QUASICLASSICAL APPROXIMATION FOR NONSTATIONARY PROBLEMS

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The quasiclassical formula for the barrier penetration is extended to include the case of time-dependent potentials. Calculation of the penetration reduces to a determination of the subbarrier "trajectory" which formally satisfies the equations of classical mechanics and the complex initial condition (the "time" \( t \) for subbarrier motion is imaginary). It is shown that the calculations can be considerably simplified by applying the general formulas to the problem of ionization of atoms in the field of a strong light wave. Formula (47) is obtained for the probability of ionization of a bound level with an orbital angular momentum \( l \) in the field of a wave with an arbitrary elliptic polarization. Some qualitative features of subbarrier motion of a particle in a rapidly varying field are considered in Sec. 4 (\( \omega \gg \omega_1 \) where \( \omega \) is the barrier oscillation frequency and \( \omega_1 \) is the frequency for tunneling through the fixed barrier). In contrast to the case of a stationary field the tunneling probability for \( \omega \gg \omega_1 \) is mainly determined by that part of the subbarrier trajectory which is adjacent to the exit point of the particle from the barrier. In this case the effective width of the barrier decreases with increasing frequency \( \omega \), and consequently the penetration increases.

1. INTRODUCTION

The probability that a particle will pass through a potential barrier is usually calculated in the quasiclassical approximation (under the condition that the penetrability of the barrier is small and the well known conditions under which the semiclassical approach is valid are satisfied). Problems with time-varying barriers, in which the barrier can change while the particle passes through it, have recently become important. An example of such a problem is that of ionization of an atom by the field of a strong light wave (laser pulse focused with the aid of a lens). The theory of this process was considered by a number of workers \([1-5]\), who used rather complicated computation methods. Yet the "insinuation" of an electron through a sufficiently broad and smooth barrier has a quasiclassical character in both a constant and an alternating field, and the lack of a convenient method for calculating the tunneling probability is due only the fact that no quasiclassical approximation has been developed for alternating fields.

The purpose of the present paper is to extent the quasiclassical method to include the nonstationary case. In Sec. 2 we derive formulas (7) and (24) for the probability \( w \) of tunneling through a periodic potential barrier \( V(x, t) \) of arbitrary form and of frequency \( \omega \), and for the momentum spectrum of the emitted particles. The probability \( w \) is determined by a function \( \tilde{W} \) which is calculated along the complex subbarrier particle "trajectory". \( \tilde{W} \) is closely related here to the classical action (see (19)). In Sec. 3 we consider the application of these formulas to the problem of level ionization in an alternating electric field. We derive formula (47) for the most general case (level with orbital angular momentum \( l \) in the field of an elliptically polarized wave), a case not considered in earlier papers \([4,5]\). It turns out here that formula (24) gives not only the correct exponential for the ionization probability, but also the exact form of the pre-exponential factor. A qualitative description of the tunnel effect in a homogeneous field \( V(x, t) = -Fx \cos \omega t \) is the subject of Sec. 4. It is shown that a transition to an imaginary "time" explains many features of the subbarrier motion in an alternating field, such as the narrowing of the barrier with increasing frequency \( \omega \), which was observed in \([5]\). It is shown that when \( \omega \gg \omega_1 \) the tunneling probability is determined essentially by a small section of the subbarrier trajectory near the point of emergence.
2. TUNNEL EFFECT IN THE QUASICLASSICAL APPROXIMATION

The penetrability of a time-varying barrier can be obtained in principle by solving the Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = \{ H_0(x) + V(x, t) \} \psi(x, t) \]  

(1)

with the specified initial condition

\[ \psi(x, t)|_{t=t_0} = \psi_0(x) \exp(-iKt_0), \quad E_0 = -\frac{\hbar^2}{2m}, \quad H_{0\psi_0} = E_{0\psi_0}. \]  

(2)

Here \( H_0(x) = -\frac{\hbar^2}{2m} + V_0(x) \) is the unperturbed Hamiltonian\(^1\), and \( V(x, t) \) is the oscillating potential causing the tunnel transition from the bound state \( \psi_0(x) \) to the continuum state. The following assumptions are made with respect to \( V_0(x, t) \): \( V(x, t) \) varies periodically, \( V(x, t + T) = V(x, t) \), where \( T = 2\pi/\omega \); 2) the potential \( V(x, t) \) is a weak perturbation in the region \( kr \lesssim 1 \) in which the wave function \( \psi_0(x) \) is essentially concentrated, \(|V(x, t)| \lesssim \kappa^2 \) for \( kr \lesssim 1 \); 3) the turning points of the potential \( V_0(x) \) + \( V(x, t) \) lie in the region \( kr \gg 1 \) (for all \( t \)).

Under these conditions, the tunneling probability is determined by the remote "tail" of the wave function \( \psi(x, t) \), and there is a broad region in which the motion is quasiclassical. The perturbation \( V(x, t) \) causes the bound level to be transformed into a quasistationary state, the average lifetime of which is much longer than the atomic times \( \sim \kappa^{-2} \). Let us transform (1) into an equation for the quasistationary state. To this end we arbitrarily divide the potential \( V_0(x) \) into two parts\(^2\):

\[ V_0(x) = V_{\text{sh}}(x) + V_{\text{Coul}}, \]

where \( V_{\text{sh}}(x) \) is the short-range part of the potential and \( V_{\text{Coul}} \) is the Coulomb "tail." Introducing the Green's function \( G(x_2t_2; x_1t_1) \), which describes the motion of the electron when \( r > a \),

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\[ \{ \frac{\partial}{\partial t} + \frac{4}{2} \Delta - V_{\text{Coul}}(x_2) - V(x_2, t_2) \} G(x_2t_2; x_1t_1) = i\hbar (x_2 - x_1) \delta (t_2 - t_1), \]

(3)

we rewrite Eq. (1) with initial condition (2) in the form of an integral equation:

\[ \psi(x_2, t_2) = -i \int_{t_0}^{t_2} dt \int dx_1 G(x_2t_2; x_1t_1) V_{\text{sh}}(x_1) \psi(x_1, t_1) \]

\[ + \int dx_1 G(x_2t_2; x_1t_1) \psi(x_1, t_0). \]

(4)

The last term in (4) describes the smearing of the initial state and attenuates in proportion to \( \kappa^{-2}(t - t_0) \)^{3/2}.

The usual procedure of switching \( V(x, t) \) on adiabatically at \( t_0 = \infty \) leads to the following equation for the quasistationary state

\[ \psi(x_2, t_2) = -i \int_{t_0}^{t_2} dt \int dx_1 G(x_2t_2; x_1t_1) V_{\text{sh}}(x_1) \psi(x_1, t_1). \]

(5)

The factor \( V_{\text{sh}}(x_1) \) cuts off this integral at \( x |x_1| \lesssim 1 \), so that \( \psi(x_1, t_1) \) can be replaced by the wave function of the unperturbed atom\(^4\):

\[ \psi(x_2, t_2) = -i \int_{t_0}^{t_2} dt \int dx_1 G(x_2t_2; x_1t_1) V_{\text{sh}}(x_1) \psi_0(x_1). \]

(6)

With the aid of calculations similar to those made in \(^4\) we obtain the flux of particles going off to infinity, and obtain the tunneling probability \( w \) in the form of a sum of the probabilities of the multi-photon processes:

\[ w = \sum n \omega_n, \quad \omega_n = 2\pi \int dp \delta(p^2/2 - p_n^2/2) |F(p)|^2, \]

(7)

where \( p_n = \sqrt{2(n - \nu) \omega} \) and

\[ F(p_n) = \lim_{t \to \infty} \int_{t_0}^{t} \psi(p_n, t) \]

(8)

The quantity \( \nu \) in (7) yields the tunneling threshold (the minimum number of quanta that must be absorbed in order for an electron to become detached from the atom). To determine \( \nu \), we take account of the fact that the electron emerging from under the barrier is under the influence of the field

\[ V(x, t) = \sum_{n=1}^{\infty} V_n(x) \cos(n \omega t + a_n) \]

(the atomic potential \( V_0(x) \) can be neglected as \( r \to \infty \)). Treating the influence of \( V(x, t) \) by the

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\(^1\)We use the atomic system of units: \( e = \hbar = m = 1 \).

\(^2\)An example of a perturbation \( V(x, t) \) satisfying the conditions (1) - (3) may be the potential (26) with \( F \ll F_0 = \kappa^2 \).

\(^3\)For example, we can put:

\[ V_{\text{sh}}(x) = e^{-r/a} V_0(x), \quad V_{\text{Coul}}(x) = (1 - e^{-r/a}) V_0(x) \approx -\kappa e^{-1}(1 - e^{-r/a}). \]

Here \( \kappa a \gg 1 \); \( a \) plays the role of the continuity point and drops out from the final formulas. It is necessary to separate \( V_{\text{Coul}} \) because the Coulomb interaction distorts the asymptotic form of the wave function at arbitrarily small distances from the atom, and must be taken into account when subbarrier motion is considered.

\(^4\)A rigorous justification for this substitution is given in the appendix.
Kapitza method\cite{6,7}, we replace $V(x, t)$ by an effective time-independent potential:

$$V_{\text{eff}} (x) = \frac{1}{4\omega^2} \sum_{n=1}^{\infty} \frac{f_n^2(x)}{n^2}, \quad f_n(x) = -\frac{\partial V_n}{\partial x}. \tag{9}$$

Since $V_{\text{eff}}(x) > 0$, the electron can go off to infinity only if $V_{\text{eff}}(x)$ is bounded when $|x| \to \infty$. In this case

$$\nu = \frac{\omega_0}{\omega} \left[ 1 + \frac{1}{2}(\omega_0) \sum_{n=1}^{\infty} \frac{f_n^2}{n^2} \right], \quad f_n \to \lim_{n \to \infty} f_n(x) \tag{10}$$

(we have confined ourselves here for simplicity to one-dimensional motion). This leads to a limitation on the potential $V(x, t)$:

$$|V(x, t)| \leq C r \quad \text{for} \quad r \to \infty. \tag{11}$$

If $V(x, t)$ contains terms that are linear in $x_1$ as $r \to \infty$ (see, e.g., (26)), this will affect the values of $\nu$ and $p_n$. The increase of $\nu$ is proportional to the kinetic energy, averaged over the period, of an electron moving to infinity in a potential $V(x, t)$.

The partial probability $w_n$ of tunneling with absorption of $n$ quanta of energy $\hbar \omega$ is determined, in accordance with (8), by the quantities $F(p_n)$. It is possible to derive for them simple quasiclassical formulas. The wave function $\psi(p_0, t)$ corresponding to a state with a definite average momentum $p_n$ at infinity, satisfies an equation similar to (6), in which $G(x_2 t_2; x_1 t_1)$ must be replaced by the Green's function in the mixed $(p, x)$ representation:

$$G(p_2 t_2; x_1 t_1) = \frac{1}{(2\pi)^{1/2}} \int e^{-ipx} G(x_2 t_2; x_1 t_1) dx_2. \tag{12}$$

Using for $G(x_2 t_2; x_1 t_1)$ the quasi-classical approximation\cite{8,10}:

$$G(x_2 t_2; x_1 t_1) \sim \frac{\theta(t_2 - t_1)}{[2\pi(t_2 - t_1)]^{1/2}} e^{-i\theta x_2 - i\theta x_1}, \tag{13}$$

$$S(x_2 t_2; x_1 t_1) = \int_{t_1}^{t_2} \left( \frac{1}{2} x^2 - V_{\text{Coul}}(x) \right) dt \tag{14}$$

and calculating the integral in (12) by the saddle-point method, we obtain the quasiclassical asymptotic expression for $G(p_2 t_2; x_1 t_1)$:

$$G(p_2 t_2; x_1 t_1) \sim \frac{1}{(2\pi)^{1/2}} e^{iW(p_2 t_2; x_1 t_1)}, \tag{15}$$

where

$$W(p_2 t_2; x_1 t_1) = S(p_2 t_2; x_1 t_1) - p_0 x_2. \tag{16}$$

The action $S$ in (15) is calculated along the classical trajectory defined by the conditions $x(t_1) = x_1$ and $p(t_1) = p_1$. The quantities $x_2$ and $p_2$ are not independent variables and are determined from (16). By varying $W$ at fixed values of $t_1$ and $t_2$ we obtain\footnote{With allowance for the well known formula $\delta S = p_1 \delta x_2 - \delta p_1 x_1$ (see [7]).} $\delta W = -x_2 \delta p_2 - p_1 \delta x_1$, whence

$$\frac{\partial W}{\partial p_2} = -x_2, \quad \frac{\partial W}{\partial x_1} = -p_1. \tag{17}$$

For the total derivative $dW/dt$, taken along the trajectory, we get from (15): $dW/dt_1 = dS/dt_1 = -L(t_1)$; on the other hand,

$$\frac{\partial W}{\partial t_1} = \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x_1} x_1 \frac{\partial W}{\partial t} - \frac{\partial W}{\partial p_1} p_1. \tag{18}$$

Comparing these expressions, we get

$$\frac{\partial W}{\partial t_1} = p_1 x_1 - L(t_1) = H(t_1). \tag{19}$$

Substituting formula (15) for $G$ in (6), we arrive at an integral containing a rapidly oscillating exponential:

$$\psi(p_2 t_2) = \frac{e^{-iE \delta t}}{(2\pi)^{1/2}} \int dt \int dx_2 e^{iW(p_2 t_2; x_1 t_1)V_{\text{Coul}}(x_1)\delta p_0(x_1)}, \tag{20}$$

where

$$W(p_2 t_2; x_1 t_1) = S(p_2 t_2; x_1 t_1) - p_0 x_2 = \int L(t) + E \delta t - p_0 x_2 \tag{21}$$

$S$ is the so-called reduced action\cite{7}. The main contribution to the integral (18) is made by the saddle point. The saddle-point conditions with respect to the variables $t_1$ and $x_1$ have, with allowance for (16) and (17), the form

$$H(t_1) = E_0 = -\nu^2/2, \quad p(t_1) = 0. \tag{22}$$

Among all the paths that contribute, according to Feynman\cite{8,9}, to $\psi(p_2 t_2)$, the only ones that "survive" in the quasiclassical case are those lying in the vicinity of the classical trajectory. The specific feature of our problem is that there exists no real trajectory satisfying Newton's equations, the initial conditions (20), and the condition $p(t_2) = p_0$, since the passage of the particle through the potential barrier is impossible in classical mechanics. This causes the "initial instant" $t_1^0$ to go off to the complex plane.\footnote{\textsuperscript{5}}

Nonetheless, the formal apparatus of classical mechanics continues to operate.

The saddle point $x_1(t_1^0) = x(t_1^0)$ lies in the region $\kappa r \gg 1$ (see [11]), making it possible to substitute in (18) $\delta p_0(x)$ in the form

$$\delta p_0(r) = C \kappa^{3/2}(\kappa r)^{3/2} e^{-\kappa r} Y_{3/2}(rt), \quad \lambda = \kappa r / x. \tag{23}$$

The factor $(\kappa r)^{3/2}$ is the contribution of the Coulomb "tail" to the action $S$ at the point $r$. Indeed,
\[ S(r) = \frac{t}{t_n} \int r \left( 1 - \frac{r_n^2}{r} \right)^{3/2} dr = i \chi \left( \frac{r}{r_0} \right)^{1/2} \]

\[ \sim r_0 \text{Arc}
\]

\[ \left( \frac{r}{r_0} \right) \approx i(\kappa r - \lambda \ln \kappa r + \text{const}) \text{for } r \gg r_0. \quad (22) \]

Here \( r_0 = 2\lambda /\kappa, \quad \varphi_0(r) \sim r^{-1} \exp[iS(r)] \) for \( \kappa r \gg 1. \)

We shall combine the term \((\kappa r - \lambda \ln \kappa r)\) in the argument of the exponential with the corresponding terms in the function \( W \), after which the action \( S \) in (14) should be calculated with the exact atomic potential \( V_0(x) \), and the continuity parameter \( a \) drops out. As a result we get

\[ \lim_{t \rightarrow \infty} \left[ \frac{F(t)}{2t_2 - t_0} \right] = -\frac{\alpha}{2\pi} C_{\alpha} \left[ \frac{\kappa}{1(\delta W/\delta\beta)^2} \right] \gamma \]

\[ \times Y_{\text{ins}}(\alpha_0) \exp \{ i[\mathcal{W}(p_2; x_1^0, t_1^0) - E_m] \} \quad (23) \]

(only the zeroth harmonic of (18), which increases linearly with \( t_2 \), is of importance in the calculation of this limit).

All the saddle points lying in the strip \( 0 \leq \text{Re } t \leq T \) \( (T = 2\pi /\omega) \) contribute to (23). Let us consider those saddle points with the smallest values of \( \text{Im } \mathcal{W} \) and denote their number by \( g \) (thus, \( g = 2 \) for the field (26)). Assuming that we can neglect the interference between the individual saddle points, we get:

\[ F(p) = \frac{\kappa}{2\pi} C_{\alpha} \left[ \frac{\text{Im } \mathcal{W}}{1(\delta W/\delta\beta)^2} \right] \gamma \]

\[ \times Y_{\text{ins}}(\alpha_0) \exp \{ -2i\text{Im } \mathcal{W}(p_2; x_1^0, t_1^0) \} \}, \quad (24) \]

where \( \alpha_0 \) is a (complex) unit vector specifying the direction of the classical trajectory at the turning point (20).

Formulas (7) and (24) determine the tunneling probability \( W_0 \) and the momentum spectrum of the emerging particles. We emphasize that (24) contains only quantities pertaining to the classical trajectory of the particle, and the value of \( |F(p)|^2 \) depends only on the subbarrier section of the trajectory. A highly illustrative description of the subbarrier motion is obtained by going over to imaginary time (see Sec. 4). For subbarrier motion in a constant field we have \( p_0 = 0 \) and \( W \sim S \); in addition, by virtue of the energy conservation law, we have \( S = \int p^2 dt \). Replacing in (7) the summation over \( n \) by integration, we obtain the penetrability of the static barrier

\[ D \sim \exp \left\{ -2 \left[ \int_{x_1}^{x_2} p^2 dx \right] \right\} \quad p^2 = 2(E - V(x)) < 0. \quad (25) \]

Here \( x_1 \) and \( x_2 \) are the classical turning points, and the integral is taken over the extremal trajectory that minimizes \( \text{Im } S(x_1, x_2) \). In the one-dimensional case, the question of finding the extremal trajectory drops out, and (25) goes over into the well-known formula \((11)\) for the coefficient of transmission through the barrier.

In the adiabatic case \((\omega \ll \omega t)\), the extremal trajectory \( x(t) \) is close to the corresponding trajectory \( x_0(t) \) in a constant field. Therefore we can easily get with the aid of (19) and (24) the general form of the adiabatic correction to the tunneling probability (see \((12)\)).

3. IONIZATION OF ATOMS BY AN ELECTRIC FIELD

To calculate with the aid of (7) and (24), it is necessary to specify more concretely the perturbation \( V(x, t) \) that causes the tunneling. We put

\[ V(x, t) = -F(t)x, \quad F(t) = \{F\cos\omega t, eF\sin\omega t, 0\}, \quad (26) \]

which corresponds to the problem in which the atom is ionized by the field of a light wave with ellipticity \( \epsilon (-1 \leq \epsilon \leq 1) \). Using the equation of motion \( \ddot{x} = -\nabla V_0(x) + F(t) \) and the condition (20), we transform formula (19) into

\[ \mathcal{W}(p, \beta; x_1, t_1) = \int_{t_1}^{t_2} \left( \frac{1}{2} p^2 + r_0^2 \right) dt. \quad (27) \]

where \( U_0(x) = V_0 - x_1 \beta V_0 / \partial x_1 \). In (27) we put \( t_2 = 0 \), so that \( \text{Im } \mathcal{W} \) does not change after the particle emerges from under the barrier. Let us find the most probable value of the momentum \( p = p_0 \) on emerging from under the barrier, and the form of \( \mathcal{W}(p) \) near \( p = p_0 \). Going over to the dimensionless variables \(^6\)

\[ \xi = \frac{x_0}{x}, \quad \tau = \omega t, \quad k = (p - p_0) / x, \]

we write out the expansion

\[ \xi(\tau) = \xi_0(\tau) + \xi_1(\tau) + \xi_2(\tau) + \ldots \quad \xi_n \sim k^n. \quad (28) \]

Here \( \xi_0(\tau) \) is the extremal trajectory that minimizes \( \text{Im } \mathcal{W} \), and \( \xi_n(\tau) \) are the corrections for it \((k \text{ is assumed to be a small parameter})\). Using the equations of motion and formula (27), we obtain a chain of equations for the determination of \( \xi_n(\tau) \). Cutting this chain off at \( n = 2 \), we arrive at the following results; \((12)\)

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\(^6\) The "time" to for the subbarrier trajectory is pure imaginary, and therefore \( t \) takes on real values.
1) The most probable momentum $p_0$ is given by the condition
$$\Re \langle \hat{x}_0 \rangle \tau_0 = 0 \quad \text{or} \quad \Im \langle \hat{x}_0 \rangle (0) = 0. \quad (29)$$

2) The expansion of $\hat{W}(p)$ in the vicinity of the minimum point $p = p_0$ is
$$\hat{W}(p) = \frac{\omega_0}{\omega} f(p_0) + a_{ij} k_i k_j + \ldots, \quad (30)$$
where
$$f(p_0) = \int_{-\tau_0}^{\tau_0} \left\{ 1 - \frac{2 + 2p_0^2}{\omega} U_0 \left( \frac{\hat{x}_0}{\omega} \right) \right\} d\tau, \quad (31)$$
and the factor $a_{ij}$ does not depend on $k$. Putting
$$a_{ij} k_i k_j = -\hat{x}_0 \hat{\xi}_0 \langle \hat{x}_0 \rangle (0), \quad (31a)$$
Owing to the factor $\omega_0^2/\omega > 1$, the momentum distribution is narrow, so that the retention in (30) of only terms quadratic in $k_1$ is justified.

Thus, to determine the ionization probability it is sufficient to find the extremal trajectory $\xi(t)$ and the quantities $a_{ij}$ connected with the correction $\xi(t)$. They satisfy the following equations:
$$\ddot{\xi}_0(t) = -\frac{1}{\gamma} f(\xi) - \nabla U(\xi), \quad \dot{\xi}_0(0) = 0, \quad (33)$$
$$\ddot{\xi}_0(t) = -\frac{\partial U}{\partial \xi_j} (\xi_j) (0), \quad \dot{\xi}_0(0) = - ik_1, \quad (34)$$
where
$$f(\xi) = \{\chi \tau_0 - \gamma \chi \tau_0 \}, \quad U(\xi) = -\chi^2 V_0(\xi/\omega).$$

The trajectory $\xi(t)$ can be obtained for an arbitrary atomic potential $V_0(\tau)$ only by numerical calculation. We confine ourselves here to the simplest case of a $\delta$-potential ($V_0(\tau) = 0$ with $\tau > 0$), which was already considered earlier [3-4].

In this case the condition (20) is somewhat modified:
$$x(\tau_0) = 0, \quad p^2(\tau_0) = -\chi^2 \text{ or } \xi^2(-\tau_0) = 1. \quad (35)$$

The extremal trajectory $\xi(t)$ is determined from (33) and (26).\(^7\)

\(^7\)It satisfies the equation $\ddot{\xi}_0 = -f(\tau)/\gamma$. If the $i$-th component of the force $f_1(\tau)$ is real, then we get from the conditions (29), (33), and (35) that $f_1(\tau) = 0$; on the other hand if $f_1(\tau)$ is pure imaginary, then $\ddot{\xi}_0(0) = 0$. For the field (25) of elliptically polarized light it follows therefore that the momentum $p_0$ is directed along the $y$ axis, with $\xi^2(-\tau_0) = 0$. The condition leads directly to expression (45) for $p_0$.\(^8\)

\(^8\)The most probable momentum of the emitted electrons differs from zero when $0 < |\psi| < 1$ and is directed along the $y$ axis:
$$\xi(0) = \left\{ \begin{array}{ll}
\frac{1}{\gamma} (\gamma - \chi \gamma), & \text{if } \xi^2(-\tau_0) = 0, \\
\frac{e}{\gamma} \left( \frac{\sh T_0}{T_0} - \frac{\sh T}{T} \right), & \text{otherwise}.
\end{array} \right. \quad (36)$$

where $\tau_0 = \tau_0(\gamma, \epsilon)$ is the positive root of the equation
$$\sh^2 T_0 - \epsilon^2 \left( \left( \frac{\sh T_0 - \sh T}{T_0} \right)^2 \right) = 0. \quad (37)$$

The solution of Eq. (34) for $\xi(t)$ is
$$\xi(t) = a + i(\omega_0 k + k_1),$$
and the initial condition at $\tau = \tau_0$ assumes in the case of a $\delta$-potential the form
$$\xi(t) = \lambda_0(\tau), \quad (38)$$
From this we get $a_{ij} = \lambda_0 + \tau_0 \delta_{ij}$. Formula (32) for the pre-exponential factor simplifies, when account is taken of the condition (35), to
$$f(\tau) = \chi T_0^2 \left( 1 - \epsilon^2 + \epsilon^2 \sh T_0 \right). \quad (39)$$

Substituting expression (36) for $\xi(t)$, we arrive at the following formula for the momentum spectrum of the electrons emitted upon ionization of a bound level with orbital angular momentum $l$:
$$F(p) = 2 \pi D(y, \epsilon) Y_{lm}(\hat{n}_0) \left| 2 \pi \right|^{2} \exp \left( -\frac{2 \pi \omega_0}{\omega} \right) \left| f(y, \epsilon) \right|$$
where
$$f(y, \epsilon) = \left( 1 + \frac{1 + \epsilon^2}{2 \gamma} \right) \left( 1 - \epsilon^2 + 2 \epsilon \sh T_0 \right) \left( \frac{\sh T_0}{\gamma} \right). \quad (40)$$

$$\begin{align*}
\Delta &= \frac{\tau_0}{\sh T_0} \left( 1 - \epsilon^2 - \sh T_0 \right), \\
\Delta &= \frac{\epsilon^2 \left( 1 - \sh T_0 / \tau_0 \right)}{\sh T_0}, \quad (42)
\end{align*}$$

$$c_{\tau} = \tau_0 \left( 1 + \epsilon^2 \left( 1 - \sh T_0 / \tau_0 \right) \right), \quad (43)$$

The unit vector $\hat{n}_0$ defines the direction of the initial velocity. Since $\xi^2(0)$ is imaginary, this vector is complex. Putting $\hat{n}_0 = n_0(\theta, \varphi)$ and $\vartheta = \psi$, we get from (36)
$$\th \psi = \left( \frac{\xi^2}{\xi} \right)_{\psi \rightarrow \infty} = \epsilon (\th T_0 - 1 / \tau_0). \quad (44)$$
\[ p_0 = (0, \pm k_0, 0), \quad k_0 = \frac{\epsilon \, \text{sh} \tau_0}{\gamma / \tau_0} \]  \hspace{1cm} (45)

(see Fig. 2 of [5]). This causes the most probable number of absorbed photons \( n_0(\gamma, \epsilon) \) (with \( \epsilon \neq 0 \)) to exceed the ionization threshold \( \nu = (\omega_0 / \omega) [1 + (1 + \epsilon^2)^{2/2 \gamma^2}] \):

\[
\frac{n_0}{\nu} - 1 = \frac{\epsilon^2}{1 + (1 + \epsilon^2)^{2/2 \gamma^2}} \left( \frac{\text{sh} \tau_0^2}{\gamma \tau_0} \right)^2
= \begin{cases} 
\frac{\epsilon^2}{(1 + 1/2 \gamma^2) (\text{Arsh} \gamma)^2} & \text{for } |\epsilon| < 1 \\
(2 \tau_0 \text{ch} \tau_0 - 1)^{-1} & \text{for } |\epsilon| = 1
\end{cases}
\]  \hspace{1cm} (46)

(see Fig. 1).

Greatest interest attaches to the average ionization probability for unpolarized atoms. The corresponding averaging is effected with the aid of the formula \( Y_{l m^*}(n) \sim Y_{l - m}(n^*) \) and the addition theorem for spherical functions:

\[
\frac{1}{2l + 1} \sum_{m = -l}^{l} |Y_{lm}(n_0)|^2 = \frac{1}{4 \pi} P_l(\text{ch} 2 \psi),
\]

\[
\psi = \text{Arsh} \left\{ \frac{e}{(\text{ch} \tau_0 - \frac{1}{\tau_0})} \right\}.
\]

The parameter \( \psi \) increases monotonically with increasing \( \gamma \); when \( \gamma \gg 1 \) we have

\[
\text{ch} 2 \psi \approx \frac{1 + \epsilon^2}{1 - \epsilon^2 + 2 \epsilon^2 / \ln 2 \gamma}.
\]

When \( l = 0 \), formulas (40)-(43) coincide with the corresponding expressions (23)-(28) from [5], where, however, less convenient variables were used.\(^8\) As seen from the foregoing, all the quantities in the formulas for the ionization probability can be expressed in simple fashion in terms of \( \tau_0 \). The dependence of \( \tau_0 \) on the ellipticity \( \epsilon \) of the light, obtained by numerically solving Eq. (37), is shown in Fig. 2 (\( \tau_0 = \sinh^{-1} \gamma \) when \( \epsilon = 0 \)). We note that the variable \( \tau_0 \) has a simple physical meaning: \( \tau_0 / \omega \) is the total time of motion of the particle under the barrier.

Substituting (40)-(43) into (7) and integrating in the \((p_x, p_z)\) plane, we obtain the partial probability of ionization with absorption of \( n \) photons:

\[
w_n = \frac{C_n \nu^2}{\pi} \left( \frac{\omega}{2 \omega_0} \right)^{\nu_0} \frac{P_l(\text{ch} 2 \psi)}{\text{ch}^2 \psi} R_n
= \exp \left\{ -\frac{2 \omega_0}{\omega} f(\gamma, \epsilon) \right\},
\]  \hspace{1cm} (47)

where

\[
R_n = \left[ \frac{(n - \nu)}{2} \right]^{\nu_0} J_n,
\]

\[
J_n = \int dx \exp \left\{ -\left[ a(1 - x^2) + b(x - x_0)^2 \right] \right\} I_0(c(1 - x^2)).
\]  \hspace{1cm} (48)

The dimensionless constant \( C_n \) is defined in (21), \( I_0(z) \) is a Bessel function of imaginary argument, and

\[
a = (n - \nu) \left[ 1 + \frac{(1 - \epsilon^2)}{\Delta} \right] \nu_0,
\]

\[
b = 2(n - \nu) \left[ 1 + \frac{\epsilon^2(1 - \theta \tau_0 / \tau_0)^2}{\Delta \theta \tau_0} \right] \nu_0,
\]

\[
c = (n - \nu) \Delta^{-1} \tau_0 \left[ (n_0 - \nu) / (n - \nu) \right].
\]  \hspace{1cm} (49)

In the case when \( \epsilon = 0 \) (linear polarization), the integral \( J_n \) can be determined exactly:

\[
J_n = \int dx \exp \left\{ -\frac{2 \omega_0}{\omega} f(\gamma, 0) \right\}.
\]
\[ R_n = e^{-\sigma(n-v)w(\sqrt{b(n-v)})}, \quad \sigma = 2(\tau_0 - \theta \tau_0), \quad b = 2\theta \tau_0, \quad \tau_0 = \text{Arsh} \gamma, \]

where \( w(x) = \exp(-x^2) \int \exp(t^2) dt \) (a plot of \( w(x) \) is given in \(^{[41]}\)). Formula (47) then takes a simpler form

\[ w_n = \omega \left| C_{1d} \right|^2 \left( \frac{\omega}{2\omega_0} - \theta \tau_0 \right)^{\frac{1}{2}} w(\sqrt{b(n-v)}) \times \exp \left\{ -\left[ \frac{2\omega_0}{\omega} f(\gamma) + a(n-v) \right] \right\}, \]

which coincides with formula (61) of \(^{[4]}\). We note that expression (51) for \( w_n \) is more convenient when the frequency \( \omega \) is fixed and \( \gamma \gg 1 \) (tanh \( \tau_0 \approx 1 \)). When \( \epsilon \approx 0 \) and \( \gamma \approx 30-50 \) the determination of the pre-exponential factor \( R_n \) calls for numerical calculations.

The dependence of \( w_n \) on the orbital angular momentum \( l \) is given by the factor \( |C_{1d}|^2 \Pi_l \cosh(2\alpha) \), which increases when \( \epsilon \to 1 \).

4. SOME FEATURES OF SUBBARRIER MOTION IN AN ALTERNATING FIELD

The probability of tunneling through a nonstationary barrier is determined by the function \( \tilde{W} \) calculated along the classical-particle trajectory (see formulas (19) and (24)). In this case the "time" \( t' \) during which the subbarrier motion effected is pure imaginary for the extremal trajectory that minimizes \( \text{Im} \tilde{W} \).

We shall show that a consistent transition to a real variable \( t = it' \) in the equations for the subbarrier motion leads to a clear picture of the passage of the particle through the barrier. We confine ourselves below to the case of a homogeneous field, \( V(x, t') = -F(t') x \), but the results are more general.

We start with linear polarization. The extremal trajectory is one-dimensional and corresponds to a momentum \( P_x = 0 \) at emergence:

\[ x(t) = F_0 e^{-\omega t} (\cosh \omega t_0 - \omega \omega t), \quad -t_0 \leq t \leq 0, \]

\[ x(-t_0) = x, \quad \omega t_0 = \tau_0 = \text{Arsh} \gamma. \]

We see from the equation of motion \( \ddot{x} = -F \cosh \omega t \) that the oscillating external field \( F(t') = F \cos \omega t' \) is transformed in the case of the subbarrier motion into a decelerating force. When \( \gamma \gg 1 \) the decelerating field increases exponentially on going deeper into the barrier. The growth of \( F(t) \) with increasing frequency \( \omega \) leads to a reduction of the effective width of the barrier \( x(0) \):

\[ \alpha x(0) = \frac{F_0}{2F} \left( 1 + \frac{1}{2} \right) = \frac{2\omega_0}{\omega} \left( 1 + \frac{1}{2} \right) \]

In actual experiments, \(^{[13,14]}\) the laser frequency \( \omega \) is fixed, and \( \gamma \gg 1 \); under these conditions, the width of the barriers in units of \( \kappa^{-1} \) (the atomic radius) is simply \( 2\omega_0/\omega \) and does not depend on the field intensity \( F \).

Figure 3 gives an idea of the buildup of the reduced action \( \tilde{S} \) during the course of the subbarrier motion (for the extremal trajectory (36) the difference between \( \tilde{W} \) and \( \tilde{S} \) vanishes, since \( p \cdot x | t=0 = 0 \)). The difference between low fre-
where
\[ H(\xi, \xi; \tau) = \frac{1}{2} \xi^2 + U(\xi) + \frac{\xi \chi \tau}{\gamma}, \quad H(-\theta_0) = \frac{1}{2}, \]
and
\[ U(\xi) = -\frac{1}{\xi^2} V_0(x). \]

With the aid of the equation of motion
\[ (\dot{\xi} = \partial U/\partial \xi - \gamma^{-1} \cosh \tau) \]
we get
\[ \frac{\text{Im} S(\xi)}{\text{Im} S(\xi_0)} = 1 - a(\gamma) \sqrt{1 - \frac{\xi}{\xi_0}} + \ldots. \]

Therefore \( \gamma \geq H \geq 0 \), and it follows from (54) that \( \text{Im} \, S(\tau, -\theta_0) \) is a monotonically increasing function of \( \tau \). Since \( \xi(\tau) = \xi(0) - \gamma \tau^2 \gamma \) when \( \tau \to 0 \), \( \text{Im} \, S \) has a root singularity as a function of \( \xi \) at the final point:
\[ \frac{dH}{d\tau} = \frac{\xi \sinh \tau}{\gamma} < 0 \quad (\tau_0 \leq \tau < 0). \]

where
\[ a(\gamma) = \frac{2}{f(\gamma)} \sinh \tau_0 \left( \frac{\tau_0^2}{2} \right) = \left\{ \frac{2}{\gamma} \frac{\gamma}{2} \right\} \ln \gamma \quad \text{for} \quad \gamma \gg 1. \]

The coefficient \( a(\gamma) \) increases together with \( \gamma \), and this explains the behavior of the curves in Fig. 3b.

We proceed to the case of elliptic polarization. The extremal trajectory is determined by (36);
\[ \xi_y \] is imaginary because the corresponding component of the force \( f \) is imaginary:
\[ \frac{d\xi_y}{d\tau^2} = \frac{1}{\chi} f(\tau), \quad f(\tau) = \left\{ \text{ch} \tau, -i \sinh \tau, 0 \right\}. \]

At the instant of emergence from under the barrier we have \( \xi_x(0) = \chi^{-1} \cosh \theta_0 - 1 \) and \( \xi_y = \xi_z = 0 \); the width of the barrier \( \xi_x(0) \) increases also with increasing ellipticity \( \epsilon \) (see formula (63) of [10]), leading to a decrease in the ionization probability. To explain this phenomenon, we note that the motion along the \( x \) axis obeys the same law as in the case of linear polarization, but the initial velocity \( \xi_x(-\theta_0) \) increases (since \( \xi_x^2(-\theta_0) = 1 \) and \( \xi_y^2(-\theta_0) < 0 \)). This leads to a corresponding increase of the time \( \tau_0 \) needed to stop the particle.

The dependence of the subbarrier motion on the ellipticity of the light \( \epsilon \) can be investigated with the aid of Eqs. (36) and (57). In the main, the influence of \( \epsilon \) leads to the appearance of a "transverse" coordinate \( \xi_y \); although \( \xi_y = 0 \) at the ends of the trajectory, we have here in the intermediate region \( \xi_y \sim \epsilon \xi_x \). Figure 3b shows that when \( \epsilon \neq 0 \) the final section of the trajectory merely assumes a more important role in the buildup of the action \( \tilde{S} \).

This property, which is characteristic of subbarrier motion in the antiadiabatic case \( \gamma \gg 1 \), points to a way of taking Coulomb interaction into account (the Coulomb correction for the case \( \gamma \lesssim 1 \) was obtained earlier [15]). Replacing the Coulomb term \( \sigma/\xi_0^2 \) in the exact equation
\[ \frac{\xi}{\gamma} = \frac{1}{\gamma} \left( \frac{\sigma}{\gamma^2} - \chi \tau \right), \quad \sigma = \left( \frac{\gamma^2}{\chi} \right) \lambda = \frac{\epsilon}{2 \chi} \gamma \]
by the constant force \( \sigma/\xi_0^2 \), we obtain an equation that can be solved analytically. This question will be considered in greater detail in another paper.

We have assumed so far that the particle emerges from under the barrier immediately upon reaching the turning point. In quantum mechanics, however, multiple reflections from the barrier boundary are also significant. This can be seen already in the simplest example:
\[ V(x) = \begin{cases} V_0 & \text{for } 0 < x < a \\ 0 & \text{for } x < 0 \text{ and } x > a \end{cases} \]

A particle with momentum \( k \) \( (k < K = \sqrt{2mV_0}) \) on the barrier can emerge from the left end can emerge from the point \( x = a \) after \( (2n + 1) \) "to and fro" passages. The amplitude of such a process is
\[ A_{2n+1} = d(k', k') \left[ a(k', k) \right]^{2n} \gamma(k', k) \tau^{2n+1} \text{Im} \tilde{S}, \]
where
\[ \tilde{S} = \int_0^a \text{dx} = k'a, \quad k' = ik = i \sqrt{K^2 - k^2} \] is the momentum of the particle under the barrier, and \( a \) and \( b \) are coefficients that take into account the reflection and refraction of the wave at the
points where the potential changes jumpwise. Summing the contributions of all the paths, we obtain the amplitude of the emerging wave:

\[ A = \sum_{n=0}^{\infty} A_{2n+1} = \frac{2ik\kappa}{(k^2 - x^2)^2 - sh^2 \alpha x + 2ik\kappa ch \alpha x}. \]

The wave function of a particle passing through the barrier is of the form \( A \exp \{ ik (x - a) \} \); the penetrability \( D \) of the barrier is equal to

\[ D = |A|^2 = \frac{4k^2\kappa^2}{(k^2 + x^2)^2 - sh^2 \alpha x + 4k^2\kappa^2}. \]

Formulas (63) and (64) coincide with the result of the accurate solution of the Schrödinger equation (see [11], p. 104). In the quasiclassical case \( \text{Im} \, \tilde{S}_0 = \kappa a \gg 1 \), and only the first term, corresponding to a single passage, "survives" in the sum (63). A similar situation obtains for an arbitrary time-constant barrier.

To assess the role of multiple reflections in an alternating field, we replace the short-range potential \( V_0(x) \) by a reflecting wall at the point \( x = 0 \). The subbarrier trajectory consists of \( n \) pieces \( \xi_k (\tau) \), \( 1 \leq k \leq n \) (see Fig. 4). The growth of the decelerating force \( F(\tau) = -F \cosh \tau \) with increasing \( \tau \) causes the amplitudes of the maxima to decrease rapidly with increasing \( n \) when \( \gamma \gg 1 \). The functions \( \xi_k (\tau) \) satisfy the equations

\[ \begin{align*}
\xi_k (0) &= 0, \\
\xi_k (\tau) &= \frac{1}{\gamma} (\tau - \tau_k), \\
\xi_k (\tau) &= -\xi_{k+1}(-\tau_k),
\end{align*} \]

(65)

The result is shown in Fig. 6c. When \( \gamma \gg 1 \), the function \( f_n(y) \), which determines the penetrability of the barrier, for \( n = 1 \) can be discarded when \( \gamma \gg 1 \), and only the first term, corresponding to a single passage, "survives" in the sum (63). A similar situation obtains for an arbitrary time-constant barrier.

In conclusion, we make one remark concerning the replacement of the field of the light wave by a homogeneous electric field \( F(t') = F \cos \omega t' \). As proposed in [14, 5], such a replacement is valid if \( \lambda \gg \kappa^{-1} (\lambda = 2\pi/k \) is the wavelength of the light). It is now clear that this condition is too weak: the external field must be homogeneous not only within the confines of the atom, but at much larger distances on the order of the width of the barrier. The correct condition follows from (53):

\[ \frac{\text{ch} \tau_k - \text{ch} \tau_k}{\tau_k - \tau_k} = \frac{2 \text{sh} \tau_k}{\tau_k - \tau_k} = \frac{2 \text{sh} \tau_k}{\tau_k - \tau_k}. \]

When a plane wave is incident on an infinite step, a reflected and a refracted wave are produced:

\[ \begin{align*}
\psi(x) &= e^{i\kappa x} + a(k_1, k_2) e^{-i\kappa x}, & \text{for } x < 0, \\
\psi(x) &= b(k_1, k_2) e^{i\kappa x}, & \text{for } x > 0.
\end{align*} \]

Here \( k_1 \) and \( k_2 \) are the wave vectors on the left and on the right of the interface \( x = 0 \), and

\[ a(k_1, k_2) = \frac{k_1 - k_2}{k_1 + k_2}, \quad b(k_1, k_2) = \frac{2k_1}{k_1 + k_2}. \]

It is seen from (55) that the sign of the potential \( V_0(x) \) is reversed on going to the imaginary "time" \( \tau \).
\[ k x_0 = \frac{2 \pi}{\lambda} \frac{x_0}{\lambda} = \sqrt{\frac{2I}{mc^2}} \sqrt{1 + \sqrt{1 + \nu^2}} \ll 1, \quad (69) \]

where \( I = \kappa^2/2 \) is the ionization potential and \( m \) is the electron mass. For atoms \((2I/mc^2)^{1/2} \sim 10^{-2}\) and condition (69) is satisfied.

### APPENDIX

To justify the approximation made on going over from (5) and (6) (a similar approximation was used in essence in [4,5]), we determine the correction to \( \psi_0(x, t) = \psi_0(x) \exp(iK^2t/2) \) by perturbation theory. We confine ourselves here to the particular case \( V(x, t) = -Fz \cos \omega t \). Putting \( \psi = \psi_0 + \psi_1 + ... \) we get the following equation for the correction \( \psi_1 \):

\[
\left( i \frac{\partial}{\partial t} - H_0 \right) \psi_1 = V(x, t) \psi_0 = -F \cos \omega t \psi_0(x, t).
\]

(A.1)

In the simplest case \( l = 0 \), (s-level), the variables in (A.1) separate, and the solution takes the form

\[
\psi(x, t) = \psi_0(x, t) \left[ 1 + \frac{F}{2F_0} \cos \theta \frac{f_\pm(\xi)}{2} e^{-i\theta} + \frac{F}{2F_0} \frac{f_\mp(\xi)}{2} e^{i\theta} \right],
\]

(A.2)

where \( F_0 = k^2 \) (field inside the atom), \( \xi = kx \), and the functions \( f_\pm(\xi) \) satisfy the equation

\[
\frac{df_\pm}{d\xi} + 2a(\xi) \frac{df_\pm}{d\xi} + \left( \pm \eta - \frac{2}{\xi^2} \right) f_\pm = -4\xi.
\]

(A.3)

Here \( \eta = \omega/\omega_0 \) and \( a(\xi) = d \ln (r \varphi_0)/d(\kappa r) \); \( a(\xi) = -1 \) for the level in the \( \delta \)-potential, and \( a(\xi) = -1 + \xi^{-1} \) for the ground state of the hydrogen atom. Equation (A.3) can be readily solved if \( \eta = 0 \) (case of constant field):

\[
f_\pm(\xi) = \begin{cases} \xi^2 & \text{for } \delta\text{-potential}, \\ \xi^2 + 2\xi^2 & \text{for hydrogen atom}. \end{cases}
\]

(A.4)

The difference between \( \varphi(x) \) and the wave function \( \varphi_0(x) \) of the free atom becomes appreciable when \( r \approx r_\star = \kappa^{-1} \sqrt{F_0/F} \). This conclusion does not depend on the particular form of the potential \( V_0(x) \); it can be shown that

\[
\frac{\varphi(r)}{\varphi_0(r)} = 1 + \frac{F}{2F_0} (\varphi r)^2 \cos \theta + O\left( \frac{F}{F_0} \varphi_0 (\frac{F}{F_0}) \right)
\]

(A.5)

when \( \kappa^{-1} \ll r \ll r_\star \), independently of \( V_0(x) \) and of the orbital angular momentum \( l \). We shall therefore confine ourselves to the case of the \( \delta \)-potential in our treatment of an alternating field.

The exact solution of (A.3) takes the form

\[
f_\pm(\xi) = \frac{2}{n_\pm^2(1 - n_\pm/2)} \int \left( 1 - n_\pm + \xi^{-1} \right) (e^{n_\pm} - 1)
\]

\[
- n_\pm \left( 1 + \left( 1 - \frac{n_\pm}{2} \right) \xi \right)^{-1},
\]

(A.6)

where \( n_\pm = 1 - \sqrt{1 + \eta} \). When \( \omega \ll \omega_0 \) the expressions for \( f_\pm(\xi) \) simplify to:

\[
f_\pm(\xi) \approx \frac{2}{n_\pm^2} \left[ e^{n_\pm} - (1 + n_\pm \xi) \right] = \sum_{i=0}^{\infty} \frac{2}{(k+2)^2} (n_\pm \xi)^k.
\]

(A.7)

From this we get

\[
\begin{aligned}
\psi(x, t) &= \left[ 1 + \frac{F}{2F_0} \frac{\varphi_0(\xi)}{2 \eta} \cos \omega t + \ldots \right.
\text{for } \xi \ll 1, \\
\psi_0(x, t) &= \left. \left[ 1 + \frac{F}{2F_0} \frac{\varphi_0(\xi)}{2 \eta} \cos \omega t + \ldots \right. \right]
\text{for } \xi \gg 1.
\end{aligned}
\]

(A.8)

The range of values of \( R \) for which the substitution \( \psi(x, t) \rightarrow \psi_0(x, t) \) is valid is given by the inequality \( r \ll r_\star \), where

\[
\gamma_\star = 2\eta \frac{\ln \left( 1 + \frac{\gamma}{\gamma_0} \right)}{\omega_0} = \left( \frac{F_0/F}{\gamma_0} \right) \gamma_0.
\]

(A.9)

(here \( \gamma = \omega k/F \) and \( \gamma_\star = \sqrt{F_0/F} \gg 1 \)). We note that \( r_\star \) assumes a value on the order of the atomic radius \( (\kappa^{-1}) \) only when \( \omega \ll \omega_0 \) and therefore, for short-range potential, the substitution of \( \psi_0 \) for \( \psi \) in (5) is valid if \( \omega \ll \omega_0 \) as was proposed in [4,5]. It is also of interest to compare \( r_\star \) with the dynamic width of the barrier \( r_0 \) defined by formula (53). In a constant field \( r_0 \gg r_\star \); when \( \gamma = \gamma_\star \) the values of \( r_0 \) and \( r_\star \) become comparable in magnitude and become of equal order of magnitude when \( \gamma \) is increased further.

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