag{4.20}
\]

we see that the condition for the smallness of the concentration signifies \( C_0 \ll c_{CR} \) and \( x_0 \ll \delta_0 \), where \( x_0 - \)radius of local state.

Solving the Schrödinger equation

\[
\frac{1}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{N(0) S(S+1) V}{2 \hbar^2} \varphi(x) \varphi(x) = \frac{E}{\hbar^2} \varphi(x), \tag{4.21}
\]

we obtain for the minimum value of \( E \)

\[
\varphi(x) = \text{const} \exp \left( - \frac{x^2}{2 \delta_0^2} \right), \quad \delta_0^2 = \frac{\varphi_0^2}{\hbar^2 T_c}, \tag{4.22}
\]

\[
T_c = T_0 - \frac{\pi^2}{2} N(0) S(S+1) V \varphi_0^2 = \frac{1}{2a} \sqrt{\frac{7S(3)}{3y}} \frac{V \varphi_0}{\delta_0}. \tag{4.23}
\]

Consequently, the radius of the local state \( x_0 \approx \sqrt{\delta_0 \varphi_0} \) will be small compared with \( \delta_0 \) when \( \delta_0 \gg \delta_0 \) (the condition \( x_0 \gg \delta_0 \) is then satisfied automatically). In this case the value of \( T_c \) is determined by the character of the behavior of the concentration near the minimum of (4.19), and not by the value of \( c(x) \) far from the minimum point.

In order for a local state to occur in the three-dimensional case, it is necessary that the value of the perturbation exceed a certain critical value. If the radius of the concentration 'well' \( \delta_0 \) is small compared with \( \xi_0 \), we can assume that the addition to the concentration \( c'(r) \) in (4.11) is proportional to a \( \delta \) function, so that the variable part of \( V(r) \) assumes the form

\[
V'(R) \approx \int c'(r) \frac{m^2 S(S+1) V_0^2}{2 \hbar^2 R} \ln \frac{R T_0 R}{v_0}. \tag{4.24}
\]

Such a potential corresponds to the 'incidence' of a particle on a scattering center \(^{1)} \) (if \( f c'(r) dr < 0 \)), since \( V'(R) \) has a singularity stronger than \( R^2 \to 0 \). However, that actually formula (4.10), together with expression (4.24), is valid only when \( R \gg v_0 / \omega_D \), when the summation over the frequencies in (4.9) can be extended to infinity.

If the fluctuation radius \( \delta_0 \) is large compared with \( \xi_0 \), then we can use formula (4.14). As can be readily shown, in this case the condition for the existence of the local state assumes the form (we omit all the dimensionless factors of order of unity)

\[
|c'| \gg \frac{c_{CR}}{\delta^2}, \tag{4.25}
\]

where \( c' \) —characteristic value of the deviation of the concentration from the mean value (in the region with characteristic dimensions \( \sim \delta_0 \)). Naturally, \( c' \) must be negative in order for a local state to occur.

The foregoing analysis pertained to the case of small concentration \( (\tau \gg \Delta^2) \). The non-exchange part of the scattering has then dropped out from the final expressions. In the case when the concentration is not small, the parameters \( \lambda \) of the generalized Ginzburg-Landau equation will depend on the total free path \( l = v_0 t^{1/3} \). The radius of the local state will then be determined also by the length of the free path relative to scattering without spin flip, and thus, a mechanism arises whereby the nonmagnetic impurities influence the critical temperature of superconductors containing paramagnetic impurities \(^{3)} \). However, this more complicated case, an analysis of which can also be carried out with the aid of Eqs. (2.14) and (2.15), is beyond the framework of the present article.

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\(^{1)} \) A. A. Abrikosov and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. 35, 1558 (1958); 36, (1959) [Sov. Phys.-JETP 8, 1090 (1959); 9, 220 (1959)].

\(^{2)} \) A. A. Abrikosov and L. P. Gor'kov, ibid. 39, 1781 (1960); 12, 1243 (1961).

\(^{3)} \) I. O. Kulik, Ekvivalentny'yi gamilton'iyan vzaimodei'stviya elektronov s primesami v sverkhprovodnikah (Equivalent Hamiltonian of Electron-Impurity Interaction in Superconductors), VINITI, No. 112, 1967.

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