BOUNDARY CONDITIONS OF THE JOSEPHSON EFFECT

A. I. LARKIN, Yu. N. OVCHINNIKOV, and M. A. FEDOROV

Moscow Physico-technical Institute

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A boundary condition for the Josephson effect is obtained in the quasiclassical approximation from the Gor'kov equations. The results of the investigation are in agreement with those of previous ones in which this effect is considered with the aid of the tunneling Hamiltonian.

AN undamped current (Josephson current) can flow in a system consisting of two superconductors separated by a thin dielectric barrier.[1] The theory of this phenomenon is usually constructed on the basis of a model tunnel Hamiltonian.[2] It is of interest, however, to consider this effect from a more general point of view, namely by using Gor'kov's equations directly.

The influence of a dielectric film on the motion of an electron is equivalent to the presence of a certain potential barrier. The scattering of the electron by the boundaries of the barrier is assumed diffuse, and the thickness of the barrier is much smaller than the dimension of the pair. Under these assumptions, we shall derive a formula which contains all the previously obtained results.[2, 3] In addition, we shall find the distribution of the current near the barrier. The equation for the Green's functions is solved by a quasiclassical method that generalizes the method of Shapoval[4] and de Gennes.[5]

The Gor'kov equation[6] is conveniently written in matrix form

$$[-\hat{H}_0 + i\omega\tau_z + \hat{V}]\,\hat{G} = \hat{\delta}(\mathbf{r}-\mathbf{r}'),$$

$$\hat{V} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix} + \frac{e}{m}(\mathbf{pA})\,\tau_z,$$

$$\hat{H}_0 = -\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} + U(\mathbf{r}) - \mu, \qquad (1)$$

where $\hat{H}_0$ is the Hamiltonian of the electron in the field of the barrier. Δ^* is obtained from the equation

$$\Delta^* = \frac{|\lambda|}{2} T \sum_\omega \mathrm{Sp}\,(\tau_x + i\tau_y)\,\hat{G}(\mathbf{r},\mathbf{r}). \qquad (2)$$

In the problem considered here, the small parameter is the coefficient of transmission through the barrier. In the zeroth approximation the system constitutes two unconnected superconductors. For each of them, Eqs. (1) and (2) have solutions with Δ independent of r, the phases Δ on opposite sides of the barrier not being connected in any manner. In the first order in the transmission coefficient, the zeroth approximation for Δ should be substituted in (1). Thus, it is necessary to solve Eq. (1) with

$$\Delta = \begin{cases} \Delta_1, & z < 0 \\ \Delta_2, & z > 0 \end{cases}.$$

The current is expressed in terms of Green's functions by means of the formula

$$\mathbf{j}(\mathbf{r}) = -\frac{e}{m} T\, \mathrm{Sp} \sum_\omega \delta(\mathbf{r}-\hat{\mathbf{r}})\,\hat{\mathbf{p}}\tau_z\,\hat{G}. \qquad (3)$$

To calculate the current we shall use the method of classical trajectories,[4, 5] which was generalized by one of the authors.[7] Expanding $\hat{G}$ in powers of $\hat{V}$, we reduce (3) to the form

$$\mathbf{j}(\mathbf{r}) = \frac{e}{m} T\,\mathrm{Sp}\sum_\omega \langle n|\,\delta(\mathbf{r}-\hat{\mathbf{r}})\,\hat{\mathbf{p}}\,\tau_z\,|m\rangle \frac{1}{\xi_m - i\omega\tau_z}$$
$$\times\Big\{\delta_{mn} + V_{mn}\frac{1}{\xi_n - i\omega\tau_z} + V_{ml}\frac{1}{\xi_l - i\omega\tau_z}V_{ln}\frac{1}{\xi_n - i\omega\tau_z} + \ldots\Big\}, \qquad (4)$$

where the matrix elements are calculated from the eigenfunctions of $\hat{H}_0$. Introducing the time-dependent operators

$$\hat{V}(t) = \exp(-i\hat{H}_0 t)\,\hat{V}\exp(i\hat{H}_0 t),$$

we obtain

$$\mathbf{j}(\mathbf{r}) = \frac{e}{m} T\,\mathrm{Sp}\sum_\omega \langle n|\,\delta(\mathbf{r}-\hat{\mathbf{r}}(t))\,\hat{\mathbf{p}}(t)\,\tau_z$$
$$\times\{1 + g_0(t-t_1)\hat{V}(t_1)\exp[-i\xi_n(t_1-t)] + g_0(t-t_1)\hat{V}(t_1)g_0(t_1-t_2)\hat{V}(t_2)$$
$$\times\exp[-i\xi_n(t_2-t)] + \ldots\}\frac{1}{\xi_n - i\omega\tau_z}\Big|\,n\Big\rangle, \qquad (5)$$

where

$$g_0(t) = \frac{1}{2\pi}\int (\xi - i\omega\tau_z)^{-1}\exp(-i\xi t)\,d\xi.$$

Since the factor $(\xi - i\omega\tau_z)^{-1}\exp(-i\xi t)$ has a sharp maximum near the Fermi surface, and the diagonal matrix element depends little on the energy, we can take it on the Fermi surface, and then (5) goes over into the formula

$$\mathbf{j}(\mathbf{r}) = \frac{e}{m}2\pi\nu T\,\mathrm{Sp}\sum_{\omega}\int dt'\Big\langle\delta(\mathbf{r}-\hat{\mathbf{r}}(t))\,\hat{\mathbf{p}}(t)\,\tau_z\delta(t-t') \times\Big\{g_0(t-t') + \int dt_1 g_0(t-t_1)V(t_1)g_0(t_1-t') + \ldots\Big\}\Big\rangle. \quad (6)$$

here $\nu = mp_0/2\pi^2$ is the level density on the Fermi surface, and $\langle\ \rangle$ denotes averaging over all states on the Fermi surface.

If the barrier is not transparent, then the averages over the states can be replaced, with quasiclassical accuracy, by averages over all the classical trajectories with specified energy, the corresponding operators being replaced by their classical values on the trajectories. Then, denoting the quantity in the curly brackets by $g(t, t')$, we obtain for it the following equation:

$$g(t,t') = \int g_0(t-t_1)\,[\delta(t_1-t') + V(t_1)g(t_1,t')]\,dt_1. \quad (7)$$

or, transforming (7) into a differential equation

$$\left[-i\frac{\partial}{\partial t} + i\omega\tau_z + \hat{V}\right]g(t,t') = -\delta(t-t'). \quad (8)$$

Equation (7) can be readily solved for the constant Δ in the absence of the magnetic field:

$$g(t,t') = \frac{i}{2E}[\omega\tau_z - E\,\mathrm{sign}(t-t') - i\hat{V}]\exp(-E|t-t'|), \quad (9)$$

Here $E = (\omega^2 + |\Delta|^2)^{1/2}$. We note that we can obtain the results of Shapoval[4] by solving (7) by expanding in the magnetic field, and we obtain the results of de Gennes[5] by using an expansion in Δ.

In first order in the barrier penetrability, an electron moving along any trajectory on one side of the barrier, can go over to a trajectory situated on the opposite side of the barrier. This corresponds to replacement of several neighboring operators in (6) by their classical values, taken on trajectories located on the other side of the barrier. The quantity $g_0(t-t')$ which connects the neighboring operators Δ, taken on trajectories situated on opposite side of the barrier, are replaced by

$$g_0(t-t_\delta)Dg_0(t_\delta - t), \quad (10)$$

where t_δ is the time and D the amplitude of passage through the barrier.

Summing the expression in the curly brackets in (6) we obtain

$$\mathbf{j}_1(\mathbf{r}) = \frac{e}{m}2\pi\nu T\,\mathrm{Sp}\sum_{\omega}\langle\langle |D|^2\tau_z g_1(t,t_\delta)g_2(t_\delta,t_\delta)g_1(t_\delta,t)\,\mathbf{p}_0\rangle\rangle. \quad (11)$$

The double brackets $\langle\langle\ \rangle\rangle$ denote averaging over all trajectories, and the indices 1 and 2 pertain respectively to the left and right superconductors. Substituting in (11) the expression for $g(t, t')$ from (9), we obtain

$$\mathbf{j}_1(\mathbf{r}) = \frac{e}{m}\pi\nu|\Delta_1\Delta_2|\sin\alpha \times T\sum_{\omega}\frac{\langle\langle |D|^2\mathbf{p}_0\,\mathrm{sign}\,t_0\exp[-2|t_0|(\omega^2+|\Delta_1|^2)^{1/2}]\rangle\rangle}{(\omega^2+|\Delta_1|^2)^{1/2}(\omega^2+|\Delta_2|^2)^{1/2}}, \quad (12)$$

where t_0 is the time during which the electron moves from the observation point to the barrier, and $\alpha = \alpha_2 - \alpha_1$ ($\Delta_1 = |\Delta_1|\exp(i\alpha_1)$, $\Delta_2 = |\Delta_2| \times \exp(i\alpha_2)$).

We see from (11) that the current density depends on the coordinates and does not satisfy the current conservation law. This is connected with the fact that Δ was assumed independent of the coordinates in expression (1) for the current. In fact, owing to the transparency of the barrier, a decrease in Δ takes place, and this makes a contribution to the current. The initial equations were gauge-invariant, and therefore this change in Δ can be obtained from the current-conservation law. For this purpose we can seek Δ in the form

$$\Delta = \Delta_0\exp[i\int g(z)\,dz].$$

The current j_2 connected with the inconstancy of Δ can be obtained by the method described above:

$$j_2 = \frac{e}{m}\pi\nu T p_0|\Delta_1|^2\sum_{\omega}\frac{1}{E_1^2}\Big\{\int_0^\infty g(x)\left(\frac{2Ex}{v}\right)^2\Big[\left(\frac{x-r}{x}\right)^2 \times\Gamma\left(-2,\frac{2E|x-r|}{v}\right) - 2\left(\frac{2Er}{v}\right)^2\Gamma\left(-2,\frac{2Er}{v}\right) \times\Gamma\left(-2,\frac{2Ex}{v}\right)\Big]\Big\}dx, \quad (13)$$

where $\Gamma(\alpha, z)$ is the incomplete gamma function and v is the velocity on the Fermi surface. The function $j_2(z)$ is obtained from the condition that the total current

$$j_z(z) = j_1 + j_2 \quad (14)$$

must be conserved. For example, without allowance for the magnetic field produced by the current, the current would flow through the entire sample and $j_z = \mathrm{const}$. The influence of the magnetic field causes the current flowing on the surface to be

$$j_x(x,z) = j_x(x,0)\varphi(z).$$

In the limiting London case $\varphi(z) = \exp[-\kappa z]$, and then it follows from the conservation law that

$$j_z(z) = j_z(0)\exp[-\varkappa z],$$

and the function g(z) can be determined from (12)–(14). We note that near the surface of the barrier the addition to Δ is always proportional to the penetrability and therefore when calculating the current j_2 due to this addition we can assume that the barrier is not transparent. As a result j_2 vanishes on the surface of the barrier. Thus, the total current through the surface, equal to $j_1(0) + j_2(0)$, can be calculated from (12), in which we put $t_0 = 0$. As a result we obtain

$$\mathbf{j} = \frac{\pi \nu e}{m}|\Delta_1\Delta_2|\sin\alpha\cdot T\sum_\omega \frac{\langle\langle \mathbf{p}_0 \operatorname{sign} t_0 |D|^2\rangle\rangle}{(\omega^2+|\Delta_1|^2)^{1/2}(\omega^2+|\Delta_2|^2)^{1/2}}. \tag{15}$$

If the barrier separates two normal metals and a voltage V is applied to it, then the current through the barrier is

$$j_n = \frac{eVp_0\nu}{m}\int_0^1 z|D(z)|^2 dz. \tag{16}$$

Introducing the resistance of the barrier by means of the formula

$$R^{-1} = \frac{ep_0\nu}{m}\int_0^1 z|D(z)|^2 dz \tag{17}$$

and substituting it in the expression for the current in (15), we obtain

$$j = \frac{\pi}{R}|\Delta_1\Delta_2|\sin\alpha\cdot T\sum_\omega [(\omega^2+|\Delta_1|^2)^{1/2}(\omega^2+|\Delta_2|^2)^{1/2}]^{-1/2} \tag{18}$$

As $T \to 0$ the sum goes over into an integral and

$$j(0)_{T\to 0} = \frac{2}{R}\frac{|\Delta_1\Delta_2|}{|\Delta_1|+|\Delta_2|}\sin\alpha\cdot K\left(\left|\frac{|\Delta_1|-|\Delta_2|}{|\Delta_1|+|\Delta_2|}\right|\right), \tag{19}$$

where K is a complete elliptic integral. Formula (19) coincides with the result given by Anderson.[2]

In the particular case[3] when $|\Delta_1| = |\Delta_2|$

$$j = \frac{\pi}{2}\frac{|\Delta|}{R}\sin\alpha\cdot\tanh\frac{|\Delta|}{2T}. \tag{20}$$

In conclusion we note that formula (20) does not depend on the character of the scattering of the electrons by the surface of the barrier, since the transmission coefficient is averaged in the same way over the angles of the trajectories when calculating the current through the barrier, in the case of both normal and superconducting metals.

The results were obtained independently of the form of the trajectories, and therefore the final expression (20) does not depend on the presence of nonmagnetic impurities.

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77