SOVIET PHYSICS JETP VOLUME 22, NUMBER 6 JUNE, 1966

THE EQUATION OF MOTION FOR THE "HALVED" S-MATRIX AND ITS CONSEQUENCES

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Submitted to JETP editor June 1, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1812–1823 (December, 1965)

A formal equation is derived for the "halved" S-matrix in quantum field theory, satisfying the integrability condition. For the example of a self-interacting, renormalizable scalar field expressions have been obtained for the Heisenberg field operator and its derivative. Medvedev's equations for the Heisenberg "current-like" operators are solved.

1. INTRODUCTION

THERE has been a recent tendency in papers on axiomatic quantum field theory[1, 2] to utilize concepts from different systems of axioms,[3, 4] and also to resort, within reasonable limits, to apparatus appropriated from the Lagrangian formalism. At the same time it has been impossible to carry through this program consistently,[1, 2] due to the fact that a non-unitary relation was found (cf. also [5, 6]) between the operators in the Heisenberg picture and those in the asymptotic (more correctly, in the in-) picture:

$$\mathbf{F}(x) = S^{+}T_W(\mathcal{F}_{in}(x)S), \tag{1}$$

where T_W is the Wick T-product.[1, 2, 6] Our investigation has the purpose to show that the program proposed by Medvedev[1, 2] can be successfully realized if one makes use in addition to the Lagrangian formalism also of concepts from the Hamiltonian formalism, in particular its main quantity, the "halved" S-matrix.

In previously published papers[6] the following formal expression for the "halved" S-matrix has been introduced:[1)]

$$S(\sigma, -\infty) = T_D \exp\left\{-i\int_{-\infty}^{\sigma} H_I{}^{\mathrm{int}}(z; \sigma')\,dz\right\}, \tag{2}$$

where T_D is the Dyson T-product,[1, 2, 6] and $H_I^{\mathrm{int}}(z; \sigma)$ is the interaction Hamiltonian. By means of this expression one can circumvent the non-unitary character of relations of the type (1), by introducing in the interaction picture an operator $\mathfrak{F}^{\mathrm{int}}(x; \sigma)$ which is a unitarily related to the Heisenberg operator $\mathbf{F}(x)$. Specifically, we have,[6]

$$\mathbf{F}(x) = S^{+}(\sigma, -\infty)\,\mathcal{F}^{\mathrm{int}}(x; \sigma)\,S(\sigma, -\infty), \tag{3}$$

where in general $\mathfrak{F}^{\mathrm{int}}(x; \sigma) \neq \mathfrak{F}_{\mathrm{in}}(x)$, although of course $\mathfrak{F}^{\mathrm{int}}(x; \sigma)$ is also a polynomial in normal products of the field operators $\varphi_{\mathrm{in}}(x)$ and their derivatives.

The transition from (1) to (3) can be realized by making use of a previously derived analog of Wick's theorem,[6] which allows to express the T_W-product in terms of T_D-products and by first defining $H_I^{\mathrm{int}}(x; \sigma)$ according to the identity[6]

$$S = T_W \exp\left\{i\int_{-\infty}^{\infty} L_I{}^{in}(x)\,dx\right\} \equiv T_D \exp\left\{-i\int_{-\infty}^{\infty} H_I{}^{\mathrm{int}}(z; \sigma)\,dz\right\}. \tag{4}$$

Expressions of the form $T_D(\mathfrak{F}_{\mathrm{in}}(x)S)$ have to be understood in the sense that the S-matrix itself is expressed in them in terms of T_D-products, i.e., the integration region in this expression can be decomposed into an infinite number of "time-slices."

It is natural that the first problem that arises is that of the mathematical significance of a matrix $S(x^0, -\infty)$ of the form (2). Recently the point of view[7, 8] according to which the "halved" S-matrix is not a properly unitary operator, but a "pseudounitary" or "improperly unitary" operator has been widespread. This has the following meaning.[6]

In distinction from the total S-matrix, which is a truly unitary operator in the Hilbert space of in-vectors, the "halved" S-matrix is probably truly unitary in a larger space, which includes the indicated Hilbert space. In general, the matrix $S(x^0, -\infty)$ takes the in-vectors out of the in-space into the remaining part of this larger space, in which, in particular, the concept of particle number does not exist. The vacuum in the space of in-vectors, for instance, is taken by the matrix $S(x^0, -\infty)$ into a vector which is no longer the vacuum for any field operator algebra (in particular

[1)]In the following we choose for σ the plane x^0 = const.

for finite values of Z_3). In this connection, although two field operator algebras can be related via a formally unitary operator $S(x^0, -\infty)$ and may satisfy the same canonical commutation relations, the well-known theorem of Haag which implies that such theories are trivial, cannot be applied here, since the "pseudounitary" character of the matrix $S(x^0, -\infty)$ implies that only one of the two theories possesses a vacuum. The corresponding field operators naturally belong to inequivalent representations.

So far a complete solution of the problem of finding the mathematical meaning of the operator $S(x^0, -\infty)$ has not yet been solved, due to technical difficulties. One is therefore tempted to go along a different way, which is in general characteristic for the axiomatic direction. One might first try to guess the solution, study its properties and only afterwards investigate what equations lead to this solution and give a rigorous foundation to the whole procedure. In following this path, we shall leave aside in the present investigation all problems connected with the mathematical meaning of the matrix $S(x^0, -\infty)$, and assume that there exists an expression of the form (2) having all the required properties. In this connection, the following bears only a formal character and does not pretend to be mathematically rigorous.

Hitherto we had defined the "halved S-matrix" in a postulational manner "cutting in two" a correspondingly written total S-matrix (cf. (2) and (4)). The problem of the arbitrariness involved in such an operation and the way to remove it has remained open, since any matrix of the form

$$\tilde{S}(x^0, -\infty) = \exp\left[-i\alpha(x^0)\right] S(x^0, -\infty), \tag{5}$$

where $\alpha(x^0)$ is an arbitrary hermitian local operator and $S(x^0, -\infty)$ is of the form (2) can also be called a "halved" S-matrix.

In this connection the necessity of finding an equation satisfied by the "halved" S-matrix makes itself felt. This would permit to relate the problem of determining the arbitrariness of this matrix with solutions of this equation. Since in the axiomatics adopted here[4, 1, 2] the principal method of singling out space-time points is by taking variations with respect to $\varphi_{in}(y)$ such equations must necessarily be functional derivative equations. The equation we are interested in can be found in the following manner.[6, 1, 2]

On the one hand, if one defines the Heisenberg field operator $\mathbf{A}(x)$ in the spirit of Eq. (1), we have[6, 1, 2]

$$\mathbf{A}(x) = S^+ T_W(\varphi_{in}(x) S) = \varphi_{in}(x) - \int D^{ret}(x-y)\, \mathbf{j}(y)\, dy, \tag{6}$$

$$\mathbf{j}(y) = iS^+ \delta S / \delta\varphi_{in}(y). \tag{7}$$

On the other hand, one could require that the same operator $\mathbf{A}(x)$ should satisfy the equality

$$\mathbf{A}(x) = \tilde{S}^+(x^0, -\infty)\, \varphi_{in}(x)\, \tilde{S}(x^0, -\infty)$$
$$= \varphi_{in}(x) - i \int D(x-y)\, \tilde{S}^+(x^0, -\infty)\, \frac{\delta \tilde{S}(x^0, -\infty)}{\delta\varphi_{in}(y)}\, dy, \tag{8}$$

where $\tilde{S}(x^0, -\infty)$ is a matrix of the form (5). Then comparison of (6) and (8) yields the equation[1, 2]

$$\int D(x-y) \left\{ i\tilde{S}^+(x^0, -\infty)\, \frac{\delta \tilde{S}(x^0, -\infty)}{\delta\varphi_{in}(y)} - \theta(x^0 - y^0)\, \mathbf{j}(y) \right\} dy = 0, \tag{9}$$

It is sufficient, but not necessary, in order that this equation be satisfied, that

$$i\, \frac{\delta \tilde{S}(x^0, -\infty)}{\delta\varphi_{in}(y)} = \tilde{S}(x^0, -\infty)\, \theta(x^0 - y^0)\, \mathbf{j}(y). \tag{10}$$

As has been shown in [1, 2] the integrability condition for such an equation is of the form

$$i\left(\frac{\delta^2 \tilde{S}(x^0, -\infty)}{\delta\varphi_{in}(z)\, \delta\varphi_{in}(y)} - \frac{\delta^2 \tilde{S}(x^0, -\infty)}{\delta\varphi_{in}(y)\, \delta\varphi_{in}(z)} \right)$$
$$= \tilde{S}(x^0, -\infty)\left[\theta(z^0 - x^0) - \theta(y^0 - x^0)\right] \mathbf{\Lambda}_2(y, z) = 0, \tag{11}$$

where $\mathbf{\Lambda}_2(y, z)$ is a Heisenberg "current-like" operator, which is quasilocal in the explicit variables.[1, 2] This condition is satisfied only in theories without derivative couplings (and without counterterms) and is violated in any nontrivial local theory. Thus, in such theories, the "halved" S-matrix cannot satisfy a simple equation of the form (10). The problem whether the weaker condition (9) can be satisfied, and thus the representation (8) is possible remains unanswered.[1, 2]

In order to progress in the solving of this problem it is proposed below to give up Eq. (10) and to find out what equation is formally satisfied by the postulated "halved" S-matrix of the form (2). After this it will be possible to return to the problem of the possibility of representing the operator $\mathbf{A}(x)$ and other Heisenberg operators in the form (8).

2. DERIVATION OF THE EQUATION OF MOTION FOR THE "HALVED" S-MATRIX

We shall start (cf. also [10]) from the formal analogy with the situation encountered in the Tomonaga–Schwinger equation, where in the presence of derivative couplings the equation is no longer true with only $-L_I^{in}(x)$ in the right-hand side, but where this leads only to a modification of the equation (replacement of $-L_I^{in}(x)$ by $H_I^{int}(x;\sigma)$) and does not lead to giving up the "halved" S-matrix.

In order to find such a modified equation for our case, we vary both sides of Eq. (2) with respect to $\varphi_{in}(y)$. We obtain

$$i\frac{\delta S(x^0,-\infty)}{\delta\varphi_{in}(y)} = iT_D\Big[S(x^0,-\infty)\times\Big(-i\int_{-\infty}^{\infty}\theta(x^0-z^0)\frac{\delta H_I^{int}(z)}{\delta\varphi_{in}(y)}dz\Big)\Big]. \tag{12}$$

Expanding the T_D-product in (12), taking into account $\theta(x^0-z^0)$, we have

$$i\frac{\delta S(x^0,-\infty)}{\delta\varphi_{in}(y)} = \int_{-\infty}^{\infty} S(x^0,z^0)\frac{\delta H_I^{int}(z)}{\delta\varphi_{in}(y)}\times S(z^0,-\infty)\theta(x^0-z^0)dz = S(x^0,-\infty)\int_{-\infty}^{\infty}\theta(x^0-z^0)\times \mathrm{j}'(z,y)dz, \tag{13}$$

$$\mathrm{j}'(z,y) = S^+(z^0,-\infty)\frac{\delta H_I^{int}(z)}{\delta\varphi_{in}(y)}S(z^0,-\infty). \tag{14}$$

Since we start from definition (2), the integrability condition for Eq. (13) must automatically be satisfied for any theory in which (2) is meaningful. Indeed, it follows from (13) that

$$i\Big(\frac{\delta^2 S(x^0,-\infty)}{\delta\varphi_{in}(u)\delta\varphi_{in}(y)} - \frac{\delta^2 S(x^0,-\infty)}{\delta\varphi_{in}(y)\delta\varphi_{in}(u)}\Big)$$

$$= S(x^0,-\infty)\Big\{\int_{-\infty}^{\infty}dz\,\theta(x^0-z^0)S^+(z^0,-\infty)\times\Big[\frac{\delta^2 H_I^{int}(z)}{\delta\varphi_{in}(u)\delta\varphi_{in}(y)} - \frac{\delta^2 H_I^{int}(z)}{\delta\varphi_{in}(y)\delta\varphi_{in}(u)}\Big]S(z^0,-\infty) - i\int dz\,dt\,\theta(x^0-z^0)\theta(x^0-t^0)[T_D(\mathrm{j}'(z,y)\mathrm{j}'(t,u)) - T_D(\mathrm{j}'(z,u)\mathrm{j}'(t,y))]\Big\}, \tag{15}$$

which vanishes due to the locality of $H_I^{int}(z)$ and the symmetry of the T_D-product.in u and y (if one takes into account the substitution $z \leftrightarrow t$ in the last term). In the following it will be shown that the transition from the matrix $S(x^0, -\infty)$ of the form (2) to a matrix $\widetilde{S}(x^0, -\infty)$ of the form (5) cannot influence the validity of the integrability condition.

In particular, when

$$\frac{\delta H_I^{int}(z)}{\delta\varphi_{in}(y)} = -\frac{\delta L_I^{in}(z)}{\delta\varphi_{in}(y)} = j_{in}(z)\delta(z-y), \tag{16}$$

it follows from (13) that

$$i\frac{\delta S(x^0,-\infty)}{\delta\varphi_{in}(y)} = S(x^0,-\infty)\theta(x^0-y^0)\mathrm{j}(y), \tag{17}$$

in agreement with Medvedev's results.[1,2] In the general case the expression for $\delta H_I^{int}(z)/\delta\varphi_{in}(y)$ may contain terms involving derivatives of $\delta(z-y)$. Indeed, it is easy to see that if an arbitrary local operator M(z), which is a sum of normal products of field operators $\varphi_{in}(z)$ and their derivatives, is first represented in the form of an expansion in normal products of the fields φ_{in} only (no derivatives), and if one takes variations with respect to $\varphi_{in}(y)$ and then carries over the derivatives from the coefficient functions of the product back onto the fields φ_{in}, one obtains the following general formula:

$$\frac{\delta M(z)}{\delta\varphi_{in}(y)} = \sum_{i=0}^{\varkappa}\Big[\frac{\partial^i}{\partial y^i}\delta(y-z)\Big]\partial M(z)\Big/\partial\Big(\frac{\partial^i\varphi_{in}(z)}{\partial z^i}\Big). \tag{18}$$

For $M(z) \equiv H_I^{int}(z)$ only time derivatives are important from our point of view. Therefore we reconstruct the series (18) in terms of time-derivatives only and introduce the notation

$$\partial H_I^{int}(z)\Big/\partial\Big(\frac{\partial^i\varphi_{in}(z)}{\partial z_0^i}\Big) = \tilde{j}_i^{\,int}(z). \tag{19}$$

Substituting now the general expression for $\delta H_I^{int}(z)/\delta\varphi_{in}(y)$ into (14) and carrying out the integration with respect to z in (13), we have, finally

$$i\frac{\delta S(x^0,-\infty)}{\delta\varphi_{in}(y)} = S(x^0,-\infty)\sum_{i=0}\frac{\partial^i}{\partial y_0^i}[\theta(x^0-y^0)\tilde{\mathrm{j}}_i(y)]$$

$$= S(x^0,-\infty)\Big\{\theta(x^0-y^0)\sum_{i=0}^{\varkappa}\frac{\partial^i\tilde{\mathrm{j}}_i}{\partial y_0^i} + \sum_{i=1}^{\varkappa}(-1)^i\delta^{(i-1)}(x_0-y_0)\sum_{n=i}^{\varkappa}C_i^n\frac{\partial^{n-i}\tilde{\mathrm{j}}_n}{\partial y_0^{n-i}}\Big\}, \tag{20}$$

where C_i^n is the number of combinations of n objects taken i times ($C_i^n = \binom{n}{i}$ is the binomial coefficient),

$$\tilde{\mathrm{j}}_i(y) = S^+(y^0,-\infty)\tilde{j}_i^{\,int}(y)S(y^0,-\infty), \tag{21}$$

and $\tilde{j}_i^{int}(y)$ is defined by (19).

Thus, in the right hand side of (20) there appear delta-like terms in the time variable, in addition to the terms with $\theta(x^0-y^0)$ which are present in the simplest case (17). We thus observe a complete analogy with the ordinary Tomonaga-Schwinger equation, where in the presence of derivative couplings the equation is somewhat modified due to the occurrence of additional terms. In that case however such terms are not too essential, and for many theories they disappear from the total S-matrix.[5] This allows one to hope that, due to the fact that the additional terms in the right hand side of (20) can contribute to the "halved" S-matrix only on the surface x_0 = const, they are somehow a reflection of a unitary arbitrariness of type (5) in the definition of that quantity, and are inessential from the point of view of the total S-matrix. We shall return to this question later.

3. A NEW REPRESENTATION FOR THE HEISENBERG CURRENT OPERATOR

In order to have a convenient way of comparing the right-hand side of (20) with the expression $\theta(x^0 - y^0)\mathrm{j}(y)$, we derive here a new representation for the Heisenberg current operator. We shall start from the definition (7) and the representation of the S-matrix in terms of T_D-products (cf. (4)) and we make use of the unitarity and group property of the matrix $S(x^0, -\infty)$ of the form (2). Then

$$\mathrm{j}(y) = iS^+(x^0, -\infty)\left\{S^+(\infty, x^0)\frac{\delta S(\infty, x^0)}{\delta\varphi_{in}(y)} + \frac{\delta S(x^0, -\infty)}{\delta\varphi_{in}(y)} \times S^+(x^0, -\infty)\right\}S(x^0, -\infty), \tag{22}$$

where the "halving" point x^0 has been chosen arbitrarily.

Transforming (22) with the help of (13) we obtain

$$\mathrm{j}(y) = S^+(x^0, -\infty)\int_{-\infty}^{\infty} dz\left\{\theta(z^0 - x^0)S^+(z^0, x^0) \times \frac{\delta H_I^{\mathrm{int}}(z)}{\delta\varphi_{in}(y)}S(z^0, x^0) + \theta(x^0 - z^0)S(x^0, z^0) \times \frac{\delta H_I^{\mathrm{int}}(z)}{\delta\varphi_{in}(y)}S^+(x^0, z^0)\right\}S(x^0, -\infty). \tag{23}$$

Substituting here the expression for $\delta H_I^{\mathrm{int}}(z)/\delta\varphi_{in}(y)$, taking into account (18) and (19) and carrying out the z-integration, the result is

$$\mathrm{j}(y) = \sum_{i=0}^{\varkappa}\frac{\partial^i}{\partial y_0{}^i}\{[\theta(y^0 - x^0) + \theta(x^0 - y^0)]S^+(y^0, -\infty) \times \tilde{j}_i{}^{\mathrm{int}}(y)S(y^0, -\infty)\} = \sum_{i=0}^{\varkappa}\frac{\partial^i}{\partial y_0{}^i}\tilde{\mathrm{j}}_i(y), \tag{24}$$

where $\tilde{\mathrm{j}}_i(y)$ are of the form (21). It should be noted that the transformations involving θ-functions carried out here and in the preceding section may seem somewhat formal. However the results obtained confirm that they are legitimate. Comparing (24) multiplied by $\theta(x^0 - y^0)$ with the right hand side of (20) one notes that the difference of the two expressions consists in the fact that in (20) the functions $\theta(x^0 - y^0)$ are also differentiated, whereas in $\theta(x^0 - y^0)\mathrm{j}(y)$ the derivatives do not act on $\theta(x^0 - y^0)$. This difference may, in principle, appear already in the first order derivative.

Let us write the above expressions for the special case of a neutral scalar self-interacting field with the effective Lagrangian:[11]

$$L_I{}^{in}(x) = gZ_1 : \varphi_{in}{}^4(x) : + {}^1/_2(Z_3 - 1) : \varphi_{in}(x)K_x\varphi_{in}(x) : - {}^1/_2 Z_3\delta m^2 : \varphi_{in}{}^2(x) :, \tag{25}$$

where $K_x = \Box_x - m^2$. This theory will be the principal object of our investigation in the remainder of this paper, but this does not restrict the generality of the results obtained. As has been shown previously,[6] in this case (taking into account the term with δm^2)

$$H_I{}^{\mathrm{int}}(x) = -\frac{gZ_1}{Z_3{}^2}:\varphi_{in}{}^4(x): - \left(1 - \frac{1}{\sqrt{Z_3}}\right):\varphi_{in}(x)K_x\varphi_{in}(x): + \frac{1}{2}\delta m^2:\varphi_{in}{}^2(x):, \tag{26}$$

whence

$$\tilde{j}_0{}^{\mathrm{int}}(x) = -\frac{4gZ_1}{Z_3{}^2}:\varphi_{in}{}^3(x): - \left(1 - \frac{1}{\sqrt{Z_3}}\right)\left(2K_x\varphi_{in}(x) + \frac{\partial^2\varphi_{in}}{\partial x_0{}^2}\right) + \boldsymbol{\delta m^2\varphi_{in}(x)}, \tag{27}$$

$$\tilde{j}_2{}^{\mathrm{int}}(x) = \partial H_I{}^{\mathrm{int}}(x)\Big/\partial\left(\frac{\partial^2\varphi_{in}(x)}{\partial x_0{}^2}\right) = \left(1 - \frac{1}{\sqrt{Z_3}}\right)\varphi_{in}(x), \tag{28}$$

$$\mathrm{j}(x) = \tilde{\mathrm{j}}_0(x) + \frac{\partial^2}{\partial x_0{}^2}\tilde{\mathrm{j}}_2(x). \tag{29}$$

The equation of motion (20) for the matrix $S(x^0, -\infty)$ takes in this theory the form

$$i\frac{\delta S(x^0, -\infty)}{\delta\varphi_{in}(y)} = S(x^0, -\infty)\left\{\theta(x^0 - y^0)\mathrm{j}(y) - \delta(x^0 - y^0)\frac{\partial\tilde{\mathrm{j}}_2}{\partial y^0} - \frac{\partial}{\partial y^0}[\delta(x^0 - y^0)\tilde{\mathrm{j}}_2(y)]\right\} \tag{30}$$

or the equivalent form

$$i\frac{\delta S(x^0, -\infty)}{\delta\varphi_{in}(y)} = S(x^0, -\infty)\left\{\theta(x^0 - y^0)\mathrm{j}(y) - 2\delta(x^0 - y^0)\frac{\partial\tilde{\mathrm{j}}_2}{\partial y^0} + \left[\frac{\partial}{\partial x^0}\delta(x^0 - y^0)\right]\tilde{\mathrm{j}}_2(y)\right\}. \tag{31}$$

It is easy to understand that in going over from a matrix $S(x^0, -\infty)$ satisfying (30) or (31) to $\tilde{S}(x^0, -\infty)$ of the form (5), one might in principle simplify the right hand side of the equation by an appropriate choice of the operator $\alpha(x^0)$. However, even after such simplification, the equation for $\tilde{S}(x^0, -\infty)$ does not reduce to a trivial equation of the form (17), since for the theory under consideration such an equation does not satisfy the integrability condition,[1, 2] and as shall be shown below, the unitary transformation (5) cannot have any influence whatsoever on this condition.

4. THE PROBLEM OF DEFINING THE HEISENBERG CURRENT OPERATOR

In this section we shall make use of the formulas derived above in order to rederive in a different manner the previously obtained[6] expressions for the renormalized Heisenberg field operator $\mathbf{A}(x)$ of the form (6) and its time derivative $\dot{\mathbf{A}}(x)$. These expressions were obtained previously by applying an analog of Wick's theorem, and it was established in [6] that

$$\mathbf{A}(x) = Z_3^{-1/2} S^+(x^0, -\infty)\varphi_{in}(x) S(x^0, -\infty), \qquad (32)$$

$$\dot{\mathbf{A}}(x) = S^+ T_W(\dot{\varphi}_{in}(x) S) = Z_3^{-1/2} S^+(x^0, -\infty)\dot{\varphi}_{in}(x) S(x^0, -\infty) \qquad (33)$$

for any renormalizable theory and in particular for the theory with $L_I^{in}(x)$ of the form (25).

In the case under consideration we shall not look for a representation of the form (8) for $\mathbf{A}(x)$, since the possibility of obtaining it is still not cleared up. Instead we shall, as before,[6] develop the idea put forward by Kirzhnits,[12] that one might attempt to represent $\mathbf{A}(x)$ in the form

$$\mathbf{A}(x) = S^+(x^0, -\infty)\varphi_{in}(x) S(x^0, -\infty) + \chi(x), \qquad (34)$$

where $S(x^0, -\infty)$ is of the form (12) and the operator $\chi(x)$ is assumed to have the representation (3), i.e.

$$\chi(x) = S^+(x^0, -\infty)\chi^{\text{int}}(x) S(x^0, -\infty). \qquad (35)$$

Transforming the first term in (34) in the same manner as in (8), and comparing with (6), we obtain for $\chi(x)$[12]

$$\chi(x) = \int dy\, D(x-y)\left\{ iS^+(x^0, -\infty)\frac{\delta S(x^0, -\infty)}{\delta\varphi_{in}(y)} - \theta(x^0 - y^0)\mathbf{j}(y)\right\}. \qquad (36)$$

Taking into account Eq. (20) and the expressions for the current $\mathbf{j}(y)$ of the form (24) we obtain in the general case

$$\chi(x) = \int dy\, D(x-y)\sum_{i=0}^{\varkappa}\left\{\frac{\partial^i}{\partial y_0{}^i}[\theta(x^0 - y^0)\tilde{\mathbf{j}}_i(y)] - \theta(x^0 - y^0)\frac{\partial^i}{\partial y_0{}^i}\tilde{\mathbf{j}}_i(y)\right\} = \sum_{i=1}^{\varkappa}(-1)^i\int dy\,\delta(y^0 - x^0) \times \frac{\partial^{i-1}}{\partial y_0{}^{i-1}}\left[D(x-y)\sum_{n=i}^{\varkappa} C_i^n \frac{\partial^{n-i}\tilde{\mathbf{j}}_n}{\partial y_0{}^{n-i}}\right]. \qquad (37)$$

We thus obtain in general that $\chi(x) \neq 0$; however, if the curly bracket in (37) can be nonzero already in the presence of the first order derivative, the expression for $\chi(x)$ itself will be nonvanishing only beginning with the second derivative, since the contribution of the first derivative to $\chi(x)$ vanishes, due to the presence of the factor $D(x-y)\,\delta(x^0 - y^0) = 0$ (some terms may also vanish in the case of higher order derivatives).

In the case which interests us (taking into account (30))

$$\chi(x) = \int dy\, D(x-y)\left\{-\delta(x^0 - y^0)\frac{\partial\tilde{\mathbf{j}}_2(y)}{\partial y^0} - \frac{\partial}{\partial y^0}[\delta(x^0 - y^0)\tilde{\mathbf{j}}_2(y)]\right\} = -\tilde{\mathbf{j}}_2(x). \qquad (38)$$

Hence, taking into account (21), (28) and (35)

$$\chi^{\text{int}}(x) = (Z_3^{-1/2} - 1)\varphi_{in}(x), \qquad (39)$$

in agreement with (32).

One can, of course, obtain an expression for $\dot{\mathbf{A}}(x)$ of the form (33) either by differentiating (32) or by differentiating (34). It is however interesting to proceed independently, starting only from the definition of $\dot{\mathbf{A}}(x)$. Then, on the one hand

$$\dot{\mathbf{A}}(x) = \dot{\varphi}_{in}(x) - \int dy\,\dot{D}(x-y)\,\theta(x^0 - y^0)\,\mathbf{j}(y)\,dy, \qquad (40)$$

and on the other hand we shall look for $\dot{\mathbf{A}}(x)$ in the form

$$\dot{\mathbf{A}}(x) = S^+(x^0, -\infty)\dot{\varphi}_{in}(x) S(x^0, -\infty) + \eta(x), \qquad (41)$$

where $\eta(x)$ has a representation of the form (3). Then, equating (40) and (41) we obtain

$$\eta(x) = \int dy\,\dot{D}(x-y)\left\{iS^+(x^0, -\infty)\frac{\delta S(x^0, -\infty)}{\delta\varphi_{in}(y)} - \theta(x^0 - y^0)\,\mathbf{j}(y)\right\}, \qquad (42)$$

where, if we would require a unitary relation between $\dot{\mathbf{A}}(x)$ and $\dot{\varphi}_{in}(x)$, it would be necessary to put $\eta(x)$ equal to zero.

The terms with $\delta(x^0 - y^0)$ and its even order derivatives did not contribute to the expression (36) and similarly, the odd order derivatives of $\delta(x^0 - y^0)$ do not contribute to (42). In the particular case we are interested in we have (taking (30) into account)

$$\eta(x) = \int dy\,\dot{D}(x-y)\left\{-\delta(x^0 - y^0)\frac{\partial\tilde{\mathbf{j}}_2}{\partial y^0} - \frac{\partial}{\partial y^0}[\delta(x^0 - y^0)\tilde{\mathbf{j}}_2(y)]\right\} = -\frac{\partial\tilde{\mathbf{j}}_2}{\partial y^0}, \qquad (43)$$

from where it follows easily that

$$\eta^{\text{int}}(x) = (Z_3^{-1/2} - 1)\dot{\varphi}_{in}(x), \qquad (44)$$

in agreement with (33). Thus the use of the analog of Wick's theorem which was derived previously[6] and which relates the T_W- and T_D-products has been completely vindicated.

5. SOLUTION OF THE EQUATIONS OF MOTION FOR THE "CURRENT-LIKE" OPERATORS

Medvedev[13] (cf. also [1,2]) has proposed a representation for the S-matrix in the axiomatic approach in which the Heisenberg-picture "current-like" operators $\Lambda_\nu(x_1, \ldots, x_\nu)$ play an essential role. He has also derived equations of motion for these operators, which in the asymptotic in-picture have the form

$$\frac{\delta \Lambda_\nu(x_1, \ldots, x_\nu)}{\delta \varphi_{in}(y)} = i\theta(x_i^0 - y^0)[\Lambda_1(y), \Lambda_\nu(x_1, \ldots, x_\nu)] + \Lambda_{\nu+1}(y, x_1, \ldots, x_\nu), \tag{45}$$

where $\Lambda_1(x) = \mathbf{j}(x)$ of the form (7).

As has been stressed in [1,2] if each of the operators Λ_ν would be unitarily related to its in-transform by means of the matrix $S(x^0, -\infty)$, and the latter would satisfy an equation of motion of the form (17), then one could get rid of the nonlinear terms in (45) and it would take the simple form

$$\frac{\delta \Lambda_\nu^{in}(x_1, \ldots, x_\nu)}{\delta \varphi_{in}(y)} = \Lambda_{\nu+1}^{in}(y, x_1, \ldots, x_\nu). \tag{46}$$

This would imply: a) the true quasilocality of the operators Λ_ν^{in}; b) the possibility of deriving all Λ_ν by variations of one operator

$$\Lambda_0^{in} = \int_{-\infty}^{\infty} L_I^{in}(\xi)\, d\xi.$$

However it is known[1, 2, 6] that this is not so and therefore the problem of solving (45) remains open. These equations can be solved in principle by admitting, in agreement with (3), a representation of Λ_ν of the form

$$\Lambda_\nu(x_1, \ldots, x_\nu) = S^+(x_i^0, -\infty)\Lambda_\nu^{int}(x_1, \ldots, x_\nu) S(x_i^0, -\infty) \tag{47}$$

and making use of the equation (20) for the matrix $S(x_i^0, -\infty)$.

As an example we consider in the general case the first equation of (45). Substituting into it an expression of the type (47) for $\Lambda_2(x, y)$, the expression (24) for the current and carrying out the variation, we obtain

$$\sum_{i=0}^{\varkappa} \frac{\partial^i}{\partial x_0{}^i}\left[S^+(x^0, -\infty)\frac{\delta \tilde{j}_i^{\,int}(x)}{\delta \varphi_{in}(y)} S(x^0, -\infty)\right] + i\sum_{i,j=0}^{\varkappa} \frac{\partial^i}{\partial x_0{}^i}\frac{\partial^j}{\partial y_0{}^j}\left\{\theta(x^0 - y^0)[\tilde{\mathbf{j}}_j(y), \tilde{\mathbf{j}}_i(x)]\right\}$$

$$= i\theta(x^0 - y^0)\sum_{i,j=0}^{\varkappa} \frac{\partial^i}{\partial x_0{}^i}\frac{\partial^j}{\partial y_0{}^j}[\tilde{\mathbf{j}}_j(y), \tilde{\mathbf{j}}_i(x)] + S^+(x^0, -\infty)\Lambda_2^{int}(x, y) S(x^0, -\infty). \tag{48}$$

It is easy to see that the nonlinear terms cancel out in this equation and only terms which can in principle be represented in the form (3) remain. Thus in general, (48) can be reduced to an equation for operators in the interaction picture. However this equation will not have the simple form (46) even for the operator Λ_2^{int}, since different equal-time commutators will also contribute to it.

We will continue our further investigation not in the general form, but for the theory with $L_I^{in}(x)$ of the form (25). Substituting in (45) the representation (47) and making use of the equation of motion (30), we obtain that

$$\Lambda_{\nu+1}^{int}(y, x_1, \ldots, x_\nu) = \frac{\delta \Lambda_\nu^{int}(x_1, \ldots, x_\nu)}{\delta \varphi_{in}(y)} - i\delta(y^0 - x_i^0)\left[\frac{\partial \tilde{j}_2{}^{int}(y)}{\partial y^0}, \Lambda_\nu{}^{int}(x_1, \ldots, x_\nu)\right]. \tag{49}$$

If one takes into consideration the fact that the current $\mathbf{j}(x)$ admits the representation (29), it turns out that Eq. (49) can be applied in the case $\nu = 0$ also, and

$$\Lambda_0{}^{int} = \int_{-\infty}^{\infty} d\xi H_I{}^{int}(\xi).$$

Thus (49) implies the genuine quasilocality of Λ_ν^{int} and also the possibility of determining all Λ_ν^{int} from a given local Hamiltonian $H_I^{int}(x)$ (cf. (28)).

Knowing a $H_I^{int}(x)$ in the form (26) and a $\tilde{j}^{int}(x)$ in the form (28) one can find a concrete expression for Λ_ν^{int} in the theory we are interested in. For $\nu = 1, 2, 3, 4$ (the other $\Lambda_\nu^{int} = 0$)[2)] we obtain

$$\Lambda_1{}^{int}(y) = -\frac{4gZ_1}{Z_3^{5/2}} : \varphi_{in}{}^3(y) : + \left(\frac{1}{Z_3^{1/2}} - 1\right)\varphi_{in}(y) K_y + \left(\frac{1}{Z_3^{1/2}} - 1\right)\frac{1}{Z_3^{1/2}} K_y \varphi_{in}(y) + \frac{\delta m^2}{Z_3^{1/2}}\varphi_{in}(y), \tag{50}$$

$$\Lambda_2{}^{int}(x, y) = -\frac{12gZ_1}{Z_3{}^3} : \varphi_{in}{}^2(x) : \delta(x - y) + \left(\frac{1}{Z_3} - 1\right) K_x \delta(x - y) + \frac{\delta m^2}{Z_3}\delta(x - y), \tag{51}$$

$$\Lambda_3{}^{int}(x, y, z) = -\frac{24gZ_1}{Z_3^{7/2}}\varphi_{in}(x)\delta(x - y)\delta(x - z), \tag{52}$$

$$\Lambda_4{}^{int}(x, y, z, u) = -\frac{24gZ_1}{Z_3{}^4}\delta(x - y)\delta(x - z)\delta(x - u). \tag{53}$$

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2) This result is in agreement with the assertion of Medvedev and Polivanov[14], that in renormalizable theories the number of matrix elements with nonnegative degrees of growth is finite (not larger than four).

It should be noted that in the expression (50) itself as well as in (26) the term involving $K_x\varphi_{in}$ can be omitted.[6] We have kept it, however to the very end, since the variations were taken with respect to the operators $\varphi_{in}(y)$ and this term gave a nonvanishing contribution to (51).[3)] It should also be noted that in addition to the principal term $\delta\Lambda_\nu^{int}(x_1, \ldots, x_\nu)/\delta\varphi_{in}(y)$ Eq. (49) also contains a term involving $\delta(x^0 - y^0)$ which very probably can be eliminated by means of some unitary transformation. This problem will be investigated further.

6. CONCLUSION

We have obtained the equation of motion (13) for the matrix $S(x^0, -\infty)$ of the form (2). In general this equation does not have the simple form (17). We have succeeded to discuss with the aid of this equation a series of interesting problems regarding the structure of the mathematical apparatus of quantum field theory. The corresponding equation for the "halved" S-matrix of the more general form (5) is

$$i\frac{\delta\widetilde{S}(x^0, -\infty)}{\delta\varphi_{in}(y)} = \widetilde{S}(x^0, -\infty)\left\{\int_{-\infty}^{\infty}\theta(x^0 - y^0)\,\mathbf{j}'(z, y)\,dz + \mathbf{N}(x, y)\right\},$$

$$\mathbf{N}(x, y) = i\widetilde{S}^+(x^0, -\infty)\left[\frac{\delta}{\delta\varphi_{in}(y)}\exp(-i\alpha(x^0))\right] \times \exp(i\alpha(x^0))\,\widetilde{S}(x^0, -\infty). \quad (54)$$

We now show that the transformation from $S(x^0, -\infty)$ to $\widetilde{S}(x^0, -\infty)$ cannot influence the validity of the integrability condition of the equation of motion which was valid before (cf. (15)).[4)] The appearance of the supplementary term in the right hand side of (54) leads to additional terms in the integrability condition:

$$\frac{\delta\mathbf{N}(x,y)}{\delta\varphi_{in}(z)} - \frac{\delta\mathbf{N}(x,z)}{\delta\varphi_{in}(y)} + i[\mathbf{N}(x,y)\mathbf{N}(x,z)] + i\int_{-\infty}^{\infty}du\,\theta(x^0-u^0)[\mathbf{j}'(u,y),\mathbf{N}(x,z)] - i\int_{-\infty}^{\infty}du\,\theta(x^0-u^0) \times [\mathbf{j}'(u,z),\mathbf{N}(x,y)]. \quad (55)$$

But, according to (54)

$$\frac{\delta\mathbf{N}(x,y)}{\delta\varphi_{in}(z)} = i\int_{-\infty}^{\infty}du\,\theta(x^0-u^0)[\mathbf{j}'(u,z),\mathbf{N}(x,y)] - i\mathbf{N}(x,y)\mathbf{N}(x,z) + i\widetilde{S}^+(x^0,-\infty) \times\left[\frac{\delta^2\exp(-i\alpha(x^0))}{\delta\varphi_{in}(z)\,\delta\varphi_{in}(y)}\right]\exp(i\alpha(x^0))\,\widetilde{S}(x^0,-\infty). \quad (56)$$

Therefore the additional terms in the integrability condition are equal

$$-i\mathbf{N}(x,y)\mathbf{N}(x,z) + i\mathbf{N}(x,z)\mathbf{N}(x,y) + i[\mathbf{N}(x,y),\mathbf{N}(x,z)] + i\widetilde{S}(x^0,-\infty) \times\left[\frac{\delta^2\exp(-i\alpha(x^0))}{\delta\varphi_{in}(z)\,\delta\varphi_{in}(y)} - \frac{\delta^2\exp(-i\alpha(x^0))}{\delta\varphi_{in}(y)\,\delta\varphi_{in}(z)}\right] \times\exp(i\alpha(x^0))\,\widetilde{S}(x^0,-\infty),$$

i.e., will vanish identically, if one takes into account the locality of $\alpha(x^0)$.

Thus, Eq. (54) cannot have the trivial form (17), but owing to the presence of the supplementary term $\mathbf{N}(x, y)$ its right-hand side can be partially simplified. It is not excluded that this simplification is of such a nature as to lead to the weaker condition (9), which guarantees a unitary relation between the operators $A(x)$ and $\varphi_{in}(x)$. The realization of such a simplification and a clarification of the physical significance of the relation between $\mathbf{A}(x)$ and $\varphi_{in}(x)$ will be the object of another paper.

In conclusion I express my profound gratitude to B. V. Medvedev for continuous attention to this work and valuable remarks. I also express my gratitude to V. Ya. Fainberg, D. A. Kirzhnits and A. V. Astakhov for a useful discussion.

[3)]We note that the second term in (50) does not vanish even on the energy shell (from the viewpoint of the current operator). (Private communication by B. V. Medvedev.)

[4)]The idea of this reasoning is due to B. V. Medvedev.

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Translated by M. E. Mayer
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