RADIATION LINE WIDTH IN THE JOSEPHSON EFFECT

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It is shown that the radiation line width in the Josephson effect is determined by voltage fluctuations at the contact. The line width is determined as a function of the contact parameters and characteristics of the external circuit.

1. A Josephson element consists of two superconductors separated by a thin dielectric layer. A dc voltage V on the contact produces an alternating current and radiation at a frequency $\omega = 2eV/\hbar$. A certain radiation line width is observed in the experiment[1].

If external sources maintain the dc voltage at the contact constant, then the line width is determined by the thermal fluctuations of the voltage. We obtain below the dependence of the line width on the parameters of the Josephson contact. In an external magnetic field, the direct current flowing through the contact has resonance maxima at definite values of the voltage[1-5]. We obtain the width of these maxima and the radiation line width near them. We show that the Nyquist formula for the fluctuations of the current through the contact is applicable only if the voltage on the contact is much lower than the temperature. We obtain the current and voltage fluctuations and the radiation line width also for the case when the voltage is comparable with or higher than the temperature.

2. In the case of slow variation of the voltage on the contact, the Josephson current through the contact is determined by the formula[6-8]

$$j(t) = j_s \sin\left[\frac{2e}{\hbar}\int^{t} V(t')\,dt'\right]. \tag{1}$$

In our case, the voltage is equal to the sum of the dc voltage V_0, determined by the external source, and the ac voltage $V_1(t)$, determined by the thermal fluctuations. In experiments one usually sets a constant average contact voltage; the resultant alternating current leads to the appearance of alternating voltage harmonics. However, the frequency of these harmonics is large compared with the line width and they can therefore be neglected.

The average values of the thermal fluctuations of V_1 is zero, and the mean square value at $V_0 \ll T$ is determined by the Nyquist formula

$$\varphi(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\tau\,\overline{V_1(t)V_1(t+\tau)}\,e^{i\omega\tau} = \frac{T_{\rm eff}}{\pi}\operatorname{Re}Z(\omega). \tag{2}$$

The bar denotes hear averaging over the time T.

In our case the circuit consists of a capacitance and two resistances connected in parallel. The capacitance of the contact is equal to

$$C = S\varepsilon/4\pi d, \tag{3}$$

where S is the area of the contact, d the thickness of the dielectric layer, and ϵ its dielectric constant. One of the resistances is the external-circuit resistance $R_{\rm ext}$ and the other is the resistance of the contact to the normal current. If the contact voltage V_0 is small compared with the temperature, then, with logarithmic accuracy, we have in the case of identical superconductors on both sides of the contact[8]

$$R = R_N \operatorname{ch}^2\frac{\Delta}{8T}\left[1 + \frac{\Delta}{2T}\ln\left(\frac{T}{V_0}\right)^2\right]^{-1}. \tag{4}$$

Here R_N is the resistance of the contact in the normal state. This expression is valid only in the absence of an external magnetic field, whose influence will be discussed later. In formula (2)

$$\frac{1}{Z} = -i\omega C + \frac{1}{R_{\rm eff}}, \quad \frac{1}{R_{\rm eff}} = \frac{1}{R} + \frac{1}{R_{\rm ext}},$$
$$T_{\rm eff} = R_{\rm eff}\left(\frac{T_{\rm ext}}{R_{\rm ext}} + \frac{T}{R}\right), \tag{5}$$

where T is the temperature of the contact, and $T_{\rm ext}$ is the temperature of the external circuit. Thus, formula (2) can be represented in the form

$$\varphi(\omega) = \frac{|Z|^2}{\pi}\left(\frac{T}{R} + \frac{T_{\rm ext}}{R_{\rm ext}}\right) = \frac{1}{\pi}\frac{T_{\rm eff}R_{\rm eff}}{1 + (\omega C R_{\rm eff})^2}. \tag{6}$$

3. Let us find the radiation line shape. The radiation intensity in the frequency interval $(\omega, \omega + d\omega)$ is proportional to the expression

$$I(\omega) \sim \int_{-\infty}^{+\infty} e^{i\omega\tau}\,\overline{j(t)j(t+\tau)}\,d\tau \sim \int_{-\infty}^{+\infty} d\tau\,e^{-i\delta\omega\tau}\,\overline{\exp\left[\frac{2ie}{\hbar}\int_0^{\tau} V_1(t+t')\,dt'\right]} \tag{7}$$

where $\delta\omega = \omega - 2eV_0/\hbar$. As before, the bar denotes averaging over the time t.

When averaging the exponential, we make use of the fact that for the large times τ under consideration the thermal noise can be regarded as white noise. This means that the average of the product of any even number of factors V breaks up into a sum of product of all the possible paired averages. As the results we obtain

$$I(\omega) \sim \int_{-\infty}^{+\infty} d\tau\,e^{+i\delta\omega\tau}\exp\left[-\frac{2e^2}{\hbar^2}\int_0\int dt_1\,dt_2\,\overline{V_1(t+t_1)V_1(t_1+t_2)}\right]$$
$$= \int_{-\infty}^{+\infty} d\tau\,e^{i\delta\omega\tau}\exp\left[-\frac{8e^2}{\hbar^2}\int d\omega\,\varphi(\omega)\frac{\sin^2(\omega\tau/2)}{\omega^2}\right]. \tag{8}$$

Substituting here the expression (6) for $\varphi(\omega)$, we get

$$I(\omega) \sim \int_{-\infty}^{+\infty} d\tau\,e^{i\delta\omega\tau}\exp\left\{-\frac{4e^2R_{\rm eff}T_{\rm eff}}{\hbar^2}\left[|\tau| + R_{\rm eff}C\left(e^{-|\tau|/R_{\rm eff}C} - 1\right)\right]\right\}$$
$$= \operatorname{Re}\frac{2R_{\rm eff}C}{x + i\delta\omega R_{\rm eff}C}\Phi(1, x + 1 + i\delta\omega R_{\rm eff}C, x), \tag{9}$$

where $x = 4e^2 R_{eff}^2 T_{eff} C/\hbar^2$ and Φ is the confluent hypergeometric function.

To consider the limiting cases, we note that the times τ which are of importance in the integral (9) are of the order of the reciprocal of the line width. If $\Gamma \ll (R_{eff}C)^{-1}$, then

$$I(\omega) \sim \frac{1}{\pi}\frac{\Gamma}{(\delta\omega)^2+\Gamma^2}, \quad \Gamma = \frac{4e^2}{\hbar^2} R_{\text{eff}} T_{\text{eff}}. \tag{10}$$

In this case the line has a Lorentz shape. For small fluctuation frequencies, $\sim\Gamma$, the "resistance" of the capacitance $(\Gamma c)^{-1}$ is large and therefore the capacitance does not enter into the final answer.

In the other limiting case $\Gamma \gg (R_{eff}C)^{-1}$ we have

$$I(\omega) \sim \frac{1}{\sqrt{\pi}\,\Gamma}\exp\left[-\left(\frac{\delta\omega}{\Gamma}\right)^2\right], \quad \Gamma = \frac{2e}{\hbar}\sqrt{\frac{2T_{\text{eff}}}{C}}. \tag{11}$$

In this case the line has a Gaussian shape and the width is determined by the capacitance and does not depend on the resistance. In the case when the time $(R_{eff}C)$ required to establish equilibrium is large formula (11) can be obtained by statistical averaging of the monochromatic line

$$I(\omega) \sim \int dV_1 \exp\left(-\frac{CV_1^2}{2T_{\text{eff}}}\right)\delta\left(\delta\omega - \frac{2eV_1}{\hbar}\right). \tag{12}$$

We note that formula (12) contains T_{eff}, that is, the temperature of the resistance that shunts the capacitance.

4. The expression (4) for the resistance of the contact to the normal current is valid only if the contact is not in an external magnetic field. A magnetic field strongly influences the current-voltage characteristic of the contact, and by the same token the line width. As shown in many papers[1-5], the direct current through the Josephson contact has resonance maxima at voltages corresponding to the frequencies of the standing waves in the contact. The width of these maxima was as a rule introduced phenomenologically. Estimates show that it is determined principally by the absorption of the high-frequency radiation in the superconductor.

In order to find the dependence of the current of the voltage, we write down the current-conservation equation for a certain point of the contact in the form[3-5]

$$\left[\hat{\lambda}^{-1}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - \frac{\varepsilon}{dc^2}\frac{\partial^2}{\partial t^2}\right]\varphi = \frac{8\pi e}{\hbar c^2} j(x)$$

$$= \frac{4\pi}{Rc^2}\frac{\partial\varphi}{\partial t} + \frac{8\pi e}{\hbar c^2} j_s \sin\varphi, \quad \varphi = \frac{2e}{\hbar}\int^t V(t_1)\,dt_1; \tag{13}$$

here $\hat{\lambda}$ is the sum of the penetration depths, $\hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_2$. It is important that the penetration depth $\lambda(\omega)$ depends on the frequency and is an integral operator in time in Eq. (13).

As usual[3,4], we seek a solution in the form

$$\varphi = \frac{2e}{\hbar}V_0 t - \frac{2e\lambda(0)H_0}{\hbar c}x + \frac{4\pi e V_0 \lambda(0)}{Rc^2\hbar}x^2 + \Phi(x,t), \tag{14}$$

where R is given by formula (4). Neglecting small losses to radiation, we can assume that $\partial\Phi/\partial x = 0$ on the boundary. Solving the equation linearized with respect to Φ, we obtain for the dc component of the current the expression

$$j = \frac{V_0}{R} + \sum_{n=-\infty}^{+\infty}\frac{4\pi e j_s^2 S|a_n|^2}{c^2\hbar}Q_n\left\{\left[\frac{4e^2V_0^2\varepsilon}{\hbar^2 dc^2} - \left(\frac{\pi n}{l}\right)^2 \operatorname{Re}\frac{1}{\lambda}\right]^2 + Q_n^2\right\}^{-1}, \tag{15}$$

where

$$Q_n = \frac{8\pi e V_0}{\hbar R c^2} + \left(\frac{\pi n}{l}\right)^2 \frac{\operatorname{Im}\lambda(2eV_0/\hbar)}{|\lambda(2eV_0/\hbar)|^2},$$

$$a_n = \frac{1}{l}\int_0^l \cos\left(\frac{\pi n x}{l}\right)\exp\left[\frac{2ie\lambda(0)H_0}{\hbar c}x\right]dx.$$

The first term in Q_n is connected with the normal current through the contact and is usually small compared with the second term, which is determined by the absorption of the field of frequency $2eV_0/\hbar$ in the superconductor. At very low temperatures, both terms of Q_n decrease exponentially, and the resonance broadening connected with the radiation may become appreciable.

At voltages V_0 close to the resonance values, the second term in (15) becomes larger than the first. In this case it is necessary to replace R^{-1} in (5), (9), and (10) by $\partial j/\partial V_0$. If the resistance is not strongly shunted by the capacitance or by an external resistance, then the line width decreases near resonance.

5. The Nyquist formula used above for the contact current and voltage fluctuations is applicable only if the voltage on the contact is small compared with the temperature. Let us derive an expression valid for the fluctuations in the general case.

As usual, we describe the penetration of the electrons through the contact by a tunnel Hamiltonian;

$$\hat{T} = \sum_{\nu_1\nu_2}(T_{\nu_1\nu_2}a^+_{\nu_1}a_{\nu_2} + T^*_{\nu_1\nu_2}a^+_{\nu_2}a_{\nu_1}). \tag{16}$$

The operator of the current through the contact is

$$\hat{j} = \frac{\partial N_1}{\partial t} = -i\sum_{\nu_1\nu_2}(T_{\nu_1\nu_2}a^+_{\nu_1}a_{\nu_2} - T^*_{\nu_1\nu_2}a^+_{\nu_2}a_{\nu_1}). \tag{17}$$

Neglecting the higher powers of T, we obtain for the low-frequency current fluctuations

$$(j^2)_{\omega=0} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\tau}{2\pi}\overline{\langle\hat{j}(t)\hat{j}(t+\tau)+\hat{j}(t+\tau)\hat{j}(t)\rangle}$$

$$= \frac{1}{2\pi^2 R_N}\operatorname{Re}\iiint_{-\infty}^{+\infty} d\xi_1\,d\xi_2\frac{d\omega}{2\pi}G_1(\omega,\xi_1)G_2\left(\omega-\frac{eV_0}{\hbar},\xi_2\right). \tag{18}$$

Using dispersion relations[9] that express the Green's functions in terms of their imaginary parts, we can express, using the method developed in[8], the resultant integral in terms of the density of states;

$$\rho_s = \operatorname{Im}\frac{1}{\pi}\int G^R\frac{d^3p}{(2\pi)^3},$$

$$(j^2)_{\omega=0} = \frac{1}{4\pi R_N}\int_{-\infty}^{+\infty}dx\left[1-\operatorname{th}\left(\frac{x}{2T}\right)\operatorname{th}\left(\frac{x-eV}{2T}\right)\right]\frac{\rho_s(x)\rho_s(x-eV)}{\rho_N^2}. \tag{19}$$

For the normal metal we get from (19)

$$(j^2)_{\omega=0} = \frac{V}{2\pi R_N}\operatorname{cth}\left(\frac{eV}{2T}\right). \tag{20}$$

In the limiting case $eV \ll T$ we obtain the Nyquist formula, and in the opposite limiting case $eV \gg T$ it is necessary to substitute in the Nyquist formula $eV/2$ in place of the temperature. A similar result is obtained in these limiting cases also for a superconducting contact. Thus, when $eV \gg T$ it is necessary to replace T by $eV/2$ in formulas (5), (6), and (9).

All the formulas derived above were obtained for the width and shape of the radiation line. A similar shape will also be possessed by the absorption line, that is, by the dependence of the direct current flowing through the contact on the frequency of the incident monochromatic radiation.

Thus, the line width turns out to be dependent on the parameters of the contact and the external circuit. If the resistance to the external circuit is sufficiently large, then in most cases the line will have the form (11) and is determined by the capacitance of the contact. A possible exception is the case when the contact is in an external magnetic field. Near resonance, the effective resistance of the contact decreases strongly, and it is necessary to substitute in (5) $R^{-1} = \partial\bar{j}/\partial V_0$ where the dependence of $\bar{j}$ on V_0 is determined by formula (15). In this case the line width is inversely proportional to the slope of the current-voltage characteristic.

The line width decreases in the case when the contact is shunted by a sufficiently small external resistance or by a large capacitance. Broadening of the width of the peak with decreasing external resistance was observed experimentally[10].

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