# THE SCATTERING MATRIX FOR FINITE TIME INTERVALS IN THE WAVE FUNCTION SPACE OF INTERACTING PARTICLES

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Using as an example a nonrelativistic system with fixed particle number, the author shows, by means of a method which differs slightly from the usual one, how to construct an operator $S(t, t_0)$ which transforms the exact solution $\Psi_{int}(t_0)$ in the interaction (Dirac) picture at time $t_0$, into the exact solution $\Psi_{int}(t)$ at time t. In the space of these state vectors the S-matrix is unitary and the coefficient functions $S(t, t_0)$ can be expressed in terms of the wave functions of the problem, including the bound-state wave functions.

THE problem of constructing the S-matrix in the presence of bound states in addition to processes of "scattering type" has been considered in several papers [1,2]. It has been shown that attempts to take into consideration bound states lead to violation of the unitarity of the S-matrix, if the latter is defined in the wave-function space of the free particles. In the present paper, which is a sequel to [3], where the one particle problem was considered, we show that using the space of state vectors which are eigenvectors of the total Hamiltonian, one can write down a unitary matrix $S(t, t_0)$ which describes transitions between finite times.

We shall investigate the case where the Hamiltonian has the form

$$H = H_0 + H_1,$$
$$H_0 = \int d^3\mathbf{k} k^0 a^{(+)}(\mathbf{k}) a^{(-)}(\mathbf{k}),$$
$$H_1 = \int d\mathbf{k}'\, d\mathbf{k}''\, d\mathbf{k}_1'\, d\mathbf{k}_1'' a^{(+)}(\mathbf{k}') a^{(+)}(\mathbf{k}'')$$
$$\times \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}_1' - \mathbf{k}_1'')$$
$$\times u(|(\mathbf{k}' - \mathbf{k}'') - (\mathbf{k}_1' - \mathbf{k}_1'')|) a^{(-)}(\mathbf{k}_1') a^{(-)}(\mathbf{k}_1''), \quad (1)$$

and the wave vector satisfies the equation:

$$i\hbar \frac{\partial}{\partial t} \Psi|0\rangle = H\Psi|0\rangle. \quad (2)$$

Here the operators $a^{(+)}$ and $a^{(-)}$ satisfy the usual commutation relations:

$$[a^{(-)}(\mathbf{k}), a^{(+)}(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'), \quad [a^{(\pm)}(\mathbf{k}), a^{(\pm)}(\mathbf{k}')] = 0. \quad (3)$$

(Note that the introduction of several kinds of particles will only complicate the problem, but does not affect the principle of the method.)

The wave vector satisfying Eq. (2) for a system of n particles has the form

$$\exp(-iE_n{}^i t) \Psi^{(+)}_{E_n^i}|0\rangle = \exp(-iE_n{}^i t) \int d\mathbf{k}_1, \ldots d\mathbf{k}_n$$
$$\times \chi_{E_n^i}(\mathbf{k}_1 \ldots \mathbf{k}_n) a^{(+)}(\mathbf{k}_1) \ldots a^{(+)}(\mathbf{k}_n)|0\rangle, \quad (4)$$

where i is the label of the energy state and n denotes the number of particles. The coefficient function $\chi_{E_n^i}$ satisfies the equation

$$(E_n{}^i - p_1{}^0 - \ldots - p_n{}^0) \sum \chi_{E_n^i}(\mathbf{p}_1 \ldots \mathbf{p}_n)$$
$$= \sum{}' \int d\mathbf{k}_1\, d\mathbf{k}_1'' \delta(\mathbf{p}_{s_{n-1}} + \mathbf{p}_{s_n} - \mathbf{k}_1' - \mathbf{k}_1'')$$
$$\times u(|\mathbf{p}_{s_{n-1}} - \mathbf{p}_{s_n} - \mathbf{k}_1' + \mathbf{k}_1''|) \chi_{E_n^i}(\mathbf{p}_1 \ldots \mathbf{p}_{n-2}, \mathbf{k}_1', \mathbf{k}_1'') \quad (5)$$

(the sum $\Sigma$ runs over all permutations of $p_1 \ldots p_n$ in $\chi_{E_n^i}$ and the sum $\Sigma'$ runs over all permutations of $p_1 \ldots p_n$, $k_1'$, $k_1''$, for which both $k_1'$ and $k_1''$ are arguments of $\chi_{E_n^i}$).

In addition we note that the quantity $\Psi^{(+)}_{E_n^i}$ and its adjoint $\Psi^{(-)}_{E_n^i}$ satisfy the orthogonality relations

$$\langle 0|[\Psi^{(-)}_{E^i}, \Psi^{(+)}_{E^j}]|0\rangle = \delta_{nm}\delta(\mathbf{K}^i - \mathbf{K}^j)\delta_{\omega_i\omega_j}. \quad (6)$$

Here $\mathbf{K}^i$ denotes the center of mass momentum of the system and $\omega_i$ are all other quantum numbers describing the state. Equation (6) is a consequence of the self-adjointness of the operator H. For equal particle numbers (6) implies

$$[\Psi^{(-)}_{E_n^i}, \Psi^{(+)}_{E_n^j}]_-|0\rangle = \delta(\mathbf{K}^i - \mathbf{K}^j)\delta_{\omega^i\omega^j}|0\rangle. \quad (7)$$

Using (7) it is easy to show that $\chi_{E_n^i}(\mathbf{k}_1 \ldots \mathbf{k}_n)$ is a symmetric function of its arguments and satisfies the relations

$$\sum_{E_n k} \chi^*_{E_n^k}(\mathbf{p}_1 \ldots \mathbf{p}_n)\chi_{E_n^k}(\mathbf{k}_1 \ldots \mathbf{k}_n)$$
$$= \frac{1}{(n!)^2}\sum \delta(\mathbf{k}_1 - \mathbf{p}_1)\ldots\delta(\mathbf{k}_n - \mathbf{p}_n) \qquad (8)$$

where the sum in the right hand side is over all permutations of $\mathbf{p}_1 \ldots \mathbf{p}_n$.

In the interaction (Dirac) picture (1), (2) and (4) will be replaced by

$$i\frac{\partial \Psi^{(+)}_{\text{int}}(t)}{\partial t}|0\rangle = H_{\text{int}}\Psi^{(+)}_{\text{int}}(t)|0\rangle, \qquad (9)$$

where the Hamiltonian is

$$H_{\text{int}} = \int d\mathbf{k}_1' \, d\mathbf{k}_1'' \, d\mathbf{k}' \, d\mathbf{k}'' \, a^{(+)}(\mathbf{k}') a^{(+)}(\mathbf{k}'')$$
$$\times \delta(\mathbf{k}' + \mathbf{k}'' - \mathbf{k}_1' - \mathbf{k}_1'') \, u[|\mathbf{k}' - \mathbf{k}'' - \mathbf{k}_1' + \mathbf{k}_1''|]$$
$$\times \exp[i(k_{01}' + k_{01}'' - k_0' - k_0'')] \, a^{(-)}(\mathbf{k}_1') a^{(-)}(\mathbf{k}_1''),$$

and the wave function

$$\Psi_{\text{int}}^{(+)}(t)|0\rangle = e^{iH_0 t} e^{-iE_n^i t}\Psi^{(+)}_{E_n^i}|0\rangle$$
$$= \exp(-iE_n{}^i t)\Psi^{(+)}_{E_n^i}(t)|0\rangle = \exp(-iE_n{}^i t)$$
$$\times \int d\mathbf{k}_1 \ldots d\mathbf{k}_n \chi_{E_n^i}(\mathbf{k}_1 \ldots \mathbf{k}_n) \exp[i(k_1^0 + \ldots + k_n^0)t]$$
$$\times a^{(+)}(\mathbf{k}_1)\ldots a^{(+)}(\mathbf{k}_n)|0\rangle. \qquad (10)$$

From (7) we derive the relations for $\Psi^{(-)}(t)$, $\Psi^{(+)}(t)$:

$$\Psi^{(-)}_{E_n^i}(t)\Psi^{(+)}_{E_n^j}(t)|0\rangle = \delta(\mathbf{K}^i - \mathbf{K}^j)\delta_{\omega^i\omega^j}|0\rangle. \qquad (11)$$

We now construct the matrix $S(t, t_0)$. This is the operator which transforms the wave vector at time $t_0$ into the wave vector at time t and can be represented formally as:

$$S(t, t_0) = e^{iH_0 t}e^{-iH(t-t_0)}e^{-iH_0 t_0}. \qquad (12)$$

Another definition of this operator is through the chain of expressions

$$S(t - t_0)|0\rangle = |0\rangle,$$
$$S(t - t_0)e^{-iE_2^k t_0}\Psi^{(+)}_{E_2^k}(t_0)|0\rangle = e^{-iE_2^k t}\Psi^{(+)}_{E_2^k}(t)|0\rangle,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S(t - t_0)e^{-iE_n^k t_0}\Psi^{(+)}_{E_n^k}(t_0)|0\rangle = e^{-iE_n^k t}\Psi^{(+)}_{E_n^k}(t)|0\rangle, \qquad (13)$$
$$\cdots\cdots\cdots$$

Multiplying these equations respectively by

$$\langle 0|, \langle 0|a^{(+)}(\mathbf{p}_1)a^{(+)}(\mathbf{p}_2), \ldots, \langle 0|a^{(+)}(\mathbf{p}_1)\ldots a^{(+)}(\mathbf{p}_n),$$

we obtain equations for the coefficient functions of the S-operator such that the first equation contains only $S^{00}$, the second equation contains only $S^{00}$ and $S^{22}$, the third involves $S^{00}$, $S^{22}$ and $S^{33}\ldots$ and the n-th equation involves $S^{00}$, $S^{22}$, $S^{33}$, ..., $S^{nn}$ [1].

The first equation has the solution $S^{00} = 1$. In order to solve the second equation we substitute $S^{00}$ into it and solve for $S^{22}$, then we multiply the expression so obtained by $\chi_{E_2^k}(\mathbf{s}_1, \mathbf{s}_2)$ and sum over $E_2^i$. As a result we obtain

$$\sum_{E_2^i}\chi^*_{E_2^i}(\mathbf{s}_1, \mathbf{s}_2)e^{-iE_2^i(t-t_0)}\chi_{E_2^i}(\mathbf{p}_1, \mathbf{p}_2)$$
$$= \sum_{E_2^i}\chi^*_{E_2^j}(\mathbf{s}_1, \mathbf{s}_2)\chi_{E_2^i}(\mathbf{p}_1, \mathbf{p}_2)\exp[i(p_1^0 + p_2^0)(t - t_0)]$$
$$+ \int d\mathbf{k}_1 d\mathbf{k}_2 \sum S^{22}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{k}_1, \mathbf{k}_2, t, t_0)$$
$$\times \exp\{i(k_1^0 + k_2^0)t_0 - i(p_1^0 + p_2^0)t\}$$
$$\times \sum_{E_2^i}\chi^*_{E_2^i}(\mathbf{s}_1\mathbf{s}_2)\chi_{E_2^i}(\mathbf{k}_1\mathbf{k}_2).$$

(The first sum in the second term on the right is over the permutations of $\mathbf{p}_1$ and $\mathbf{p}_2$.)

If we make use of (8) we obtain

$$\sum S^{22}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{s}_1, \mathbf{s}_2, t, t_0) = \sum_{E_2^i}\chi^*_{E_2^i}(\mathbf{s}_1, \mathbf{s}_2)\exp\{-i(s_1^0 + s_2^0)t_0$$
$$- iE_2^i(t - t_0)\}\exp\{i(p_1^0 + p_2^0)t\}\chi_{E_2^i}(\mathbf{p}_1, \mathbf{p}_2)$$
$$- \frac{1}{2!}\sum\delta(\mathbf{p}_1 - \mathbf{s}_1)\delta(\mathbf{p}_2 - \mathbf{s}_2)$$

where the summation is over all permutations of $\mathbf{p}_1$ and $\mathbf{p}_2$, $\mathbf{s}_1$ and $\mathbf{s}_2$. This yields for $S^{22}$ the expression

$$S^{22} = \sum_{E_2^i}\chi^*_{E_2^i}(\mathbf{s}_1, \mathbf{s}_2)\exp[-i(s_1^0 + s_2^0)t_0 - iE_2^i(t - t_0)$$
$$+ i(p_1^0 + p_2^0)t]\chi_{E_2^i}(\mathbf{p}_1, \mathbf{p}_2) - \frac{1}{2!}\delta(\mathbf{p}_1 - \mathbf{s}_1)\delta(\mathbf{p}_2 - \mathbf{s}_2). \qquad (14)$$

The other coefficient functions are obtained in a similar manner, i.e., the equation for $S^{nn}$ is obtained by substituting into (13) the expressions for $S^{00}$, $S^{22}$, ..., $S^{n-1,n-1}$, multiplying by $\chi^*_{E_n^i}$, summing over $E_n^i$ and making use of the orthogonality relation (8).

We will not carry out the involved computation here, but write down the final expression for $S^{nn}$:

$$S^{nn}(\mathbf{k}_1 \ldots \mathbf{k}_n, \mathbf{k}_1' \ldots \mathbf{k}_n', t, t_0)$$
$$= \sum_{E_n^i} q_{E_n^i}(\mathbf{k}_1 \ldots \mathbf{k}_n, \mathbf{k}_1' \ldots \mathbf{k}_n', t, t_0)$$

[1]The superscripts in $S^{ii}$ denote that the corresponding quantity is the coefficient function in the S-matrix of the part which has the operator structure

$$\underbrace{a^{(+)}\ldots a^{(+)}}_{i}\;\underbrace{a^{(-)}\ldots a^{(-)}}_{i}.$$

$$+ a_n^{n-1} \sum_{E^i_{n-1}} \delta(\mathbf{k}_n - \mathbf{k}_{n-1})$$

$$\times q_{E^i_{n-1}}(\mathbf{k}_1 \ldots \mathbf{k}_{n-1}, \mathbf{k}_1' \ldots \mathbf{k}'_{n-1}, t, t_0)$$

$$+ a_2^{n-2} \sum_{E^i_{n-2}} \delta(\mathbf{k}_n - \mathbf{k}_n')\,\delta(\mathbf{k}'_{n-1} - \mathbf{k}_{n-1})$$

$$\times q_{E^i_{n-2}}(\mathbf{k}_1 \ldots \mathbf{k}'_{n-2}, \mathbf{k}_{1'} \ldots$$

$$\ldots \mathbf{k}'_{n-2}, t, t_0) + \ldots + a_n^2 \sum_{E_2^i} \delta(\mathbf{k}_n' - \mathbf{k}_n) \ldots \delta(\mathbf{k}_3 - \mathbf{k}_3')$$

$$\times q_{E_2^i}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_1', \mathbf{k}_2', t - t_0) + a_n^0 \delta(\mathbf{k}_1 - \mathbf{k}_1')\,\delta(\mathbf{k}_n - \mathbf{k}_n'). \quad (15)$$

Here

$$q_{E_l^i}(\mathbf{k}_1 \ldots \mathbf{k}_l, \mathbf{k}_1' \ldots \mathbf{k}_l', t, t_0) = \chi^{*}_{E_l^i}(\mathbf{k}_1 \ldots \mathbf{k}_l)$$

$$\times \exp\{-i(k_1^0 + \ldots + k_l^0)t_0\}$$

$$\times \exp\{-iE_l^i(t - t_0)\} \exp\{i(k_1^{0'} + \ldots + k_n^{0'})t\}$$

$$\times \chi_{E_l^i}(\mathbf{k}_1' - \mathbf{k}_l').$$

The $a_j^i$ satisfy the following recursion relations:

$$a_n^0 = -\frac{a_{n-1}^0}{1!} - \frac{a_{n-2}^0}{2!} - \ldots - \frac{a_2^0}{(n-2)!} - \frac{1}{n!},$$

$$a_2^0 = -\frac{1}{2}, \qquad a_n^k = -\frac{a_{n-1}^k}{1!} - \frac{a_{n-2}^k}{2!} - \ldots - \frac{a_k^k}{k!},$$

$$k \geqslant 2, \quad a_k^k = 1.$$

If we consider the n-th term of the S-matrix ($S^{11} = 0$ and we do not count it) it is easy to observe that the part corresponding to

$$\sum_{E_n^i} q_{E_n^i}(\mathbf{k}_1 \ldots \mathbf{k}_n, \mathbf{k}_1' \ldots \mathbf{k}_n', t, t_0)$$

determines completely the behavior of an n-particle system. In effect, making use of (11) we obtain

$$S^n(t - t_0)\, e^{-iE_n^s t_0}\, \Psi^{(+)}_{E_n^s}(t_0)\,|0\rangle = \sum_{E_n^i} \Psi^{(+)}_{E_n^i}(t)\, e^{-iE_n^i(t-t_0)}$$

$$\times \Psi^{(-)}_{E_n^i}(t_0)\, e^{-iE_n^s t_0}\, \Psi^{(+)}_{E_n^s}(t_0)\,|0\rangle = e^{-iE_n^s t}\, \Psi^{(+)}_{E_n^s}(t)\,|0\rangle.$$

The other elements of $S^{nn}$ are produced by the unitarity requirement for the S-matrix and cancel out in a trivial manner when $S(t, t_0)$ acts upon $\Psi_{int}(t_0)$. Similar non-square-integrable terms have also appeared in other investigations which however were carried out by means of the resolvent method (cf. e.g. [4]).

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[1] H. Ekstein, Phys. Rev. **101**, 880 (1956).
[2] F. J. Belinfante and C. Møller, Dan. Mat. Fys. Medd. **28**, Nr. 6 (1954).
[3] V. V. Alekseev and V. I. Grigor'ev, Vestnik MGU, Ser. Fiziki, Astr. No. 1, 42 (1965).
[4] L. D. Faddeev, JETP **39**, 1459 (1960), Soviet Phys. JETP **12**, 1014 (1961).

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