AUTOMODULATION OF THE RADIATION FROM A LASER WITH A TWO-MODE RESONATOR

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Submitted to JETP editor June 3, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) 49, 1535-1543 (November, 1965)

The oscillations of a laser with two close frequency resonator modes are considered. Possible monochromatic processes and their stability as well as nonlinear intensity oscillations ("spikes") are investigated. It is shown that such oscillations may become sustained, since the phase space of the averaged equations of the system contains a stable limit cycle. The shape and amplitude of the corresponding spikes are found.

1. FUNDAMENTAL EQUATIONS

In an earlier paper[1] (henceforth cited as I) we considered the model of a solid-state laser, whose oscillations are governed by the fields of two resonator modes. It was assumed that the mode frequencies are not too close to each other (their resonant bands do not overlap) and the field of each mode oscillates at its own frequency. In real resonators, such an assumption is valid for axial modes having an identical transverse field structure, but modes with different transverse structures can include degenerate or 'quasidegenerate' modes, which have quite close frequencies. The singularities of the interaction of such modes provide, in particular, one of the most likely explanations of the undamped automodulation of laser radiation[2,3]. However, in view of the complexity of the approximate equations used in the cited papers, the results have been restricted to an investigation of the stability of the monochromatic oscillations and several numerical solutions. The reason for the occurrence of automodulation, and its dependence on the parameters of the system, remain unclear. (Thus, Fleck and Kidder[2] relate these causes with the inhomogeneity of the additional illumination field, whereas Basov, Morozov, and Oraevskiĭ[3] obtained similar effects also for homogeneous illumination.)

From this point of view, the model assumed in I has many advantages. The corresponding equations are relatively simple and admit of a sufficiently complete qualitative investigation. At the same time, the results obtained in the single-mode idealization[4,5] make it possible to assume that these equations describe correctly many essential features of real processes.[1)]

The oscillations of the mode fields with arbitrary frequencies, orthogonal in the absence of an active medium, are described within the framework of the model in question by the following system of averaged equations (see I):

$$\dot{Q}_\lambda + Q_\lambda[\tau_\lambda^{-1} + i(\omega - \omega_\lambda)] = -2^{-1} i\omega\gamma_\lambda^*\sigma,$$
$$\dot{\sigma} + \sigma[T_2^{-1} + i(\omega - \omega_0)] = \frac{i}{\hbar} N \sum_{\lambda=1,2} \gamma_\lambda Q_\lambda,$$
$$\dot{N} + T_1^{-1}(N - N_0) = \frac{2i}{\hbar} \sum_{\lambda=1,2} (\sigma\gamma_\lambda^* Q_\lambda^* - \sigma^*\gamma_\lambda Q_\lambda). \quad (1)$$

Here $Q_\lambda(t)$ ($\lambda = 1, 2$) are the complex amplitudes of the mode fields (the field of the λ-th mode is proportional to $Q_\lambda \exp(i\omega t) + Q^*_\lambda \exp(-i\omega t)$), $\sigma(t)$ the amplitude of the off-diagonal density-matrix element of the active medium ($\rho_{12} = \sigma(t) \exp(i\omega t)$), ω the constant frequency (arbitrary for the time being) over the period of which the averaging is carried out, N the population difference of the energy levels of the medium, ω_λ the frequencies of the mode fields, τ_λ their damping times in the absence of active medium, γ_λ coefficients that depend on the spatial configuration of the mode fields, $T_{1,2}$ the relaxation times of the medium, and M_0 a parameter that depends on the illumination field and on the temperature.

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1) The equations used below do not describe only effects connected with the non-uniform de-excitation of the active medium by the resonator field. The results that follow indicate that this non-uniformity is not obligatory for the existence of undamped automodulation (although it possibly facilitates its occurrence). For traveling-wave lasers the system (1) is apparently valid without any fundamental limitations at all.

To simplify further derivations, we assumed that T_2 is much smaller than all the other characteristic times contained in (1), so that the derivative of σ in the second equation can be neglected.[2)] If we assume further that the mode frequencies are close to the center of the line of the medium, then we can neglect also the quantity $i(\omega - \omega_0)$.[3)]

Then, putting $Q_\lambda = |Q_\lambda| \exp(i\varphi_\lambda)$ (that is, introducing real amplitudes and phases) and going over to dimensionless variables, we obtain after separating the real and imaginary part in (2)

$$\dot{u}_1 = G_1[u_1(n-1) + u_2 n \cos\psi], \quad (2a)$$

$$\dot{u}_2 = G_2[u_2(pn-1) + u_1 pn \cos\psi], \quad (2b)$$

$$\dot{n} = \alpha_1 - n(1 + u_1^2 + u_2^2 + 2u_1u_2\cos\psi), \quad (2c)$$

$$u_1\dot{\varphi}_1 = G_1[\Delta_1 u_1 - nu_2 \sin\psi], \quad (3a)$$

$$u_2\dot{\varphi}_2 = G_2[\Delta_2 u_2 + pnu_1 \sin\psi]. \quad (3b)$$

We put here

$$\psi = \varphi_1 - \varphi_2, \quad u_\lambda = (4T_1T_2/\hbar^2)^{1/2}|\gamma_\lambda Q_\lambda|, \quad \Delta_\lambda = \tau(\omega_\lambda - \omega),$$

$$\alpha_\lambda = \omega_0 T_2 N_0 |\gamma_\lambda|^2 \tau_\lambda / 2\hbar, \; n = \alpha_1 N / N_0,$$

$$G_\lambda = T_1/\tau_\lambda, \; p = \alpha_2/\alpha_1, \; t' = t/T_1.$$

It must be noted that if we are not interested in the phase variations of each of the modes separately, and only in the phase difference, then the order of the system can be reduced further. Indeed, from (3a) and (3b) it follows that

$$\dot{\psi} = G_1\left[2\delta - n\left(\frac{u_2}{u_1} + \frac{pG_2}{G_1}\frac{u_1}{u_2}\right)\sin\psi\right], \quad \delta = \frac{1}{2}\tau_1(\omega_1 - \omega_2) \quad (2d)$$

(for concreteness we assume that $\delta > 0$). Equations (2a)-(2d) constitute a complete system.

If δ is sufficiently large ($\delta \gg 1$), then $\dot{\psi}$ is also large, and consequently the terms containing ψ explicitly oscillate rapidly. Carrying out a second averaging in the system (2)-(3) and discarding oscillating terms, we can obtain the equations investigated in I (where account is taken also of terms due to the finite width of the line of the medium). In the general case, however, phase relations between the mode fields noticeably influence the process under consideration.

2. EQUILIBRIUM POSITIONS AND THEIR STABILITY

We investigate first a symmetrical case, when $\tau_1 = \tau_2 = \tau$ and $\alpha_1 = \alpha_2 = \alpha$. It is easy to see that the system (2) has, in addition to a zero position, also the following equilibrium positions (corresponding to monochromatic oscillations of the laser):[4)]

a) symmetrical, in which

$$\omega = (\omega_1 + \omega_2)/2, \quad \cos\psi = (1-\delta^2)/(1+\delta^2),$$
$$n = (1+\delta^2)/2, \quad u_1^2 = u_2^2 = {}^1/_2[\alpha - (1+\delta^2)/2]; \quad (4)$$

b) two "lateral," in which

$$\omega = \frac{\omega_1 + \omega_2}{2} \pm \frac{\delta}{2\tau}\sqrt{1-\delta^{-2}}, \quad \psi = \pm\frac{\pi}{2}, \quad n = 1,$$

$$u_1^2 = \frac{\alpha - 1}{2}[1 \pm \sqrt{1-\delta^{-2}}], \quad u_2^2 = \frac{\alpha-1}{2}[1 \mp \sqrt{1-\delta^{-2}}]. \quad (5)$$

It is clear from (5) that the "lateral" equilibrium positions exist only when $\delta > 1$. For the existence of a symmetrical position (but not for its stability) it is sufficient to satisfy the conditions for self-excitation of the system (see (7)).

Let us investigate now the stability of the obtained equilibrium positions. Linearizing the system (2) about the corresponding stationary values, and assuming the deviations from them to be proportional to $\exp(\beta t')$, we can obtain the following results.

For the zero equilibrium position ($u_1 = u_2 = 0$, $n = \alpha$) small field perturbations correspond to values

$$\beta_{1,2} = G\{(\alpha - 1) \pm \sqrt{\alpha^2 - \delta^2} + i\tau((\omega_1 + \omega_2)/2 - \omega)\}. \quad (6)$$

It follows therefore that when $\alpha > \delta$ the self-excitation condition is of the form

$$\alpha > (1+\delta^2)/2, \quad (7a)$$

and when $\alpha < \delta$

$$\alpha > 1. \quad (7b)$$

In the first case the oscillation frequency is $(\omega_1 + \omega_2)/2$ (it must be recognized that the fields are proportional to $\exp[(i\omega + \beta T_1^{-1})t]$. In the second case, an equal growth increment is possessed by oscillations with two frequencies

$$(\omega_1 + \omega_2)/2 \pm \sqrt{\delta^2 - \alpha^2}/\tau.$$

[2)]With the possible exception of the initial stage of the oscillations, in which "fast" processes with characteristic time T_2 are possible. After a time of the order T_2 these processes reduce to those considered below [1,6].

[3)]Incidentally, allowance for the frequency displacement from the center of the line entails no fundamental difficulty.

[4)]Frequencies of monochromatic oscillations of this system coincide with those obtained by Lugovoĭ for a molecular generator. However, even the investigation of the stability leads here to entirely different results (in view of the different order of the quantity T_1).

We note that in the linear approximation we can transform (1) to independent equations for orthogonal modes having the complex frequencies (6) (in a resonator containing a medium). In the nonlinear case, however, these equations turn out again to be coupled, and no essential simplification of the problem is attained in this manner.

An investigation of the stability of the symmetrical equilibrium position reduces to a determination of four characteristic roots, one of which

$$\beta_1 = -G(1-\delta^2). \tag{8a}$$

Consequently, for the given position to be stable, we must at any rate satisfy the condition $\delta < 1$, that is, there should be no other nonzero equilibrium positions. Condition (8) ensures stability of the difference $u_1 - u_2$.

The remaining roots satisfy a third-order equation, which can be investigated with the aid of the ordinary stability criteria: recognizing that for a solid state laser $G = (T_1/\tau) \sim 10^4\text{–}10^6 \gg 1$, we can easily obtain also explicit expressions for these roots:

$$\beta_2 \approx -G(1-\delta^2) = \beta_1, \tag{8b}$$

$$\beta_{3,4} \approx \pm i\left[\frac{2G[2\alpha-(1+\delta^2)]}{1-\delta^4}\right]^{1/2} + \frac{1}{(1-\delta^2)^2}\left[\frac{\alpha(\delta^4+4\delta^2-1)}{1-\delta^2} - \delta^2(1+\delta^2)\right] \tag{8c}$$

(the expressions for $\beta_{2,3,4}$ are not valid if $1-\delta^2 \to 0$). Thus, if the conditions

$$\delta^2 > \sqrt{5}-2 \approx 0.23, \quad \alpha(\delta^4+4\delta^2-1) > \delta^2(1-\delta^2) \tag{9}$$

are simultaneously satisfied, the monochromatic oscillations are unstable ($\mathrm{Re}\,\beta_{3,4} > 0$). For a limited process this means the presence of a periodic or at least quasiperiodic solution (undamped automodulation).

The stability of the "lateral" equilibrium positions, existing when $\delta > 0$ is determined in the case of $G \gg 1$ by the following characteristic roots:

$$\beta_{1,2} \approx \pm i\sqrt{2G(\alpha-1)} - (\alpha\delta^2-1)/2(\delta^2-1), \tag{10a}$$

$$\beta_{3,4} \approx \pm 2iG\sqrt{\delta^2-1} + (\alpha-1)/(\delta^2-1). \tag{10b}$$

It follows from (10b) that both lateral equilibrium positions are always unstable, thus again evidencing the possibility of establishment of a limit cycle. When $\delta^2 \gg 1$ we have $\beta_{3,4} \approx \pm 2i\delta G$, that is, the intensity oscillates with a "beat" frequency $(\omega_1-\omega_2)$ (cf. I).

Inasmuch as the real parts of the complex roots in (8c) and (10) are small, the question naturally arises whether the character of the stability will change under small changes of the symmetry of the system. To answer this question it is necessary to investigate the linearized system (2) with account taken of small differences in the parameters α_1 and α_2. An analysis shows that the difference changes noticeably the real part of only the roots $\beta_{3,4}$ in (10b). Namely:

$$\mathrm{Re}\,\beta_{3,4} \approx \frac{\alpha_1-1}{\delta^2-1} \mp \frac{G\delta^2(\alpha_1-\alpha_2)}{\alpha_1\sqrt{\delta^2-1}}. \tag{11}$$

If $\alpha_1 > \alpha_2$, then the plus and minus signs pertain to the equilibrium positions for which $u_1 < u_2$ and $u_1 > u_2$, respectively. Thus, the difference between the quantities $\alpha_{1,2}$ "aggravates" the instability of one of the positions, whereas the other becomes stable if the difference $\alpha_1 - \alpha_2$ is sufficiently large. For large δ this denotes that the oscillations of the "higher-Q" mode, which has larger α, are stable (see I).

3. NONLINEAR INTENSITY OSCILLATIONS

The foregoing stability investigation offers evidence that a limit cycle can occur in phase space of the considered variables, that is, at periodic automodulation of the laser emission. For a more complete study of the generation processes, and particularly to determine the limit-cycle parameters, it is necessary to investigate essentially nonlinear motions far from the equilibrium positions. Greatest interest is attached in this case, from the point of view of allowance for the phase relations, to the case of modes that are close in frequency ($\delta < 1$), a case which will be considered first.

An investigation of the nonlinear equations (2) is made easy by the fact that all of them, except (2c), contain a large parameter G. In view of this, the processes described by them can be broken up into "fast" (with a characteristic time of the order of τ) and "slow."[5)] The fast processes are described by Eqs. (2a), (2b), and (2d) with constant parameter n.[6)] Putting $(u_1+u_2)^2 = v$ and $(u_1-u_2)^2 = w$, we can write them in the form

$$\dot v = 2Gv[n(1+\cos\psi)-1],$$
$$\dot w = 2Gw[n(1-\cos\psi)-1],$$
$$\dot\psi = 2G\left[\delta - \frac{v+w}{v-w}n\sin\psi\right]. \tag{12}$$

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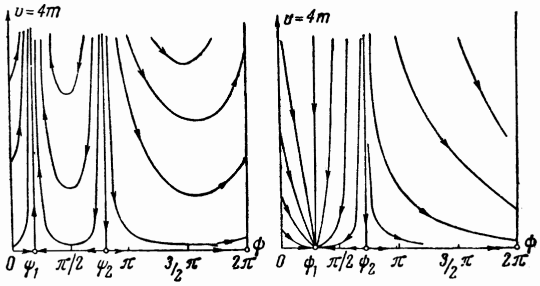
5)This circumstance was first used by Bespalov and Gaponov [5] to analyze the equations of a single-mode laser.

6)Such a description is legitimate up to the region of very high field amplitudes ($u\lambda^2 \geq G$), at which n can change just as rapidly as the remaining variables. In this region, which is not considered in what follows, there can enter trajectories corresponding to "giant" pulses in Q-switched lasers.



a b

FIG. 1. Trajectory of "fast" motions for $u_1 = u_2 = \text{const}$. a) $n + \sqrt{n^2-\delta^2} > 1$, b) $n + \sqrt{n^2-\delta^2} < 1$. Values of $\psi_{1,2}$ satisfy the equation $n \sin\psi = \delta$ ($\psi_1 < \pi/2$; $\psi_2 > \pi/2$).

It is easy to carry out a qualitative analysis of the course of the trajectory in the phase space of the system (12) (that is, in the space of variables v, w, and ψ, bounded by the planes $\psi = 0$, $\psi = 2\pi$, $w = 0$, and $w = v$). This analysis shows that when $n > \delta$ all the trajectories converge to the plane $w = 0$, and on this plane they converge to the vicinity of the straight line

$$w=0, \quad \sin\psi=\delta/n, \quad \psi<\pi/2. \tag{13}$$

Further, v increases without limit along this line if $n > n_p$, and decreases to zero if $n < n_p$, where $n_p = (1+\delta^2)/2$ is the value of n in the symmetrical equilibrium position (5). The picture of the "fast" trajectories in the plane $w = 0$ is shown in Fig. 1.

On the other hand, if $n < \delta$ at the initial instant, then the "fast" motions do not lead to any definite values of ψ and v. Further, however, n decreases monotonically in accordance with (2c) to a value larger than δ, after which the system again "relaxes" to the line (13).

Thus, as a result of the foregoing, the amplitudes of the mode fields become equal. To investigate the "slow" processes we must put in (2) $u_1 = u_2 = u$, and the problem again reduces to an integration of a third-order system:

$$\dot m = 2Gm[n(1+\cos\psi)-1], \quad \dot\psi = 2G(\delta - n\sin\psi), \quad \dot n = \alpha - n[1+2m(1+\cos\psi)], \tag{14}$$

where $m = u^2$. The trajectories (14) of the "fast" motions in phase space (which represent a layer in the space of the variables m, ψ, and n, bounded by the planes $\psi = 0$ and $\psi = 2\pi$), coincide with those shown in Fig. 1 (where $v = 4m$). It is obvious that any quasiperiodic motion should include also the "slow" sections of trajectories, lying near the lines of intersection of the surface formed by the lines (13) with the surfaces $m = 0$ and $n(1 + \cos\psi) = 1$. Let us consider these sections separately.

Near the plane $m = 0$ the equation for n does not contain m and ψ, and can be readily integrated, after which we can determine also the change in ψ and m. The corresponding motion again arrives at the surface (13), near which the "slow" motions also occur.

To study the changes in m, it is convenient to eliminate ψ from (14); as a result we get the equation

$$\frac{\alpha-n}{2G}\left[\frac{d\gamma}{dn}-\frac{1+\gamma}{n}\right]=(2n(1+\gamma)-(1+\gamma)^2)^{1/2}\times[(2n(1+\gamma)-(1+\gamma)^2)^{1/2}-\delta^2],$$
$$\gamma=\frac{\alpha-n}{2Gm}\frac{dm}{dn}. \tag{15}$$

The left side of (15) contains the parameter G^{-1} and is small for "slow" motions (neglect of this side is equivalent to substitution (13)). The corresponding integral (15) can be obtained in general form; we consider further trajectories on which the variation of n relative to the equilibrium value n_p is small, as is usually the case for real initial conditions.[5] 7) Then, putting $\eta = n - n_p$, we obtain, accurate to quantities of order η^4 and G^{-1},

$$\ln\frac{m}{m_a}=\frac{2G}{(\alpha-n_p)(1-\delta^2)}[(\eta^2-\eta_a^2)+A(\eta^3-\eta_a^3)], \tag{16}$$

where

$$A=\frac{2}{3}\left[\frac{1}{\alpha-(1+\delta^2)/2}-\frac{2\delta^2}{(1-\delta^2)^2}\right].$$

According to (16), when η increases from a certain initial value $\eta_a < 0$ to another value $\eta_b > 0$, the intensity of the field decreases sharply (exponentially) in the interval $\eta_a < \eta < 0$, and increases just as sharply for $0 < \eta < \eta_b$. If $G\eta_b^2 \gg 1$, then at the end of the given stage m grows with constant η; this is valid also after the motion leaves the region of small m. The change in η becomes appreciable again only after a sufficiently long time, when m has time to increase to large values. Therefore, in the analysis of the subsequent motion, we can neglect the terms which do not contain m in the right side of the third equation of (14). As a result, eliminating ψ, we can arrive at a second-order equation

$$\frac{m}{G}\left(\frac{n d\varkappa/dn}{1+\varkappa}+1\right)=n(2n(1+\varkappa)-1)^{1/2}[(2n(1+\varkappa)-1)^{1/2}-\delta(1+\varkappa)],$$
$$\varkappa=G^{-1}dm/dn. \tag{17}$$

In the region $\eta^2 \ll 1$, under consideration, the left side of (17) is small (of the order of η^2) for slow

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7) An exception are the "giant" pulses, for which the following quantitative results are anyway incorrect.

motions; then the integral (17) has, with the same accuracy as (16), the form

$$m - m_b = \frac{G}{1-\delta^2}\Big[-(\eta^2-\eta_b^2)+\frac{4}{3}\frac{(1+3\delta^2)}{(1+\delta^2)(1-\delta^2)^2}\times(\eta^3-\eta_b^3)-\frac{8\delta^2\eta_b^2(\eta-\eta_b)}{(1+\delta^2)(1-\delta^2)^2}\Big]. \qquad (18)$$

Successive application of formulas (16) and (18) (the method of point transformations) makes it possible to trace the motion along any specified trajectory in the region in question. In particular, it is easy to find the value of $\eta_{a'}$ at which the trajectory going out of the half-plane ($m = m_a \sim 1$, $\eta < 0$) at $\eta = \eta_a$ returns to this half-plane:

$$\eta_{a'}^2 = \eta_a^2 + B|\eta_a|^3,$$
$$B = \frac{4}{3}\left\{\frac{\delta^4+4\delta^2-1}{1-\delta^4}-\frac{1}{\alpha-(1+\delta^2)/2}\right\}. \qquad (19)$$

We note that when $\eta_a \ll 1$ the difference $\eta_a^2 - \eta_{a'}^2$ is small, of the order of η_a^3, that is, the motion is close to conservative. If $B < 0$, then the oscillations of the amplitude attenuate slowly in this approximation either to a symmetrical equilibrium position (if it is stable) or to a limit cycle lying in the vicinity of the equilibrium point. On the other hand, if $B > 0$, then the oscillations increase and go outside of the region of small η. The conditions for such a growth are more stringent than the conditions for the instability of the position of the equilibrium (9): for $B > 0$ it is also necessary to satisfy the inequality $\delta > \sqrt{5} - 2$, but for a given δ the value of α should be larger here.

If $|B|$ is small, then the limit cycle also consists of trajectories of the type (16) and (18). To determine this cycle it is necessary to include the discarded terms of higher orders of smallness. Exact calculation of these terms is a rather cumbersome; the result depends on the value of B. It can be shown that when $G^{-1} \ll B^2 \ll 1$ the principal role is played by the term of order η/G in the left side of (15). The suitably corrected relation (19) takes the form

$$\eta_{a'}^2 = \eta_a^2 + B|\eta_a|^3 + C|\eta_a|,$$
$$C = 16\delta^2[\alpha-(1+\delta^2)/2]/G(1-\delta^4) > 0. \qquad (20)$$

It follows from (20) that when $B < 0$ there exists in the region of small η a stable limit cycle, the width of which is $2\bar{\eta} = 2(C/|B|)^{1/2}$; with decreasing B, the cycle contracts to a small vicinity of the equilibrium point and vanishes when the equilibrium becomes stable.

When $B > 0$ the trajectories become untwisted and go out of the region of small η.[8)] The succeeding motion calls for a special analysis; recognizing that all fast processes lead to the line of intersection of the surfaces $m = 0$ and (13), we can state that in this case a "giant" limit cycle is possible in the region $|\eta| \sim 1$, and we can obtain a qualitative idea of its form. It must be noted, incidentally, that in practice the motion that begins at small m apparently always remains in the region of small η, owing to the spontaneous fluctuation processes which reduce the growth time of m.[5]

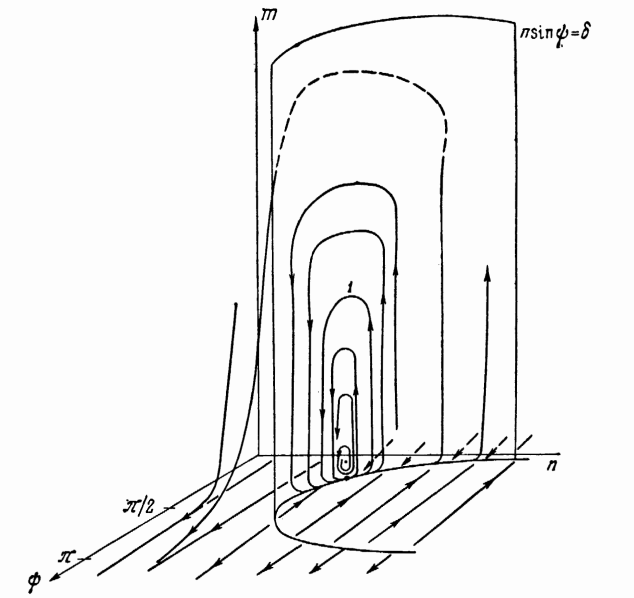

FIG. 2. Region of phase space of the system (14) containing the trajectories of the "slow" motions (1 – stable limit cycle). The "fast" trajectories in the planes n = const are not shown (see Fig. 1).

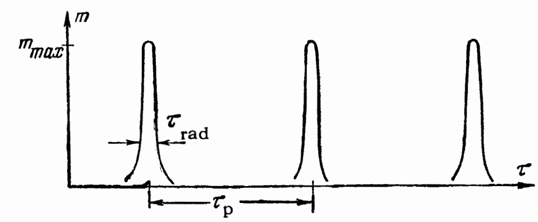

FIG. 3.

The amplitude of the "spikes" corresponding to the given $\bar{\eta}$ is, in first approximation (see (18)),

$$m_{max} \approx G\bar{\eta}^2/(1-\delta^2). \qquad (21)$$

The initial equations (14) make it is also possible to estimate the duration of the "spikes" τ_{rad} and of the pauses between them τ_p ($\tau_p \gg \tau_{rad}$, see [5]):

$$\tau_{rad} \approx \frac{1-\delta^2}{4G|\bar{\eta}|}\ln\left(\frac{G\bar{\eta}^2}{1-\delta^2}\right), \quad \tau_p \approx \frac{2|\bar{\eta}|}{\alpha-(1+\delta^2)/2}. \qquad (22)$$

Figure 2 shows the trajectories in phase space near the stable limit cycle,[9)] while Fig. 3 shows the time variation of the stationary process.

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8)Taking into account in (20) terms of order η^5, we can show that for $B > 0$ a stable limit cycle is possible, with small width ($2\bar{\eta} \sim B$) (but much larger than when $B < 0$).

9)The use of cylindrical coordinates, which are more natural for Figs. 1 and 2, leads to somewhat more complicated diagrams.

The oscillations of modes that differ greatly in frequency, when $\delta > 1$, have an entirely different character. The corresponding problem is even more complicated, but the qualitative results are clear from the limiting case $\delta \gg 1$ which is considered in I. As already indicated, although the instability of the monochromatic processes leads in this case, too, to a limit cycle in the space of the system (2), this cycle corresponds not to spikes, but to oscillations of a biharmonic type (two "almost monochromatic" modes each with its own frequency).[10] For unequal modes ($\alpha_1 \neq \alpha_2$), according to (11), harmonic oscillations are established with a frequency close to the frequency of one of the modes.

In conclusion I take this opportunity to thank A. V. Gaponov for interest in the work and a discussion of the results.

[10]Such "beats" are of interest in themselves and were observed experimentally.

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Translated by J. G. Adashko
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