ELECTRODYNAMICS OF THE INTERMEDIATE STATE

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A complete set of electrodynamic equations, averaged over volumes with dimensions considerably exceeding the layer thickness, is derived for the intermediate state of a superconductor of the first kind at low temperatures. These equations yield the dc resistance with allowance for the effect of the magnetic field of the current on the intermediate-state structure. It is shown that electromagnetic waves similar to normal metal helicons can be propagated in the intermediate state of superconductors with unequal numbers of electrons and holes. The surface impedance of the intermediate state in a varying external field is calculated.

WHEN a superconductor goes over into the intermediate state, its volume breaks up into a number of alternating layers of normal and superconducting phases.[1] The equilibrium dimensions of the layers, determined from the condition that the energy be minimal, have an order of magnitude $(L\xi_0)^{1/2}$, where L is the dimension of the body and $\xi_0 \sim 10^{-4}$ cm is the length parameter of superconductivity theory. At not too small sample dimensions ($L \gg \xi_0$), the thickness of the layers is much smaller than L.

If the superconductor is placed in an external electric field or an alternating magnetic field, then the boundaries between the phases become curved and, generally speaking, start moving. The thickness of the layers changes, too. The determination of the velocity of motion and of the layer configuration is a very complicated problem. However, since most experimentally-measured quantities characterizing the superconductor are mean values taken over a large number of layers, it is very useful to obtain in this case a macroscopic description of the intermediate state. In this description one introduces quantities that are averaged over regions whose linear dimensions greatly exceed the layer thickness. Such quantities are the magnetic induction **B** and the electric field **E**, defined as mean values of "microscopic" intensities of the magnetic and electric fields **h** and **e**. The fields **h** and **e** are quantities which have already been averaged over volumes with dimensions considerably larger than the interatomic distance, but are at the same time still small compared with the layer thickness. Owing to the very low magnetic susceptibility of the normal metal, we can ignore the difference, in the normal phase, between the field **h**, which is essentially the "microscopic" magnetic induction, and the "microscopic" magnetic intensity. In the superconducting phase **h** and **e** vanish.

The macroscopic description of the intermediate state was introduced by Peierls[2] and London.[3,4] They obtained equations that make it possible to determine the averaged fields in the state of thermodynamic equilibrium, and also in certain particular static cases in the presence of an electric field.

In this paper we obtain a complete system of macroscopic equations describing the electromagnetic properties of the intermediate state at low temperatures. These equations yield the dc resistance. In a weak current, the resistance of the intermediate state is equal to the resistance of the normal phase multiplied by its concentration. With increasing current, the influence of its magnetic field on the structure of the intermediate state becomes important. The resistance becomes dependent on the current, reaching ultimately the value characteristic of the pure normal metal.

Weakly damped electromagnetic waves with quadratic dependence of the frequency on the wave vector can propagate in the intermediate state of a sufficiently pure superconductor with unequal number of electrons and holes. In an earlier paper[5] we obtained the spectrum of these waves by a microscopic procedure. The presence of undamped oscillations exerts an essential influence on the behavior of the intermediate state in an alternating external field. They were recently observed experimentally by Maxfield and Johnson.[6]

1. FUNDAMENTAL EQUATIONS

The "microscopic" fields **h** and **e** in the normal phase satisfy Maxwell's equations

$$\mathrm{rot}\,\mathbf{h} = \frac{4\pi}{c}\mathbf{i}, \quad \mathrm{div}\,\mathbf{h} = 0, \quad \mathrm{rot}\,\mathbf{e} = -\frac{1}{c}\frac{d\mathbf{h}}{\partial t}. \qquad (1)^*$$

where **i** is the electric current density. On the interphase boundary the tangential component of the electric field and the normal component of the magnetic field should be continuous in the coordinate frame attached to the boundary.[7] Since the actual values of the electric field in the normal phase are always much lower than those of the magnetic field, and since $\mathbf{h} = \mathbf{e} = 0$ in the superconducting phase, these conditions reduce to the vanishing on the phase boundary of the component of the vector **h** which is normal to the boundary and of the tangential component of the vector $\mathbf{e} + \mathbf{V} \times \mathbf{h}/c$, where **V** is the velocity of the boundary.

Let us assume that the configuration and the velocity of the layers change little over distances of the order of their thickness. In this case the fields **h** and **e** are practically constant along the thickness of the normal layers. The latter makes it possible to obtain a simple connection between the current density **i** and the intensities **e** and **h**.

The current density should be determined by solving the kinetic equation for the electron distribution function of the normal phase. If, as usual, we seek the distribution function in the form $f = f_0 + \chi\, \partial f_0/\partial E$, where f_0 is the equilibrium function, E is the electron energy, and χ a new unknown function, then the equation for χ takes the form

$$\mathbf{v}\frac{\partial\chi}{\partial\mathbf{r}} + \Omega\frac{\partial\chi}{\partial\tau} + I\{\chi\} = -e\mathbf{v}\mathbf{e}. \qquad (2)$$

Here **v** is the electron velocity, Ω the cyclotron frequency in the field **h**, and τ the dimensionless revolution time (see [8]), e the electron charge, and $I\{\chi\}$ the collision integral. We have neglected in (2) the term containing the derivative of χ with respect to time, assuming that the characteristic frequencies of the motion are much lower than the collision frequency. This condition imposes no real limitation on the frequencies, since actually the macroscopic description becomes invalid much earlier, owing to the fact that the characteristic distance over which all the quantities change becomes comparable with the dimension of the layers.

It is necessary to add to (2) the boundary conditions on the interface with the superconducting phase. These conditions depend on the character of the electron reflection from the boundary, which in this case has a number of specific peculiarities.[9] The point is that the width of the transition layer between the phases (the order of magnitude of which is $\xi_0 \sim v/T_c$, where T_c is the temperature of the superconducting transition) and the characteristic value of the "potential energy" in the transition layer (which is equal to T_c) are such that the change in the quasimomentum of the electronic excitations ("electrons" and "holes") during reflection is much smaller than the quasimomentum itself. Indeed,

$$\delta p \sim \dot{p}(\xi_0/v) \sim (T_c/\xi_0)(\xi_0/v) \sim (T_c/pv)p \ll p. \qquad (3)$$

On the other hand, if the energy of excitation is much smaller than the energy gap Δ in the superconducting phase, then such an excitation should be reflected from the boundary, since in a superconductor there exist no excitations with energies lower than Δ. The only possibility lies here in the fact that the incident electron (hole) is converted upon reflection into a hole (electron). The quasimomentum remains practically unchanged. On the other hand the quantity $\xi(\mathbf{p}) = E(\mathbf{p}) - E_F$ reverses sign, and with it the excitation velocity which is equal to $\mathbf{v}\xi/|\xi|$.

We shall assume that the temperature of the system is small compared with the critical temperature T_c. In this case the energy of practically all the excitations is smaller than the gap Δ and, in accordance with the statements made above, their distribution function n should satisfy on the phase boundary the condition $n(\mathbf{p}, \xi) = n(\mathbf{p}, -\xi)$.[10] Since the quasimomenta of the incident and reflected excitations are practically equal, this condition retains its form even at nonzero velocity of the boundary. Going over now to the usual representation, i.e., introducing the distribution function of the electrons, $f(\mathbf{p}) = n(\mathbf{p})$ when $\xi > 0$, and $f(\mathbf{p}) = 1 - n(-\mathbf{p})$ when $\xi < 0$, we obtain the sought boundary condition for the function χ contained in (2):

$$\chi(\mathbf{p}) + \chi(-\mathbf{p}) = 0. \qquad (4)$$

As already noted, in (2) the electric field **e** can be regarded as independent of the coordinates. In this case this equation has a solution that does not depend on the coordinates and coincides with the solution corresponding to an infinite normal metal. Using the symmetry of the collision integral with respect to the inversion transformation $\mathbf{p} \to -\mathbf{p}$, we can readily verify that this solution satisfies automatically the boundary condition (4). The density of the electric current calculated with the aid

*rot ≡ curl.

of the function χ obtained in this manner will, obviously, coincide with the current density produced in an infinite normal metal placed in a magnetic field **h** and an electric field **e**. We thus conclude that the dependence of the current **i** on the electric field is determined by the static conductivity of the bulk metal in the magnetic field **h**:

$$i_i = \sigma_{ik} e_k. \tag{5}$$

It remains for us to write out the boundary condition for h_t—the magnetic-field component parallel to the interphase boundary. The latter can be easily done by using the quality of the normal forces on both sides of the boundary

$$h_t^2 / 8\pi = H_c^2 / 8\pi + F, \tag{6}$$

where H_c is the critical magnetic field and F the force acting on the boundary on the side of the electrons. In the right side of (6) there should be written, generally speaking, one more term connected with the surface tension. Simple estimates show, however, that in our case, when the radius of curvature of the boundary is much larger than the thickness of the layer, this term is negligible. Since the quasimomentum of the electronic excitations is practically unchanged by reflection, at low temperatures we can assume that the force $F = 0$. The boundary condition thus reduces to the equality $h_t = H_c$.

We now choose the origin of the coordinate frame (ξ, η, ζ) at a certain point in the middle of some normal layer. The ζ axis is directed normal to its boundaries, and the ξ axis along the magnetic field **h** at the origin. If the layer were plane-parallel and at rest, then inside the layer we would have everywhere $\mathbf{h} = \mathbf{h}_0$, where $h_{0\xi} = H_c$, $h_{0\zeta} = h_{0\eta} = 0$, and $\mathbf{e} = 0$. In our case we can linearize all the relations in terms of **e** near the origin and in terms of the deviation **h′** of the magnetic field from $\mathbf{h}_0$. The boundary conditions then yield

$$e_\xi = 0, \qquad h_\xi' = 0. \tag{7}$$

Using (7) and (5), we obtain from the first equation of (1)

$$\frac{\partial h_\eta'}{\partial \xi} = \frac{4\pi}{c} i_\zeta = \frac{4\pi}{c}(\sigma_{\zeta\eta} e_\eta + \sigma_{\zeta\zeta} e_\zeta),$$

$$\frac{\partial h_\zeta'}{\partial \xi} = -\frac{4\pi}{c} i_\eta = -\frac{4\pi}{c}(\sigma_{\eta\eta} e_\eta + \sigma_{\eta\zeta} e_\zeta). \tag{8}$$

We now go over to the averaged values. We introduce the vector **H**, the value of which at the given point (i.e., in the given physically infinitesimally small volume whose dimensions are large compared with the layer thickness but small compared with the distance over which a noticeable change takes place in the layer configuration) is equal to the field **h** in the normal layers. The absolute value of **H**, in accord with the second equation of (7), is H_c. Since $\mathbf{h} = 0$ in the superconducting regions, the magnetic induction vector **B**, defined as the average of **h** over the volume, is given by

$$\mathbf{B} = x_n \mathbf{H}, \qquad |\mathbf{H}| = H_c, \tag{9}$$

where x_n is the concentration of the normal phase.

The magnetic moment **M** per unit volume of the intermediate state, due to the currents flowing near the boundaries of the superconducting layers, is likewise related to **H** by a simple formula. Recognizing that the magnitude of these currents is determined by the jump of the magnetic field **h** on the phase boundary, we obtain (cf. [7])

$$\mathbf{M} = -x_s \mathbf{H} / 4\pi, \tag{10}$$

where $x_s = 1 - x_n$ is the concentration of the superconducting phase. Comparison of (9) and (10) shows that the quantities **B**, **M**, and **H** are connected by the relation $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, i.e., **H** is the magnetic intensity of the intermediate state.

We can readily show analogously that the average electric field **E** differs from the field **e** in the normal layers only by the factor x_n. From this, taking the first formula of (7) into account, it follows that in the intermediate state the electric field is always perpendicular to the magnetic field

$$\mathbf{EH} = 0. \tag{11}$$

From the "microscopic" equations (1) follow the usual formulas that relate **B**, **H**, and **E**:

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \operatorname{rot} \mathbf{H} = \frac{4\pi}{c}\mathbf{j}, \tag{12}$$

where $\mathbf{j} = \mathbf{i} - c \operatorname{curl} \mathbf{M}$ is the density of the conduction current.

Relations (7) and (8) make it possible to find the connection between the conduction-current component $\mathbf{j}_\perp$ perpendicular to the magnetic field, and the electric field. Comparing (7) and (8) with the last formula of (12), we obtain

$$j_{\perp\alpha} = (\sigma_{\alpha\beta} / x_n) E_\beta, \tag{13}$$

where $\sigma_{\alpha\beta}$ is the two-dimensional conductivity tensor of the normal metal placed in the field **H**, in a plane perpendicular to the magnetic field.

As to the current along the magnetic field $\mathbf{j}_{||}$, there is no universal connection for it with the electric field. Indeed, since x_n is a variable that must be determined by the equations themselves, formulas (9) specify the connection between $\mathbf{B}$ and $\mathbf{H}$. Were there also a connection between $\mathbf{j}_{\parallel}$ and the field intensities, then relations (12) would constitute a complete system of equations. Since we have the condition (11), this system would in general be overdefined. It can be said that (11) replaces the connection between $\mathbf{j}_{\parallel}$ and the field.

In order to eliminate $\mathbf{j}_{\parallel}$ from the equations, we take the vector product of the third formula of (12) and $\mathbf{H}$. Recognizing that when $|\mathbf{H}|$ = const, we have the equality

$$[\mathbf{H}, \operatorname{rot}\mathbf{H}] = -(\mathbf{H}\nabla)\mathbf{H},$$

we get

$$(\mathbf{H}\nabla)\mathbf{H} = 4\pi c^{-1}[\mathbf{j}_{\perp}\mathbf{H}]. \qquad (14)^*$$

Finally, we write out the resultant complete system of equations

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad (\mathbf{H}\nabla)\mathbf{H} = \frac{4\pi}{c}[\mathbf{j}_{\perp}\mathbf{H}],$$

$$\mathbf{EH} = 0, \quad |\mathbf{H}| = H_c, \quad \mathbf{B} = x_n\mathbf{H}, \quad j_{\perp\alpha} = \frac{\sigma_{\alpha\beta}}{x_n}E_\beta. \qquad (15)$$

We note that it is nonlinear even when $\sigma_{\alpha\beta}$ does not depend on $\mathbf{H}$. After solving the equations, we can determine the complete current density calculating $(c/4\pi)$ curl $\mathbf{H}$.

2. DIRECT CURRENT

In this section we shall calculate the dc resistance of the intermediate state. We consider a plane-parallel plate placed in a perpendicular magnetic field $\mathcal{H}$. As is well known,[7] in the absence of current the plate is in the intermediate state when $\mathcal{H} < H_c$. Since the conductivity $\sigma_{\alpha\beta}$ depends essentially on whether the parameter Ω/ν is large or small, where Ω is the cyclotron frequency in the critical field and ν is the electron collision frequency, we shall consider these two limiting cases separately.

Let at first $\Omega \ll \nu$. In this case we can use the conductivity in the absence of the magnetic field, which we assume to be isotropic. We choose the coordinate system such that the z axis is normal to the surface of the plate, the x axis coincides with the current direction, and the boundaries of the plate correspond to $z = \pm a$ (a is half the plate thickness). It is clear from symmetry considerations that all the quantities depend only on the coordinate z. The equation $\operatorname{div}\mathbf{B} = 0$ then reduces to the condition that the product $x_n H_z$ be constant and equal to the external field $\mathcal{H}$:

$$x_n H_z = \mathcal{H}. \qquad (16)$$

*$[\mathbf{H}, \operatorname{rot}\mathbf{H}] \equiv \mathbf{H} \times \operatorname{curl}\mathbf{H}$.

The equation curl $\mathbf{E} = 0$ leads to constancy of E_x and E_y, while the equation for $\mathbf{H}$ can be written in the form

$$\frac{d\mathbf{H}}{dz} = \frac{4\pi\sigma}{c\mathcal{H}}[\mathbf{EH}], \qquad (17)$$

from which, using the condition $\mathbf{E}\cdot\mathbf{H} = 0$, we get that $\mathbf{E}$ is constant. Differentiating (17) with respect to z and again using (17), we can eliminate the electric field:

$$\frac{d^2\mathbf{H}}{dz^2} + \left(\frac{4\pi\sigma|\mathbf{E}|}{c\mathcal{H}}\right)^2\mathbf{H} = 0. \qquad (18)$$

The general solution of the latter equation takes the form

$$\mathbf{H} = \mathbf{H}_1\cos\left(\frac{4\pi\sigma|\mathbf{E}|}{c\mathcal{H}}z\right) + \mathbf{H}_2\sin\left(\frac{4\pi\sigma|\mathbf{E}|}{c\mathcal{H}}z\right), \qquad (19)$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are constant vectors which, by virtue of the condition $|\mathbf{H}| = H_c$, are mutually perpendicular, with $|\mathbf{H}_1| = |\mathbf{H}_2| = H_c$. Substitution of (19) in (17) yields

$$\mathbf{E} = |\mathbf{E}|H_c^{-2}[\mathbf{H}_1\mathbf{H}_2]. \qquad (20)$$

The obtained formulas define the field in the plate in the general case, when besides the field $\mathcal{H}$ there is also an arbitrary external field lying in the xy plane. In the particular case considered here, we have from (19) and (20)

$$H_z = H_c\cos\left(\frac{4\pi\sigma E}{c\mathcal{H}}z\right), \quad H_y = -H_c\sin\left(\frac{4\pi\sigma E}{c\mathcal{H}}z\right),$$

$$E_y = E_z = 0, \quad E_x \equiv E. \qquad (21)$$

Substituting this in (16) we get the concentration of the normal phase

$$x_n = \mathcal{H}/H_c\cos\left(\frac{4\pi\sigma E}{c\mathcal{H}}z\right). \qquad (22)$$

If $\cos(4\pi\sigma Ea/c\mathcal{H}) \geq \mathcal{H}/H_c$, then the concentration obtained with the aid of (22) satisfies everywhere within the plate the necessary condition $x_n \leq 1$. This means that the entire plate is in the intermediate state. We calculate the total current in this case:

$$J \equiv \int_{-a}^{a} j_x\,dz = -\frac{c}{4\pi}\int_{-a}^{a}\frac{dH_y}{dz}dz = \frac{cH_c}{2\pi}\sin\left(\frac{4\pi\sigma E}{c\mathcal{H}}a\right). \qquad (23)$$

On the other hand, if $\cos(4\pi\sigma Ea/c\mathcal{H}) < \mathcal{H}/H_c$, then formula (22) yields $x_n > 1$ when $|z| > z_0$, where $\cos(4\pi\sigma Ez_0/c\mathcal{H}) = \mathcal{H}/H_c$. It is clear that actually at these values of z we simply have $x_n = 1$, i.e., part of the plate is in the normal state (see [4]). Formulas (21), which define H, are valid only when

$|z| < z_0$. When $|z| > z_0$, the current density is σE, and the value of the electric field in the normal region is the same as in the region occupied by the intermediate state, by virtue of the condition of continuity of the tangential component of **E**. The total current through the plate is

$$J = -\frac{c}{4\pi}\int_{-z_0}^{z_0}\frac{dH_y}{dz}dz + \sigma E\cdot 2(a - z_0)$$

$$= \frac{cH_c}{2\pi}\left[1 - \left(\frac{\mathcal{H}}{H_c}\right)^2\right]^{1/2} + 2\sigma E\left\{a - \frac{c\mathcal{H}}{4\pi\sigma E}\arccos\frac{\mathcal{H}}{H_c}\right\}. \tag{24}$$

Formulas (23) and (24) solve our problem and determine $E(J)$ for all J.

If $J \ll cH_c$, we get from (23)

$$E = \frac{x_n^{(0)}}{\sigma}\left(\frac{J}{2a}\right), \tag{25}$$

where $x_n^{(0)} = \mathcal{H}/H_c$ is the concentration of the normal phase in the absence of current. We see that the dc resistance in the intermediate state is equal to the sum of the resistances of the normal regions.[4, 11–14] The separation boundaries between the phases do not have additional electric resistivity. This fact is not trivial (since it is known[15, 11, 9] that they have a large thermal resistance) and is closely connected with the unique character of the reflection of the electronic excitations from the phase boundaries, discussed above.

If $J \gg cH_c$, then (24) yields

$$E = \sigma^{-1}(J/2a), \tag{26}$$

i.e., the resistance of the plate is the same as the resistance of the normal metal; this is perfectly natural, for in this case almost the entire volume is occupied by the purely normal phase.

Let now $\Omega \gg \nu$. We confine ourselves here to a metal with an unequal number of electrons and holes. For such metals the off-diagonal elements of the conductivity tensor $\sigma_{\alpha\beta}$, which are equal to $\pm Nec/H_c$ (see [8]), where N is the difference in the number of electrons and holes per unit volume, greatly exceeds the diagonal ones, and we can write

$$\mathbf{j}_\perp = \frac{Nec}{x_n H_c^2}[\mathbf{EH}]. \tag{27}$$

Substituting (27) in (14) and taking (9) and (11) and the condition curl E = 0 into account, we obtain

$$E_x = H_x = 0, \qquad H_y = -4\pi NeE_y\mathcal{H}^{-1}z,$$
$$H_z = (H_c^2 - H_y^2)^{1/2}, \tag{28}$$

from which we readily obtain a relation for $E_y(J)$:

$$E_y = \frac{\mathcal{H}}{2acNe}J. \tag{29}$$

This formula is valid if $J < (c/2\pi)(H_c^2 - \mathcal{H}^2)^{1/2}$. When $J > c(H_c^2 - \mathcal{H}^2)^{1/2}/2\pi$, part of the volume of the plate goes over into a purely normal state, and the character of $E(J)$ is determined in this case by the detailed structure of the conductivity tensor of the normal metal.

The formulas obtained can be generalized in trivial fashion to the case of a plate placed in an inclined magnetic field.

3. SMALL OSCILLATIONS

The general equations (15) can be used also to solve the question of the possible propagation of electromagnetic waves in the intermediate state. We put

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}', \quad |\mathbf{H}_0| = H_c, \quad x_n = x_n^{(0)} + x_n'$$

and linearize (15) with respect to the quantities $\mathbf{H}'$, x_n', and **E**. The dependence of the latter on the coordinates and on the time is given by the factor $\exp(-i\mathbf{k}\cdot\mathbf{r} - i\omega t)$. We obtain the following relations:

$$(\mathbf{kH}_0)x_n' + x_n^{(0)}(\mathbf{kH}') = 0, \quad [\mathbf{kE}] = c^{-1}\omega(\mathbf{H}_0 x_n' + x_n^{(0)}\mathbf{H}'),$$
$$\mathbf{EH}_0 = 0, \quad \mathbf{H}'\mathbf{H}_0 = 0, \quad (\mathbf{kH}_0)\mathbf{H}' = -4\pi c^{-1}i[\mathbf{j}_\perp, \mathbf{H}_0]. \tag{30}$$

We have assumed that the equilibrium values of $\mathbf{H}_0$ and $x_n^{(0)}$ do not depend on the coordinates. This is correct for the intermediate state produced in an arbitrary ellipsoid placed in a homogeneous magnetic field. Actually the results obtained from (30) pertain also to bodies of arbitrary shape, provided we consider wavelengths much shorter than the distance over which $\mathbf{H}_0$ and $x_n^{(0)}$ change noticeably.

Multiplying the second and the last equations of (30) vectorially by $\mathbf{H}_0$ and comparing the result, we obtain

$$\mathbf{E} = \frac{4\pi i\omega}{c^2(\mathbf{kH}_0)^2}x_n^{(0)}H_c^2\mathbf{j}_\perp. \tag{31}$$

If we introduce a coordinate system with z axis along H_0, then the last equation can be written in the form of a system of two equations for E_x and E_y:

$$E_x = \frac{4\pi i\omega}{c^2(\mathbf{kH}_0)^2}H_c^2(\sigma_{xx}E_x + \sigma_{xy}E_y),$$

$$E_y = \frac{4\pi i\omega}{c^2(\mathbf{kH}_0)^2}H_c^2(\sigma_{yx}E_x + \sigma_{yy}E_y). \tag{32}$$

Equating to zero the determinant of the system (32), we obtain an equation for $\omega(\mathbf{k})$:

$$\omega^2(\sigma_{xy}\sigma_{yx}-\sigma_{xx}\sigma_{yy})-i\omega\frac{c^2(\mathbf{kH}_0)^2}{4\pi H_c^2}\times(\sigma_{xx}+\sigma_{yy})+\left(\frac{c^2(\mathbf{kH}_0)^2}{4\pi H_c^2}\right)^2=0. \qquad (33)$$

If $\Omega \lesssim \nu$, then all the components of the tensor $\sigma_{\alpha\beta}$ have, generally speaking, the same order of magnitude. Then the oscillations, as seen from (33), attenuate over distances of the order of their wavelength.

When $\Omega \gg \nu$ the spectrum $\omega(\mathbf{k})$ depends essentially on whether the metal has an equal number of holes and electrons or not. In the former case, as when $\Omega \lesssim \nu$, the oscillations are rapidly damped. In the second case we have

$$\sigma_{xy}=-\sigma_{yx}=Nec/H_c \gg \sigma_{xx},\sigma_{yy}$$

and (33) yields

$$\omega=\pm(cH_c/4\pi Ne)k_z^2, \qquad (34)$$

where $k_z=(\mathbf{k}\cdot\mathbf{H}_0)/H_c$. Substituting (34) in one of the equations (32), we can readily verify that the waves are circularly polarized, and that two signs in (34) correspond to different directions of rotation of the vector $\mathbf{E}$ in a plane perpendicular to $\mathbf{H}_0$.

With the aid of (33) we can calculate also the damping of the oscillations. Retaining only the principal terms, we obtain

$$\gamma\equiv\left|\frac{\mathrm{Im}\,\omega}{\omega}\right|=\frac{\sigma_{xx}+\sigma_{yy}}{2\pi|\sigma_{xy}|}. \qquad (35)$$

Thus, there exist weakly damped oscillations of the intermediate state of a sufficiently pure superconductor with unequal numbers of electrons and holes. These oscillations are in their properties similar to helicons which exist in a normal level placed in a strong magnetic field (see, for example, [16]).

The helicon-like oscillations of the intermediate states were recently observed experimentally in indium.[6] However, the condition $T \ll T_c$ was not satisfied in the experiment, so that the experimentally observed values of the oscillation frequencies are lower than those given by (34). With decreasing temperature, as noted in [6], the frequencies increase and thus approach their theoretical values.

We present, finally, an estimate of the maximum possible oscillation frequencies. This can be done by noting that the values of the wave vector should be smaller than the reciprocal of the layer thickness, i.e., $k \lesssim 10^2-10^3\ \mathrm{cm}^{-1}$. From (34) we then get $\omega \lesssim 10^3-10^{12}\ \mathrm{sec}^{-1}$ for $H_c \sim 10^2$ and $N \sim 10^{22}$. The maximum wave propagation velocity is in this case of the order of $10-10^2$ cm/sec.

4. BEHAVIOR IN AN ALTERNATING FIELD

The presence of weakly damped oscillations greatly influences the behavior of the intermediate state in an external alternating electromagnetic field. We consider again a plane-parallel plate situated in a constant magnetic field $\vec{\mathscr{H}}$, and the origin lies in the middle of the plate. Assume that a plane electromagnetic wave propagating along the z axis is incident on the plate. Choosing the x axis along the direction of the magnetic field polarization in the incident wave, we write (15) inside the plate in the form

$$x_n^{(0)}\frac{dH_x'}{dz}=\frac{4\pi}{c}(\sigma_{yx}E_x+\sigma_{yy}E_y),\qquad \frac{dE_x}{dz}=\frac{i\omega}{c}x_n^{(0)}H_y',$$

$$x_n^{(0)}\frac{dH_y'}{dz}=-\frac{4\pi}{c}(\sigma_{xy}E_y+\sigma_{xx}E_x),\qquad \frac{dE_y}{dz}=-\frac{i\omega}{c}x_n^{(0)}H_x', \qquad (36)$$

where $x_n^{(0)}=\mathscr{H}/H_c$, and $\mathbf{H}'=\mathbf{H}-\mathbf{H}_0$, where $\mathbf{H}_0$ is the magnetic intensity in the absence of the alternating field. We have allowed for the fact that all the quantities depend only on the coordinate z by virtue of symmetry.

From (36) we obtain the following general expression for H_x' inside the plate:

$$H_x'=B_1e^{ik_1z}+B_2e^{-ik_1z}+B_3e^{ik_2z}+B_4e^{-ik_2z}, \qquad (37)$$

where B_1, B_2, B_3, and B_4 are arbitrary constants

$$k_1=q+\frac{iq(\sigma_{xx}+\sigma_{yy})}{4|\sigma_{xy}|},\qquad k_2=iq+\frac{q(\sigma_{xx}+\sigma_{yy})}{4|\sigma_{xy}|},$$

$$q=\frac{(4\pi\omega|\sigma_{xy}|)^{1/2}}{c}.$$

If we now determine H_y', E_x, and E_y with the the aid of (36) and write the ordinary continuity conditions for the tangential components of H and E at $z=\pm a$ (where a is half the plate thickness), then we can find the amplitudes of the waves passing through the plate. Simple calculations yield the following result:

$$\frac{C_1}{A_0}=i\mathscr{H}\left(\frac{\omega}{4\pi cH_c|Ne|}\right)^{1/2}\times\frac{\operatorname{sh}2qa-\sin 2qa}{\operatorname{sh}2qa\sin 2qa+2ia\operatorname{sh}2qa\cos 2qa}, \qquad (38)^*$$

*sh ≡ sinh.

$$\frac{C_2}{A_0} = eN\mathscr{H}\left(\frac{\omega}{4\pi c H_c |Ne|^3}\right)^{1/2} \times \frac{\operatorname{sh} 2qa + \sin 2qa}{\operatorname{sh} 2qa \sin 2qa + 2i\alpha \operatorname{sh} 2qa \cos 2qa}. \qquad (39)$$

Here A_0 is the amplitude of the incident wave, C_1 and C_2 are the amplitudes of H_x and H_1 in the transmitted wave, and $\alpha = (\sigma_{xx} + \sigma_{yy})qa/4|\sigma_{xy}|$. We assume that $\alpha \ll 1$, i.e., the waves in the plates attenuate little over distances on the order of a and $qa \gtrsim 1$. In the derivation of (38) and (39) we also use the fact that the parameter ω/cq, which represents the ratio of the wave velocities in the plate and in vacuum, is exceedingly small.

Formulas (38) and (39) show that the coefficient of transmission of the wave through the plate experiences resonant oscillations. The ratio of the quantities $|C_{1,2}|^2$, taken at the resonance, i.e., at $q = \pi n/2a$ ($n = 1, 2, \ldots$), and far from it, is of the order of $\alpha^{-2} \gg 1$. We note also that C_1 and C_2 are proportional to the first power of the magnetic field $\mathscr{H}$ (see [6]).

We now consider the case $\alpha \gg 1$. The waves in the plate are then damped at a distance much smaller than a, and therefore the surfaces of the plates are not correlated with each other. Choosing one of the surfaces as the plane $z = 0$, we can write the general solution of (36), which attenuates inside the plate (in the region of positive z) for $z \ll a/\alpha$, in the form

$$H_x' = Ae^{iqz} + Be^{-qz},$$

$$H_y' = -i\frac{\sigma_{xy}}{|\sigma_{xy}|}(Ae^{iqz} - Be^{-qz}),$$

$$E_y = -\frac{\omega x_n^{(0)}}{cq}(Ae^{iqz} - iBe^{-qz}),$$

$$E_x = \frac{\omega x_n^{(0)}}{cq}\frac{\sigma_{xy}}{|\sigma_{xy}|}(Be^{-qz} - iAe^{iqz}), \qquad (40)$$

where A and B are arbitrary constants. Expressing them with the aid of the first two equations of (40) in terms of H_x' and H_y' at $z = 0$, and substituting in the last two equations, we obtain the following connection between the electric field and the magnetic field on the surface

$$E_x = \frac{\omega}{cq}\frac{x_n^{(0)}}{2}(1-i)\left\{\frac{\sigma_{xy}}{|\sigma_{xy}|}H_x' + H_y'\right\},$$

$$E_y = \frac{\omega}{cq}\frac{x_n^{(0)}}{2}(1-i)\left\{-H_x' + \frac{\sigma_{xy}}{|\sigma_{xy}|}H_y'\right\}. \qquad (41)$$

If we introduce in usual fashion (see [7], p. 397) the surface impedance $\zeta_{\alpha\beta}$, then (41) yields

$$\zeta_{\alpha\beta} = x_n^{(0)}\frac{1-i}{2}\left(\frac{\omega}{4\pi|\sigma_{xy}|}\right)^{1/2}\left\{\delta_{\alpha\beta} - \frac{\sigma_{\alpha\beta}}{|\sigma_{xy}|}\right\}, \qquad (42)$$

where the indices α and β run through the values of x and y.

All the preceding formulas of this section pertain to a superconductor with $N \neq 0$ under the condition $\Omega \gg \nu$. In the opposite limiting case, putting $\sigma_{\alpha\beta} = \sigma\delta_{\alpha\beta}$, we obtain in similar manner

$$\zeta_{\alpha\beta} = (1-i)x_n^{(0)}(\omega/8\pi\sigma)^{1/2}\delta_{\alpha\beta}, \qquad (43)$$

which coincides with the impedance of the normal metal having a conductivity $\sigma/x_n^{(0)}$.

If the plate is in an oblique magnetic field $\mathscr{H}$, we obtain in lieu of (43)

$$\zeta_{\alpha\beta} = \frac{|\mathscr{H}_z| H_c}{(H_c^2 - \mathscr{H}_t^2)}\left(\frac{\omega}{8\pi\sigma}\right)^{1/2}(1-i)\left\{\delta_{\alpha\beta} - \frac{\mathscr{H}_\alpha\mathscr{H}_\beta}{H_c^2}\right\}, \qquad (44)$$

where $\mathscr{H}_z$ is the component of $\mathscr{H}$ normal to the plate, $\mathscr{H}_t^2 = \mathscr{H}_\alpha^2$. The depth of penetration of the field is then

$$\delta = \frac{c}{(2\pi\sigma\omega)^{1/2}}\left(1 - \frac{\mathscr{H}_t^2}{H_c^2}\right)^{1/2}. \qquad (45)$$

In a perpendicular field $\mathscr{H}_t = 0$, and (45) goes over into an expression corresponding to a conductor with conductivity σ/x_n and magnetic permeability x_n.

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[1] L. D. Landau, JETP **7**, 371 (1937).

[2] R. Peierls, Proc. Roy. Soc. **A155**, 613 (1936).

[3] F. London, Physica **3**, 450 (1936).

[4] F. London, Superfluids **1**, New York–London, (1950), p. 96–124.

[5] A. F. Andreev, JETP Letters **4**, 7 (1966), transl. p. 4.

[6] B. W. Maxfield and E. F. Johnson, Phys. Rev. Lett. **15**, 677 (1955).

[7] L. D. Landau and E. M. Lifshitz, Élektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Fizmatgiz, 1959.

[8] I. M. Lifshitz, M. Ya. Azbel', and M. I. Kaganov, JETP **31**, 63 (1956), Soviet Phys. JETP **4**, 41 (1957).

[9] A. F. Andreev, JETP **46**, 1823 (1964), Soviet Phys. JETP **19**, 1228 (1964).

[10] A. F. Andreev, JETP **47**, 2222 (1964), Soviet Phys. JETP **20**, 1490 (1965).

[11] N. V. Zavaritskiĭ, JETP **38**, 1673 (1960), Soviet Phys. JETP **11**, 1207 (1960).
[12] W. J. De Haas, J. Voogd, and J. M. Jonker, Physica **1**, 281 (1934).
[13] A. D. Misener, Proc. Roy. Soc. **A166**, 43 (1938).
[14] E. R. Andreev, Proc. Roy. Soc. **A194**, 80 (1948).
[15] K. Mendelssohn and J. L. Olsen, Proc. Phys. Soc. **A63**, 2 (1950); Phys. Rev. **80**, 859 (1950).
[16] É. A. Kaner and V. G. Skobov, JETP **46**, 1106 (1964), Soviet Phys. JETP **19**, 749 (1964).

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