CONTRIBUTION TO THE THEORY OF THE MOTT EXCITON IN A STRONG MAGNETIC FIELD

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The dependence of the energy of a Mott exciton on its transverse momentum is calculated in the case when the exciton is in a strong magnetic field (the distance between the Landau levels $eH/\mu\varepsilon$ exceeds the Coulomb energy $\mu e^4/c^2\varepsilon^2\hbar^2$, where $\mu$ is the reduced mass and $\varepsilon$ the dielectric constant). The dependence of the probability for exciton production on the transverse component of the photon momentum and on the electric field is also determined.

We introduce the vector operator which plays the role of the momentum of the exciton in all three of its components commute with one another.

This circumstance makes it possible to obtain the dependence of the eigenfunctions of $P(r_1 - r_2)$, where $P$ is the exciton momentum. Finally, substituting (4) in (1) we obtain the equation of relative motion of the electron and hole:

\[
\begin{align}
\frac{d\mathbf{p}}{dt} &= \mathbf{F}^\text{e} + \mathbf{F}^\text{h}, \\
\mathbf{F}^\text{e} &= e\mathbf{E} + e\mathbf{v}_\text{c}, \\
\mathbf{F}^\text{h} &= e\mathbf{E} - e\mathbf{v}_\text{c}.
\end{align}
\]

We shall henceforth confine ourselves to the case of an isotropic dispersion law for electrons and holes. Then the Hamiltonian $H$ of an exciton situated in homogeneous electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is

\[
\begin{align}
H &= \frac{1}{2m_1} \left(-i\hbar \nabla_1 + e\mathbf{A}_1\right)^2 + \frac{1}{2m_2} \left(-i\hbar \nabla_2 + e\mathbf{A}_2\right)^2 + e\mathbf{E} (r_1 - r_2) - \frac{e_2}{c} [\mathbf{r}_1 - \mathbf{r}_2], \tag{3}
\end{align}
\]

We introduce the vector operator

\[
\mathbf{P} = -i\hbar \nabla_1 - i\hbar \nabla_2 - e\mathbf{A}_2 - e\mathbf{A}_2 - \frac{e}{c} [\mathbf{R}, r_1 - r_2], \tag{4}
\]

which plays the role of the momentum of the exciton in a magnetic field. It is easy to verify that the operator $\mathbf{P}$ commutes with the Hamiltonian (1) and, moreover, all three of its components commute with one another.

This circumstance makes it possible to obtain the dependence of the eigenfunctions of $\mathbf{R}$ on the coordinate of the center of gravity of the exciton:

\[
\mathbf{R} = (m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2) / (m_1 + m_2).
\]

Without loss of generality we can choose a definite gauge for $\mathbf{A}$:

\[
\mathbf{A} = \frac{1}{2\mu} [\mathbf{H} r].
\]

Then (2) takes the form

\[
\mathbf{p} = -i\hbar \mathbf{V}_r - \frac{e}{2c} [\mathbf{H} r], \quad r = r_1 - r_2. \tag{5}
\]

The eigenfunctions of (3) are

\[
\psi_p (r_1, r_2) = \exp \left( \frac{i}{\hbar} \int \left( \mathbf{p} + \frac{e}{2c} [\mathbf{H} r] \right) \right) \psi_p (r), \tag{6}
\]

where $\psi_p (r)$ is an arbitrary function of $r$, and $\mathbf{p}$ is the exciton momentum. Finally, substituting (4) in (1) we obtain the equation of relative motion of the electron and hole:

\[
\begin{align}
\frac{d\mathbf{p}}{dt} &= \frac{-e}{2\mu} \mathbf{H} \mathbf{r} + \mathbf{E} \mathbf{p} + \frac{e^2}{8\mu\varepsilon} [\mathbf{H}^2 \mathbf{p} + \frac{e}{M\varepsilon} \mathbf{P} \mathbf{p}], \\
\mu &= m_1 + m_2, \quad M = m_1 + m_2, \quad \gamma = m_3 - m_1.
\end{align}
\]

The quantity $\mathbf{P}$ which enters in this equation has all the properties of a momentum. In particular, the average exciton velocity $\mathbf{V}$ is determined from the usual relation

\[
\mathbf{V} = \partial E / \partial \mathbf{p}. \tag{7}
\]

For the proof we note that, according to footnote 1

\[
\frac{d\mathbf{V}}{dt} = \frac{\partial E}{\partial \mathbf{p}},
\]

where $\mathbf{V} = d\mathbf{R}/dt$ is the operator of the c.m.s. velocity. On the other hand, $\mathbf{V}$ coincides with the derivative of the Hamiltonian of (5), $\mathcal{H}_\mathbf{p}$, with respect to the momentum

\[
\frac{\partial \mathcal{H}_\mathbf{p}}{\partial \mathbf{p}} = \frac{1}{\hbar} \left( \mathbf{p} + \frac{e}{c} [\mathbf{H} r] \right) = \mathbf{V}.
\]

We shall henceforth confine ourselves to the case of an electric field and momentum perpendicular to the magnetic field; the latter will be assumed directed along the z axis. The transverse components will be designated by the index $\rho$. 

449
By using the transformation
\[ \Psi(r) = \Phi(r - r_0) \exp(i\mathbf{r}'P'/2\hbar), \]
where
\[ \mathbf{P}' = \mathbf{P} + M_e \frac{e}{\hbar} \mathbf{H} \]
we can reduce (5) to the form
\[ (\mathbf{P}'^2 - \mathbf{P}''^2) = \frac{e^2}{2\hbar^2} \mathbf{P}'^2 - \frac{\hbar^2}{2\hbar^2} \frac{\nabla^2}{\nabla^2} + \frac{P_0^2}{2M_e} - \frac{\mathbf{P}''^2}{2M_e} \Phi = E \Phi. \]

In the case of a strong field we can neglect in the zeroth approximation the Coulomb interaction. Then the dependence of the energy on \( \mathbf{P} \) and \( \mathbf{F} \) is determined by the expression
\[ \frac{\mathbf{P}^2 - \mathbf{P}''^2}{2M_e} = -\frac{e^2}{2\hbar^2} \mathbf{P}'^2 - \frac{\hbar^2}{2\hbar^2} \frac{\nabla^2}{\nabla^2} + \frac{P_0^2}{2M_e} - \frac{\mathbf{P}''^2}{2M_e}. \]
From this follow the physically obvious relations for the drift velocity
\[ \mathbf{V} = \frac{e}{\hbar} [\mathbf{F} \mathbf{H}], \]
and for the dipole moment of the exciton
\[ d = \frac{Mc^2 \mathbf{F}}{\hbar} + \frac{c}{\hbar} [\mathbf{F} \mathbf{H}]. \]

As expected, the total effective mass of the free electron and hole, corresponding to motion across the field, is infinite. On the other hand, the nonzero values of the momentum (in the absence of \( \mathbf{F} \)) describe the average distance between particles
\[ P_0 = -d/e. \]

A finite transverse mass occurs only when the Coulomb interaction is taken into account.

As shown by Elliott and Loudon, the wave function of the exciton \( \Phi \) from (10) can be represented in first approximation in the small parameter of the theory \( \mu_e^4/e^2h^2/(e\mathbf{H}e/\mu_c) \) in the form
\[ \Phi(r) = \psi(r)\psi(z), \]
where \( \psi(r) \) describes free transverse motion, and \( \psi(z) \) satisfies the equation that results from the averaging of (10) with the aid of \( \psi(r) \).

We confine ourselves to an exciton in the zeroth Landau band. Then
\[ \psi(r) = \frac{1}{\sqrt{2\pi}r_0} \exp\left\{-\frac{\mathbf{r}_0^2}{2r_0^2}\right\}, \]
where \( r_0 = \sqrt{e\hbar/eH} \). The function \( \psi(z) \) satisfies the equation
\[ \left\{-\frac{\hbar^2}{2\mu_c} \frac{\nabla^2}{\nabla^2} + U(z)\right\} \psi = \lambda \psi, \]
where
\[ U(z) = -\frac{e^2}{2\mu_c\hbar^2} \int \frac{dp}{\left[(p + P_0)^2 + z^2\right]} \exp\left\{-\frac{p^2}{2z^2}\right\}. \]

The energy of the exciton (with allowance for the spin) is written here in the form
\[ E = \Delta + \frac{\mathbf{P}^2 - \mathbf{P}''^2}{2M_e} + \frac{eH}{2\mu_c} + W, \]
\[ V = \frac{p}{M} + \left( 1 - \frac{M}{M_{c}} \right) \frac{e}{c} \frac{H}{\theta} \]
\[ d = \left( 1 - \frac{M}{M_{c}} \right) \frac{c}{H} \frac{\theta}{\theta} + \left( 1 - \frac{M}{M_{c}} \right) \frac{e}{H} \frac{dH}{dP} \]

From (26) we get the polarizability at a specified exciton momentum \( P \):
\[ \alpha_{P} = \left( 1 - \frac{M}{M_{c}} \right) \frac{c}{H} \frac{\theta}{\theta} \]

Its order of magnitude is \( \varepsilon \tau_{0} \kappa_{B} \), i.e., it is very small. The polarizability at a specified velocity \( \alpha_{V} \), which coincides with the polarizability of the exciton at rest, is
\[ \alpha_{V} = \left( M_{c} - M \right) \frac{c^{2}}{\kappa_{B}} \approx M_{c} \frac{c^{2}}{\kappa_{B}} \]

Its order of magnitude is \( \varepsilon \tau_{0} \kappa_{B} \), as expected from physical considerations.

We present one more formula for the addition to the energy, expressed in terms of \( V \) and \( \theta \) :
\[ \Delta E_{v} = \frac{1}{2} M_{c} V^{2} - \frac{1}{2} \alpha_{v} \varepsilon V^{2} \approx \frac{1}{2} \left( V^{2} - \frac{c^{2}}{\kappa_{B}} \varepsilon V^{2} \right) \]

At large values of \( P' \), the dependence of the exciton energy on the momentum and on \( \varepsilon \) has a more complicated form. We present the corresponding expression for the ground state only:
\[ \Delta E_{v} = -\frac{2\hbar}{\pi \tau_{0} \kappa_{B}} \frac{\lambda P \theta}{2M} + \frac{P - P_{0}}{2M} \]

The expression is given with logarithmic accuracy. When \( P' \ll \hbar / r_{B} \), the principal term is the logarithmic one. To the contrary, when \( P' \gg \hbar / r_{B} \) the logarithmic term becomes a small addition.

In conclusion, let us discuss the dependence of the probability of exciton production on the electric field and on the momentum of the absorbed photon. As it is well known (see, e.g., (41)), this dependence is determined by a factor \( |\Psi(0)|^{2} \), where \( \Psi(x) \) is the wave function of the relative motion of the exciton \( (8) \). According to (8), (9), and (12) we have in the zeroth Landau band
\[ |\Psi_{0}(0)|^{2} = \psi_{0}(0) \psi_{0}^{*}(0) \]
\[ = \frac{1}{2\pi r_{B}^{2}} \exp \left[ -\frac{r_{B}^{2} P^{2}}{2\kappa_{B}} \right] \psi_{0}^{2}(0) \]

The expressions for \( |\Psi_{0}(0)|^{2} \) can be readily obtained by the same method.\[ 2 \]
For the ground state we have
\[ |\Psi_{0}(0)|^{2} = \frac{\lambda}{2\pi r_{B}^{2} \psi_{0}^{2}} \psi_{0}^{2}(0) \exp \left[ -\frac{r_{B}^{2} P^{2}}{2\kappa_{B}} \right] \]

and for the excited states \( (\nu = 1, 2, \ldots) \)
\[ |\Psi_{0}(0)|^{2} = \frac{1}{2\pi r_{B}^{2} \psi_{0}^{2}} \psi_{0}^{2}(0) \exp \left[ -\frac{r_{B}^{2} P^{2}}{2\kappa_{B}} \right] \]