QUASILINEAR RELAXATION OF LONGITUDINAL OSCILLATIONS OF A PLASMA

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It is shown on the basis of the equations of quasilinear theory that when $t \to \infty$ a "plateau" appears in the distribution function of resonant particles under the action of three-dimensional longitudinal oscillations. The relaxation of the distribution function of electrons moving with respect to ions under the action of unstable ion-sound oscillations is considered. It is shown that if the electron velocity is close to the critical value above which the oscillations are unstable, then the resonant electrons are slowed down during the first stage of relaxation to a velocity of the order of that of sound. Subsequently a "plateau" is formed along the direction of the beam and the oscillation spectrum becomes one-dimensional. Quasilinear relaxation under the action of one-dimensional wave packets in a magnetic field is considered in detail.

In many papers (see the reviews [1,2]), the authors have investigated the relaxation of the "background" velocity distribution function of plasma particles on the basis of the quasilinear theory for cases when the oscillation spectrum represents a one-dimensional wave packet. As shown in these papers, the reaction of the plasma oscillations on the particles leads to establishment of a "plateau" on the distribution function. We consider in the present paper, on the basis of the quasilinear theory, the change in the "background" distribution function for the case of a non-one-dimensional oscillation spectrum. We shall show that as $t \to \infty$ the system goes over into the stationary state. If the velocity-space volume Ω_v occupied by the resonant particles at the instant $t = 0$ is infinite and the oscillation energy differs from zero when $t \to \infty$, then the oscillation spectrum in the final state becomes one-dimensional, and a one-dimensional 'plateau' appears on the distribution function. On the other hand, if the volume Ω_v is finite at $t = \infty$, then a three-dimensional "plateau" is produced in the final state.

In the case of unstable ion-sound oscillations excited by electrons moving relative to the ions with a velocity u which is close to the critical velocity u^* above which the ion-sound oscillations are unstable, the relaxation process proceeds as follows: during the initial stage (within a time $\sim 1/\gamma$), the oscillation spectrum becomes almost one-dimensional, since the oscillations build up most strongly along the beam; this is accompanied by a decrease in the translational velocity of the resonant frequencies to a certain value $v_m < u^*$. As a result of the deceleration of resonant particles, the oscillations which propagate at an angle ϑ to a direction which is not very close to zero, begin to attenuate, and then a "plateau" is formed on the electron distribution function along the beam direction.

We consider also the change in the distribution function, caused by longitudinal oscillations in the magnetic field. We show that relaxation of the distribution function takes place when $t \to \infty$. On the basis of the equations derived we have investigated the deformation of the distribution function in the case of narrow one dimensional wave packets.

1. RELAXATION OF LONGITUDINAL OSCILLATIONS IN THE ABSENCE OF A MAGNETIC FIELD

The fundamental equations of the quasilinear theory are of the form [1]

$$\frac{\partial f_0^\alpha}{\partial t} = \pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \frac{\partial}{\partial \mathbf{v}} \sum_k \mathbf{k}\frac{\varepsilon_\mathbf{k}}{k^2}\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)\delta(\omega - \mathbf{kv}), \quad (1)$$

$$\frac{\partial \varepsilon_\mathbf{k}}{\partial t} = 2\gamma_\mathbf{k}\varepsilon_\mathbf{k}, \quad (2)$$

$$\gamma_\mathbf{k} = \pi\omega \sum_{\alpha=e,\,i} \Omega_\alpha^2 \int \left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)\delta(\omega - \mathbf{kv})\,d\mathbf{v} \Big/ \Big[k^2 - \sum_{\alpha=e,\,i}\Omega_\alpha^2 \times \int \frac{\mathbf{kv}}{(\omega - \mathbf{kv})^2}\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)d\mathbf{v}\Big], \quad (3)$$

where $f_0^\alpha = f_0^\alpha(\mathbf{v}, t)$ is the average distribution function of resonant particles, $\epsilon_k = k^2|\varphi_k|^2$ is the spectral energy density of the oscillations with frequency $\omega = \omega(\mathbf{k})$, $\Omega_a = (4\pi e_\alpha^2 n_{0\alpha}/m_\alpha)^{1/2}$ is the Langmuir frequency of particles with charge e_α, mass m_α, and equilibrium density $n_{0\alpha}$. The integration in the denominator of (3) is in the sense of the principal value. The form of the region occupied by the resonant particles can be determined from the equation $\omega(\mathbf{k}) = \mathbf{k}\cdot\mathbf{v}$, by assigning to the vector $\mathbf{k}$ all possible values for which $\epsilon_k(t)$ exceeds appreciably the thermal-noise level.

The distribution function of the nonresonant particles changes much more slowly than the distribution function of the resonant particles. Therefore Eq. (1) can be used also for nonresonant particles, if we put for them $\partial f_0^\alpha/\partial t = 0$.

From (1) we obtain the following conservation law for the number of resonant particles

$$\frac{\partial}{\partial t}\int f_0^\alpha\, d\mathbf{v} = 0.$$

Let us examine the change in the entropy of the resonant particles. To this end we multiply (1) by $1 + \ln f_0^\alpha$ and integrate over the volume Ω_v. We then find that

$$\frac{\partial S^\alpha}{\partial t} \equiv -\frac{\partial}{\partial t}\int f_0^\alpha \ln f_0^\alpha\, d\mathbf{v} = \pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \int d\mathbf{v} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2 f_0^\alpha}\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)^2 \delta(\omega - \mathbf{k}\mathbf{v}) \geqslant 0,$$

that is, the entropy S^α of the resonant particles increases. It is therefore natural to expect relaxation of the distribution function. Multiplying (1) by f_0^α and integrating with respect to Ω_v, we obtain

$$\frac{\partial}{\partial t}\int (f_0^\alpha)^2\, d\mathbf{v} = -2\pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \int d\mathbf{v} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2}\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)^2 \times \delta(\omega - \mathbf{k}\mathbf{v}) \leqslant 0. \quad (4)$$

It follows therefore that the left side of (4) tends to zero when $t \to \infty$, that is, the right side of (4) is also equal to zero when $t \to \infty$.

Since the quantities under the integral sign in (4) are positive, it is necessary for this purpose either to have $\epsilon_k = 0$, or

$$\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)_{\omega = \mathbf{v}\mathbf{k}} = 0 \quad (5)$$

for all $\mathbf{k}$ for which $\epsilon_k \neq 0$. Since

$$\varepsilon_{\mathbf{k}}\left(\mathbf{k}\frac{\partial f_0^\alpha}{\partial \mathbf{v}}\right)_{\omega = \mathbf{k}\mathbf{v}} = 0,$$

the state of the system becomes stationary when $t \to \infty$:

$$\partial f_0^\alpha / \partial t = 0, \qquad \partial \varepsilon_{\mathbf{k}} / \partial t = 0.$$

It follows from (5) that f_0^α is constant if $\epsilon_k = 0$ for the values of $\mathbf{v}$ determined from the equation $\omega(\mathbf{k}) = \mathbf{k}\cdot\mathbf{v}$, that is, a plateau is formed in the resonant region in the stationary state.

We shall now show that formation of a three-dimensional "plateau" is impossible if the volume Ω_v occupied by the resonant particles in the final state is infinite (this takes place for both Langmuir and ion-sound oscillations). If a three dimensional "plateau" were to be formed, then it would follow from the condition for the conservation of the number of resonant particles that $f_0^\alpha = 0$ in the final state. Since the number of the resonant particles is conserved, this means the particles have acquired infinite energy. This contradicts the energy conservation law. It follows therefore that in the final state the spectrum of the oscillations becomes one-dimensional, and a "plateau" is formed on the distribution function along the direction of oscillations when $t = \infty$.

The foregoing conclusions, naturally, are valid only if the time of nonlinear interaction of the waves is much longer than the time of quasilinear relaxation. An estimate shows [3] that for Langmuir oscillations excited by a low-density electron beam, the nonlinear interaction of the wave can actually be neglected; for ion-sound oscillations the nonlinear interaction of the waves is also insignificant (see below).

2. RELAXATION OF UNSTABLE ION-SOUND OSCILLATIONS

If the plasma electrons drift with velocity u relative to the ions, then in a strongly non-isothermal plasma ($T_e \gg T_i$) there are excited ion-sound oscillations with frequency and increment given by the following expressions (see [4]):

$$\omega(\mathbf{k}) = kv_s / (1 + a_e^2 k^2)^{1/2}, \quad (6)$$

$$\gamma_{\mathbf{k}} = \frac{\pi\omega v_e^2}{2(1 + a_e^2 k^2)}\int\left(\mathbf{k}\frac{\partial f_0}{\partial \mathbf{v}}\right)\delta(\omega - \mathbf{k}\mathbf{v})\, d\mathbf{v}; \quad (7)$$

$$a_e = (T_e / 4\pi e^2 n_0)^{1/2}, \quad v_e = (T_e / m_e)^{1/2}, \quad v_s = (T_e / m_i)^{1/2}.$$

If the plasma electrons have at the initial instant a Maxwellian velocity distribution shifted by the drift velocity u, then

$$\gamma_{\mathbf{k}} = \left(\frac{\pi m_e}{8 m_i}\right)^{1/2} \frac{kv_s}{(1 + a_e^2 k^2)^2}\left(\frac{ku}{\omega}\cos\vartheta - 1\right). \quad (8)$$

We shall show that the volume is infinite for sound oscillations $\omega \approx kv_s$ ($a_e k \ll 1$). From the condition $\omega = \mathbf{k}\cdot\mathbf{v}$ it follows that the limits of the region Ω_v are determined by the inequalities $v > v_s$ and

$$\frac{v_s}{v}\cos\vartheta_0 - \sin\vartheta_0\left(1-\frac{v_s^2}{v^2}\right)^{1/2} < \cos\theta < \frac{v_s}{v}\cos\vartheta_0 + \sin\vartheta_0\left(1-\frac{v_s^2}{v^2}\right)^{1/2},$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{u}$, and $\epsilon_k \neq 0$ when $\theta < \vartheta_0 < \pi/2$. From this we find that

$$\Omega_v = 2\pi\int_{v_s}^{\infty} v^2 dv \int \sin\theta d\theta = 4\pi\sin\vartheta_0 \int_{v_s}^{\infty} v\,(v^2 - v_s^2)^{1/2}\,dv = \infty.$$

The volume Ω_v is finite only for one-dimensional oscillations ($\vartheta_0 = 0$). It is easy to verify also that $\Omega_v = \infty$ also for oscillations with $a_e k \gtrsim 1$.

Since the perturbation increases rapidly at first:

$$\varepsilon_k(t) = \varepsilon_k(0)\exp\left(2\int_0^t \gamma_k\,dt\right),$$

the oscillations that grow fastest when $\gamma_k \gg 1$ are those propagating along the stream (it is assumed that $\epsilon_k(0)$ is not close to zero when $\vartheta \approx 0$), with wave vectors $k \approx k_m$. The quantity k_m is determined from the condition that when $\vartheta = 0$ and $k = k_m(t)$ we get $\partial\epsilon_k(t)/\partial k = 0$; if the initial perturbation of $\epsilon_k(0)$ does not have sharp maxima, then k_m can be determined from the condition

$$\frac{\partial}{\partial k}\int_0^t \gamma_k(t)\,dt = 0;$$

at the initial instant of time $k_m \sim 1/a_e$. For $\vartheta \approx 0$ and $k \approx k_m$ we obtain

$$\varepsilon_k(t) = \varepsilon_k(0)\exp\left[A(t) - B(t)\vartheta^2\right],$$

$$A(t) = \left(\int_0^t \gamma_k(t)\,dt\right)_{\vartheta=0}, \qquad B(t) = \frac{1}{2}\left(\frac{\partial^2}{\partial\vartheta^2}\int_0^t \gamma_k(t)\,dt\right)_{\vartheta=0}. \tag{9}$$

In the right side of (4) each term tends to zero when $t \to \infty$. Let us separate the largest terms with $\vartheta \approx 0$ and $k \approx k_m$. Putting

$$\eta = v^2 = v_\perp^2 + v_\parallel^2, \qquad \zeta = v_\parallel u = vu\cos\theta,$$

we find, if the initial perturbation is axially symmetrical, that

$$\sum_{\mathbf{k}} \frac{\varepsilon_k}{k^2}\left(\mathbf{k}\frac{\partial f_0}{\partial\mathbf{v}}\right)^2\delta(\omega - \mathbf{kv}) = \sum_{\mathbf{k}}\frac{\varepsilon_k}{k^2}\left(2\omega\frac{\partial f_0}{\partial\eta} + ku\frac{\partial f_0}{\partial\zeta}\right)^2\delta(\omega-\mathbf{kv}) \approx \left(2v_m\frac{\partial f_0}{\partial\eta} + u\frac{\partial f_0}{\partial\zeta}\right)^2\sum_{\mathbf{k}}\varepsilon_k\delta(\omega-\mathbf{kv}) \to 0,$$

where $v_m = v_s/(1 + a_e^2 k_m^2)^{1/2}$. From this we find that when $t \to \infty$

$$2v_m\frac{\partial f_0}{\partial\eta} + u\frac{\partial f_0}{\partial\zeta} = 0,$$

that is

$$f_0 = f_0[v_\perp^2 + (v_\parallel - v_m)^2].$$

Thus, the translational velocity of the resonant particles $\bar{v}_\parallel$ changes from a value u to v_m, that is, the main result is not the formation of a "plateau," but a change in the translational velocity of the resonant particles.

If at the initial instant the beam velocity u differs insignificantly from the critical velocity u^* at which ion-sound instability begins, then oscillations $\vartheta \approx 0$ are excited, and when $t \to \infty$ the spectrum of the oscillations becomes one-dimensional and deceleration of the beam takes place. The oscillations that were growing at small values of t become damped as a result of the deceleration of the resonant particles. On the other hand, if the beam velocity is sufficiently large at $t = 0$, then the picture presented above for the relaxation is valid only during the initial stage, so long as the buildup of oscillations with $\vartheta \sim 1$ is insignificant.

Let us examine in greater detail the behavior of the function f_0 when $\gamma t \gg 1$. Recognizing that the main contribution to the sum over $\mathbf{k}$ in the right side of (1) is made by the values $\vartheta \approx 0$ and $k \approx k_m$, we obtain

$$\frac{\partial f_0}{\partial t} = \frac{1}{8\pi^2}\left(\frac{e}{m_e}\right)^2\left(2v_m\frac{\partial}{\partial\eta} + u\frac{\partial}{\partial\zeta}\right) \times\left[\left(2v_m\frac{\partial f_0}{\partial\eta} + u\frac{\partial f_0}{\partial\zeta}\right)\int d\mathbf{k}\,\varepsilon_k\delta(\omega - \mathbf{kv})\right]. \tag{10}$$

In the case of a fast beam ($v_e \gg u \gg v_m \sim v_s$) at the initial stage of quasilinear relaxation expression (10) simplifies to

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial\zeta}\left[D(t,\eta,\zeta)\frac{\partial f_0}{\partial\zeta}\right] \tag{11}$$

$$D(t,\eta,\zeta) = \frac{u^2}{8\pi^2}\left(\frac{e}{m_e}\right)^2\int d\mathbf{k}\,\varepsilon_k(t)\,\delta(\omega-\mathbf{kv}) \approx \frac{u^2}{8\pi^2}\left(\frac{e}{m_e}\right)^2\frac{1}{v_\perp}\left\{k\varepsilon_k(0)\exp\left[A - B\left(\frac{v_\parallel - v_m}{v_\perp}\right)^2\right] \times\left[\left|\frac{B}{4}\frac{d^2}{dk^2}\ln(\varepsilon_k(0)e^A)\right|\right]^{1/2}\right\}_{k=k_m}. \tag{12}$$

From the diffusion equation (11) it follows that the function f_0 ceases to depend on the angle θ with increasing t, that is, $f_0 \to f_0(v^2)$. Thus, the translational velocity of the resonant particles decreases to values of the order of v_s, in agreement with the analysis presented above.

Let us consider now the relaxation of the electron distribution function under the influence of long-wave sound oscillations $\omega \approx kv_s$. This case is realized, for example, if at the initial instant the oscillation spectrum $\epsilon_k(0) \neq 0$ only for $a_e k \ll 1$, and also if T_e/T_i is not too large (in the latter case oscillations with $ka_e \sim 1$ have $V_{ph} \sim v_i(T_i/m_i)^{1/2}$ and attenuate strongly). In the case in question the quantity $v_m = v_s$ does not depend on the time.

Introducing new variables

$$\xi_{1,2} = \eta / 2v_s \pm \zeta / u,$$

we represent (10) in the form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \xi_1}\left[D(t, \xi_1, \xi_2)\frac{\partial f_0}{\partial \xi_1}\right], \tag{13}$$

where $D(t, \xi_1, \xi_2)$ differs from (12) by a factor $u^2/4$. The diffusion coefficient differs noticeably from zero in the region $B(v_\parallel - v_s)^2/v_\perp^2 \lesssim 1$, that is, when

$$\xi_2 + 2v_s - \delta(8v_s\xi_2/B)^{1/2} < \xi_1 < \xi_2 + 2v_s + \delta(8v_s\xi_2/B)^{1/2},$$

where $\delta \sim 1$.

It follows from the diffusion equation (13) that with increasing time the distribution function approaches a stationary state, in which $\partial f_0/\partial\xi_1 = 0$, that is, $f_0 = f_\infty(\xi_2)$. Since the location of the boundaries of the "plateau" is unknown, we estimate the magnitude of the "plateau" $f_\infty(\xi_2)$ by assuming that $B(0) \sim B(\infty)$. It is then easy to find, in the usual manner [1], the height f_∞ of the "plateau" when $\xi_2 = \text{const}$, using the conservation of the number of resonant particles with given ξ_2:

$$f_\infty = \frac{1}{(2\pi)^{3/2} v_e^3}\exp\left[-\frac{v_\perp^2 + (v_\parallel - v_s)^2}{2v_e^2}\right] \times\left\{1 + O\left[\frac{v_\perp^2(u - v_s)^2}{Bv_e^4}\right]\right\}. \tag{14}$$

We see therefore that formation of a "plateau" with respect to the variable ξ_1 corresponds to deceleration of the resonant particles: their translational velocity changes from $\bar{v}_\parallel = u$ to $\bar{v}_\parallel = v_s$. (We note that no separation of oscillations with $k \sim k_m$ takes place with increasing t for the long-wave oscillations under consideration ($a_e k \ll 1$), unlike the case of the short-wave oscillations considered above ($a_e k \gtrsim 1$).)

Let us estimate now the energy of the oscillations. In order of magnitude, we obtain for the energy of the electric field

$$\varepsilon = \frac{1}{8\pi}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}},$$

and for the change in the kinetic energy of the resonant particles and of the adiabatic particles

$$\Delta\varepsilon_{res,\,ad} = {}^1/_2\, m_e n_0 \int v^2 (f_\infty - f_H)_{res,\,ad}\, d\mathbf{v},$$

the following expressions:

$$\varepsilon(t) = \varepsilon(0)e^{A(t)},$$

$$\Delta\varepsilon_{res} = -2\Delta\varepsilon_{ad} \sim -m_e n_0 v_s (u - v_s) B^{-3/2} \approx -\varepsilon(t)\,(1 + \bar{k}^2 a_e^2)\,/\,\bar{k}^2 a_e^2,$$

where $\bar{k}$ is the mean value of the wave vector in the wave packet under consideration. From this we get

$$\varepsilon(t) \sim \bar{k}^2 a_e^2 m_e n_0 v_s (u - v_s)\,(1 + \bar{k}^2 a_e^2)^{-1} \times [\ln(\varkappa \ln^{3/2}\varkappa)]^{-3/2}, \tag{15}$$

where $\kappa = m_e n_0 v_s(u - v_s)/\epsilon(0)$. If the initial energy $\epsilon(0)$ is the energy of the thermal noise, then $\epsilon(0) = n_0 T_e/N_D$, where $N_D = (4\pi/3)n_0 a_e^3$ is the number of particles in the Debye sphere.

To estimate the energy of the oscillations we can also use a relation that follows from (1) and (2):

$$\int v_\parallel (f_0 - f_H)\, d\mathbf{v} = -\left(\frac{e}{m}\right)^2 \frac{1}{v^2}\sum_{\mathbf{k}} \frac{(1 + k^2 a_e^2) k_\parallel \varepsilon_{\mathbf{k}}(t)}{\Omega k^2}. \tag{16}$$

The integration in (16) can be carried out only for resonant electrons, since the change in the current of the adiabatic electrons is m_i/m_e times smaller than the change in the current of the resonant electrons. Replacing f_0 in the left side of this equation by (14) and integrating with respect to v within the limits $0 \leq v_\parallel \leq v_e/B^{1/2}$, we obtain for ϵ the estimate

$$\varepsilon \sim n_0 m_e v_s (u - v_s)\,/\,B^{1/2},$$

which is in good agreement with (15), if we take into account the already mentioned uncertainty in the estimate of B. It follows from (16) that the development of ion-sound oscillations leads to a decrease in the electron current.

In the foregoing analysis we did not take into account the nonlinear interaction of the waves. It is easy to verify, using the kinetic equation for the waves [2], that the relaxation time of the intensity of the ion-sound oscillations, due to the nonlinear interaction of the waves, is at least T_e/T_i times larger than the time of quasilinear relaxation.

3. LONGITUDINAL OSCILLATIONS OF A PLASMA IN A MAGNETIC FIELD

The Cerenkov instability of long-wave longitudinal oscillations in a strong magnetic field was investigated on the basis of the quasilinear theory in [5], where it was shown that these oscillations lead to the formation of a "plateau" on the distribution function along the magnetic field. The propagation of damped electromagnetic waves was considered in the quasilinear approximation in [6] where it was assumed, however, that the "background" distribution function $f_0(v, t)$ breaks up into the product $f_0(v_\parallel, t)\, f_0(v_\perp)$, where $v_\parallel$ and $v_\perp$ are the plasma-particle velocity components parallel and perpendicular to the external magnetic field B_0, an assumption which does not hold true in general. The quasilinear interaction between almost monoenergetic beams and high-frequency longitudinal plasma oscillations in a magnetic field was investigated in [7]. It was shown in [8] that the distribution function in the presence of a "plateau" in a magnetic field can be unstable. In the present section we obtain quasilinear-theory equations for longitudinal plasma oscillations without the limitations imposed in [5-7].

We put $f^\alpha = f_0^\alpha + F^\alpha$, where $f_0^\alpha = \langle f^\alpha \rangle$, $\langle F^\alpha \rangle = 0$, and the averaging is over times which are much longer than the oscillation period, and over distances which are much longer than the wavelength. Expanding F^α and the potential φ in a Fourier series

$$F^\alpha = \sum_{\mathbf{k}} F_{\mathbf{k}}^\alpha(\mathbf{v}, t)\, e^{i(\mathbf{kr}-\omega t)}, \quad \varphi = \sum_{\mathbf{k}} \varphi_{\mathbf{k}}(t)\, e^{i(\mathbf{kr}-\omega t)},$$

we obtain from the kinetic equation and Maxwell's equations

$$\frac{\partial f_0^\alpha}{\partial t} - \omega_\alpha \frac{\partial f_0^\alpha}{\partial \Phi} = -i \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial \mathbf{v}} \sum_{\mathbf{k}} \mathbf{k} \varphi_{\mathbf{k}}^* F_{\mathbf{k}}^\alpha, \tag{17}$$

$$\frac{\partial F_{\mathbf{k}}^\alpha}{\partial \psi} + i \frac{\omega - \mathbf{kv}}{\omega_\alpha} F_{\mathbf{k}}^\alpha = -\frac{i e_\alpha \varphi_{\mathbf{k}}}{m_\alpha \omega_\alpha} \left(\mathbf{k} \frac{\partial f_0^\alpha}{\partial \mathbf{v}} \right), \tag{18}$$

$$\mathbf{v} = (v_\perp, \Phi, v_\parallel), \quad \mathbf{k} = (k_\perp, \varphi, k_\parallel), \quad \omega = \omega_{\mathbf{k}} + i\delta(\mathbf{k}, \mathbf{v}, t),$$

where $\omega_\alpha = e_\alpha B_0 / m_\alpha c$, $\psi = \Phi - \varphi$, Φ and φ are the polar angles in velocity space and in wave-vector space, and $\omega_{\mathbf{k}}$ are the frequencies of the longitudinal oscillations determined from the dispersion equation of the linear theory. It is assumed that $F_{\mathbf{k}}^\alpha$, $\varphi_{\mathbf{k}}$, and f_0^α change slowly in time, so that the following inequalities are satisfied

$$\delta = \frac{\partial}{\partial t} \ln F_{\mathbf{k}}^\alpha \ll \omega_{\mathbf{k}}, \quad \gamma_{\mathbf{k}} = \frac{\partial}{\partial t} \ln \varphi_{\mathbf{k}} \ll \omega_{\mathbf{k}},$$

$$\sigma = \frac{\partial}{\partial t} \ln \frac{\partial f_0}{\partial v_i} \ll \omega_{\mathbf{k}}.$$

Integrating (17), we obtain

$$F_{\mathbf{k}}^\alpha = -\frac{e_\alpha \varphi_{\mathbf{k}}}{m_\alpha \omega_\alpha} \exp(i\lambda \sin\psi) \sum_{n=-\infty} \frac{A_n e^{in\psi}}{\beta + n},$$

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} d\psi \left(\mathbf{k} \frac{\partial f_0^\alpha}{\partial \mathbf{v}} \right) \exp(in\psi - i\lambda \sin\psi),$$

$$\lambda = \frac{k_\perp v_\perp}{\omega_\alpha};$$

$$\beta = \beta_0 + i\beta_1 = \frac{\omega_{\mathbf{k}} - k_\parallel v_\parallel}{\omega_\alpha} + \frac{i}{2\pi\omega_\alpha} \int_0^{2\pi} \delta(\psi)\, d\psi. \tag{19}$$

Substituting (19) in (17) we obtain after some transformations

$$\frac{\partial f_0^\alpha}{\partial t} - \omega_\alpha \frac{\partial f_0^\alpha}{\partial \Phi} = Q^\alpha(\mathbf{v}, t),$$

$$Q^\alpha(\mathbf{v}, t) = \frac{1}{2\pi\omega_\alpha} \left(\frac{e_\alpha}{m_\alpha} \right)^2 \frac{\partial}{\partial \mathbf{v}} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2} \mathbf{k} \left\{ \frac{\beta_1}{\beta_1^2 + (\beta_0 + n)^2} \right.$$

$$\times \int_0^{2\pi} \left(\mathbf{k} \frac{\partial f_0^\alpha}{\partial \mathbf{v}} \right) \cos[n(\psi - \psi') + \lambda(\sin\psi - \sin\psi')]\, d\psi'$$

$$- \frac{\beta_0 + n}{\beta_1^2 + (\beta_0 + n)^2} \int_0^{2\pi} \left(\mathbf{k} \frac{\partial f_0^\alpha}{\partial \mathbf{v}} \right) \sin[n(\psi - \psi')$$

$$\left. + \lambda(\sin\psi - \sin\psi')]\, d\psi' \right\}. \tag{20}$$

Expanding f_0^α and Q^α in a Fourier series in Φ:

$$f_0^\alpha(v_\perp, \Phi, v_\parallel, t) = \sum_{n=-\infty}^{\infty} f^{(n)}(v_\perp, v_\parallel, t)\, e^{in\Phi},$$

$$Q^\alpha(v_\perp, \Phi, v_\parallel, t) = \sum_{n=-\infty}^{\infty} Q^{(n)}(v_\perp, v_\parallel, t)\, e^{in\Phi} \tag{21}$$

we obtain

$$\frac{\partial f^{(n)}}{\partial t} - in\, \omega_\alpha f^{(n)} = Q^{(n)}.$$

From this we get

$$f^{(0)}(v_\perp, v_\parallel, t) = f^{(0)}(v_\perp, v_\parallel, 0) + \int_0^t Q^{(0)}(v_\perp, v_\parallel, t)\, dt,$$

$$f^{(n)}(v_\perp, v_\parallel, t) = \left[f^{(n)}(v_\perp, v_\parallel, 0) - i \frac{Q^{(n)}(v_\perp, v_\parallel, 0)}{n\omega_\alpha} \right]$$

$$\times \exp(in\omega_\alpha t) + \frac{iQ^{(n)}(v_\parallel, v_\perp, t)}{n\omega_\alpha}.$$

Since the function f_0 is averaged over times Δt such that

$$\frac{1}{\omega} \ll \Delta t \ll \frac{1}{\delta}, \ \frac{1}{\nu}, \ \frac{1}{\sigma},$$

expression (21) should not contain rapidly oscillating terms $\sim \exp(in\omega_\alpha t)$ $(n \neq 0)$ when $1/\omega_\alpha \gtrsim \Delta t$, that is, the condition $f^{(n)}(0) = iQ^{(n)}(0)/n\omega_\alpha$ should be satisfied. It follows therefore that after a sufficiently long time interval $(\omega_\alpha t \gg 1)$ we can retain in the sum (21) only one term with $n = 0$, since

$$f^{(0)} \sim Q^{(0)} t \gg f^{(n)} \sim Q^{(n)}(t) / n\omega_\alpha.$$

Substituting $f_0^\alpha \approx f^{(0)}(v_\perp, v_\parallel, t)$ in (20) and averaging over Φ we obtain after some transformations

$$\frac{\partial f_0^\alpha}{\partial t} = \frac{1}{\omega_\alpha}\left(\frac{e_\alpha}{m_\alpha}\right)^2 \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2}\left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial}{\partial v_\perp}\right) \times \left[\frac{\beta_1 J_n^2(\lambda)}{\beta_1^2 + (\beta_0 - n)^2}\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)\right] \quad (22)$$

where $J_n(\gamma)$ is a Bessel function. Equation (22) coincides with the equation obtained in [7] under the assumption that $\epsilon_k = k^2|\varphi_k|^2$ does not depend on φ and the function $f_H = f_0|_{t=0}$ does not depend on Φ.

In the case of resonant particles, for which $|\beta_0 - n| \lesssim |\beta_1|$, Eq. (22) takes the form

$$\frac{\partial f_0^\alpha}{\partial t} = \pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \left\{\frac{\partial}{\partial v_\parallel} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} J_n^2(\lambda)\frac{\varepsilon_{\mathbf{k}}}{k^2} k_\parallel \times \left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)\delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha) + \frac{\omega_\alpha}{v_\perp}\frac{\partial}{\partial v_\perp}\sum_{n=-\infty}^{\infty} n \sum_{\mathbf{k}} J_n^2(\lambda)\frac{\varepsilon_{\mathbf{k}}}{k^2}\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right) \times \delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha)\right\}. \quad (23)$$

In the case of nonresonant particles (adiabatic particles), for which $|\beta_0 - n| \gg \beta_1$, the right side of (22) is small, so that we can put in it $\sigma = 0$, $\delta = \gamma$, and $f_0^\alpha = f_H^\alpha$. We then obtain from (22)

$$f_0^\alpha = f_H^\alpha + \frac{1}{2}\left(\frac{e_\alpha}{m_\alpha}\right)^2 \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \frac{1}{k^2}[\varepsilon_{\mathbf{k}}(t) - \varepsilon_{\mathbf{k}}(0)] \times \left(k_\parallel \frac{\partial}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial}{\partial v_\perp}\right) \times \left[\frac{J_n^2(\lambda)}{(\beta_0 - n)^2}\left(k_\parallel \frac{\partial f_H^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_H^\alpha}{\partial v_\perp}\right)\right]. \quad (24)$$

Substituting the expression (19) in the Poisson equation

$$k^2 \varphi_{\mathbf{k}} = 4\pi \sum_\alpha e_\alpha n_{0\alpha} \int F_{\mathbf{k}\alpha}\, d\mathbf{v},$$

we find that the frequency ω_k is determined from the dispersion equation of the linear theory, that is, does not change with time, while γ_k is equal to

$$\gamma_{\mathbf{k}} = -\pi \sum_\alpha \Omega_\alpha^2 \sum_{n=-\infty}^{\infty} \int d\mathbf{v}\, J_n^2(\lambda)\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right) \times \delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha)\left[\sum_\alpha \Omega_\alpha^2 \sum_{n=-\infty}^{\infty} \int d\mathbf{v}\, J_n^2(\lambda) \times \left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)\frac{1}{(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha)^2}\right]^{-1}. \quad (25)$$

The integration in the denominator of (25) is over the adiabatic region; we can also integrate over the entire velocity space, and in this case the integral with respect to $v_\parallel$ is taken in the sense of principal value.

Let us consider the change in the entropy of the resonant particles. To this end we multiply (23) by $1 + \ln f_0^\alpha$ and integrate over the volume Ω_v occupied by the resonant particles. We then obtain

$$\frac{\partial S^\alpha}{\partial t} \equiv -\frac{\partial}{\partial t}\int f_0^\alpha \ln f_0^\alpha\, d\mathbf{v} = \pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \sum_{n=-\infty}^{\infty} \int d\mathbf{v} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2} \times J_n^2(\lambda)\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)^2 \frac{1}{f_0^\alpha}\delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha) \geqslant 0,$$

that is, the entropy of the resonant particles increases.

4. RELAXATION OF OSCILLATIONS IN A MAGNETIC FIELD

Multiplying (23) by f_0^α and integrating over the volume Ω_v occupied by the resonant particles, we obtain

$$\frac{\partial}{\partial t}\int (f_0^\alpha)^2 d\mathbf{v} = -2\pi\left(\frac{e_\alpha}{m_\alpha}\right)^2 \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2} \times \int d\mathbf{v}\, \delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha) \times J_n^2(\lambda)\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)^2 \leqslant 0. \quad (26)$$

It follows therefore that in the final state either $\epsilon_k = 0$, or

$$\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)_{\omega_k = k_\parallel v_\parallel + n\omega_\alpha} = 0. \quad (27)$$

Since $k_\parallel$ in (27) is generally speaking arbitrary, the condition (27) can be satisfied only if

$$\partial f_0^\alpha / \partial v_\parallel = 0, \qquad \partial f_0^\alpha / \partial v_\perp = 0,$$

that is, when a "plateau" is formed in the velocity region Ω_v determined from the condition $\omega_k = k_\parallel v_\parallel + n\omega_\alpha$ for all $k_\parallel$ for which $\epsilon_k \neq 0$. This condition defines the resonant region $\Omega_v^{(n)}$ for a given value of n. If the regions $\Omega_v^{(n)}$ do not intersect (as is the case, for example, for sufficiently narrow wave packets), then a separate "plateau" is established in each of the regions.

Let us consider now the damping of narrow wave packets ($k_1 < k < k_2$, $k_2 - k_1 \ll k$) propagating at an angle ϑ to the magnetic field. The expressions for the frequencies of the longitudinal oscillations are given in [4] (as is well known, when $B_0^2 \gg 4\pi n_0(T_e + T_i)$ there exist in the plasma three branches of long-wave oscillations, all having along the magnetic field a phase velocity much larger than the thermal velocity of the plasma particles; when $T_e \gg T_i$ and ϑ is not close to $\pi/2$, there are two oscillation branches, corresponding to a splitting sound wave; in addition, when $\vartheta \approx \pi/2$ there is a large number of oscillation branches with frequencies close to ω_α [9,10]).

The resonant region $\Omega_v^{(n)}$ corresponds to particle velocities along the magnetic field in the interval $v_{min}(n) < v_\parallel < v_{max}(n)$, where

$$v_{min(max)}(n) = \min(\max)\left[\frac{\omega(k_{1,2}) - n\omega_\alpha}{k_{1,2}\cos\vartheta}\right]$$

It is obvious that the region $\Omega_v^{(n)}$ does not overlap the region $\Omega_v^{(m)}$ if $v_{max}(m) < v_{min}(n)$, or else if $v_{max}(n) < v_{min}(m)$.

In the case of non-overlapping region and a narrow wave packet, we can retain only one term in the sum over n in the right side (23), and we can put

$$\omega_{\mathbf{k}} = \omega_1 + v_g(k - k_1), \qquad \omega_1 = \omega_{k_1},$$
$$v_g = \partial\omega_{\mathbf{k}} / \partial k|_{k=k_1}.$$

Then (23) takes the form

$$\frac{\partial f_0^\alpha}{\partial t} = \frac{1}{2}\left(\frac{e_\alpha}{m_\alpha}\right)^2\left(\frac{\omega_1 - n\omega_\alpha - k_1 v_g}{v_\parallel - v_g/\cos\vartheta}\frac{\partial}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial}{\partial v_\perp}\right)$$
$$\times J_n^2(\lambda)\frac{\varepsilon_{\mathbf{k}}}{k^2|v_\parallel|}\left(\frac{\omega_1 - n\omega_\alpha - k_1 v_g}{v_\parallel - v_g/\cos\vartheta}\frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right). \tag{28}$$

When n = 0, equation (33) simplifies to

$$\frac{\partial f_0^\alpha}{\partial t} = \frac{\partial}{\partial v_\parallel}\left(D\frac{\partial f_0^\alpha}{\partial v_\parallel}\right), \quad D = \frac{1}{2}\left(\frac{e_\alpha}{m_\alpha}\right)^2\frac{\omega^2\varepsilon_{\mathbf{k}}J_n^2(\lambda)}{|v_\parallel|^3 k^2}. \tag{29}$$

It is easy to see that (29) has solutions of the form

$$f_0^\alpha(v_\perp, v_\parallel, t) = f_H^\alpha(v_\perp) f_0^\alpha[v_\parallel, J_0^2(\lambda)t],$$

if at the initial instant

$$f_0^\alpha(v_\perp, v_\parallel, 0) = f_H^\alpha(v_\perp) f_H^\alpha(v_\parallel).$$

Obviously, the function $f_0^\alpha(v_\perp, v_\parallel, t)$ splits into a product of the distribution functions with respect to $v_\perp$ and $v_\parallel$, for arbitrary t, only for long-wave oscillations ($\lambda \ll 1$). From the one-dimensional diffusion equation (29) follows the formation of a "plateau" in the resonance region if $\epsilon_k(\infty) \neq 0$.

When $n \neq 0$ it is convenient to introduce in place of $v_\perp$ and $v_\parallel$ the new variables

$$\xi_{1,2} = (v_\parallel - v_g/\cos\vartheta)^2 \pm (\omega_1 - n\omega_\alpha - k_1 v_g)v_\perp^2/n\omega_\alpha.$$

Then (28) takes the form

$$\frac{\partial f_0^\alpha}{\partial t} = \frac{\partial}{\partial\xi_1}\left[D(\xi_1, \xi_2, t)\frac{\partial f_0^\alpha}{\partial\xi_1}\right],$$
$$D = 8\left(\frac{e_\alpha}{m_\alpha}\right)^2\frac{(\omega_1 - n\omega_\alpha - k_1 v_g)^2 J_n^2(\lambda)\varepsilon_{\mathbf{k}}(t)}{k^2|v_\parallel|}. \tag{30}$$

It is easy to verify that in the case under consideration, that of narrow packets, the number of particles is conserved in the regions

$$v_{min}(n) < v_\parallel < v_{max}(n),$$
$$\eta_m(n)\omega_\alpha/k_\perp < v_\perp < \eta_{m+1}(n)\omega_\alpha/k_\perp,$$

where $\eta_m(n)$ is the root of the Bessel function $J_n(\eta_0 = 0)$.

The diffusion coefficient $D(\xi_1, \xi_2, t)$ vanishes for fixed ξ_2 at the points $\xi_1 = \alpha_m$, where

$$\{\alpha_m\} = \{\xi_2 + 2(\omega_1 - n\omega_\alpha - k_1 v_g)$$
$$\times\omega_\alpha^2\eta_{0,1,\ldots}/n\omega_\alpha k_\perp^2;\ 2v_{1,2}^2 - \xi_2\},$$
$$v_{1,2} \equiv (\omega_1 - n\omega_\alpha - k_1 v_g)/k_{2,1}\cos\vartheta,$$
$$\alpha_0 < \alpha_1 < \alpha_2 < \ldots .$$

We shall assume that the quantities $v_{1,2}$ do not depend on the time. It is obvious that the number of particles in the interval $\alpha_m < \xi_1 < \alpha_{m+1}$ for fixed ξ_2 is conserved:

$$\frac{\partial}{\partial t}\int_{\alpha_m}^{\alpha_{m+1}} f_0^\alpha\, d\xi_1 = D\frac{\partial f_0^\alpha}{\partial\xi_1}\Big|_{\xi_1=\alpha_m}^{\xi_1=\alpha_{m+1}} = 0.$$

In the final state in the interval $\alpha_m < \xi_1 < \alpha_{m+1}$ there is formed a "plateau" of height

$$f_\infty^\alpha = \frac{1}{\alpha_{m+1} - \alpha_m}\int_{\alpha_m}^{\alpha_{m+1}} f_H^\alpha\, d\xi_1. \tag{31}$$

If the initial distribution function is Maxwellian, then

$$f_H^\alpha = \frac{1}{(2\pi)^{3/2}v_e^3}\exp\left[-\frac{v_g^2}{2v_\alpha^2\cos^2\vartheta}\right.$$
$$\left.-\left(1 - \frac{n\omega_\alpha}{\omega_1 - n\omega_\alpha - k_1 v_g}\right)\frac{\xi_2}{4v_\alpha^2}\right]$$
$$\times\exp\left[-\left(1 + \frac{n\omega_\alpha}{\omega_1 - n\omega_\alpha - k_1 v_g}\right)\frac{\xi_1}{4v_\alpha^2} - \frac{\sqrt{\xi_1 + \xi_2}\,v_g}{\sqrt{2}\,v_\alpha^2\cos\vartheta}\right].$$

For oscillations with $v_g \approx 0$ (such oscillations are, for example, long-wave oscillations with "hybrid" frequencies [4]) we obtain from (31)

$$f_\infty^\alpha = \frac{4(\omega - n\omega_\alpha)}{(2\pi)^{3/2}v_\alpha\omega(\alpha_{m+1} - \alpha_m)}\left\{\exp\left[-\frac{\omega\alpha_m}{4v_\alpha^2(\omega - n\omega_\alpha)}\right]\right.$$
$$\left.-\exp\left[-\frac{\omega\alpha_{m+1}}{4v_\alpha^2(\omega - n\omega_\alpha)}\right]\right\}\exp\left[-\frac{(\omega - 2n\omega_\alpha)\xi_2}{4v_\alpha^2(\omega - n\omega_\alpha)}\right]. \tag{32}$$

For example, if

$$\alpha_m = \xi_2 + 2(\omega - n\omega_\alpha)\,\omega_\alpha \eta_\nu^2 / nk_\perp^2,$$
$$\alpha_{m+1} = \xi_2 + 2(\omega - n\omega_\alpha)\,\omega_\alpha \eta_{\nu+1}^2 / nk_\perp^2$$

(this case is realized when $|\vartheta - \pi/2| \ll kv_e/\omega_e$), then

$$f_\infty^\alpha = \frac{2nk_\perp^2}{(2\pi)^{3/2} v_\alpha \omega\omega_\alpha} \exp\left(-\frac{\omega\omega_\alpha \eta_\nu^2}{2nk_\perp^2 v_\alpha^2}\right)$$
$$\times \left\{1 - \exp\left[-\frac{\omega\omega_\alpha}{2nk_\perp^2 v_\alpha^2}(\eta_{\nu+1}^2 - \eta_\nu^2)\right]\right\}$$
$$\times \exp\left[-\frac{v_\parallel^2}{2v_\alpha^2} + \frac{(\omega - n\omega_\alpha) v_\perp^2}{2n\omega_\alpha v_\alpha^2}\right]. \qquad (33)$$

Let us consider in greater detail the relaxation of oscillations with $k_\perp v_\perp/\omega_\alpha \ll 1$ and $v_g = 0$ when $\omega \to \omega_\alpha$. Equation (28) then takes the form

$$\frac{\partial f_0^\alpha}{\partial \tau} = \frac{1}{v_\perp}\frac{\partial}{\partial v_\perp}\left(v_\perp \frac{\partial f_0^\alpha}{\partial v_\perp}\right),$$
$$\tau = \frac{1}{8}\left(\frac{e_\alpha}{m_\alpha}\right)^2 \frac{\sin^2\vartheta}{|v_\parallel|}\int_0^t \varepsilon_{\mathbf{k}}(t)\,dt. \qquad (34)$$

Putting

$$f_0^\alpha(v_\perp, v_\parallel, t) = f_H{}^\alpha(v_\parallel) f^\alpha(v_\perp, \tau),$$

we obtain from (34)

$$f^\alpha(v_\perp, \tau) = \frac{1}{2\pi(v_\alpha^2 + 4\tau)}\exp\left[-\frac{v_\perp^2}{2(v_\alpha^2 + 4\tau)}\right]. \qquad (35)$$

In the case of a narrow packet $v_2 - v_1 \ll v_\alpha$, the function $f_H^\alpha(v_\perp, \tau)$ is deformed little and the damping decrement γ is determined by the expression for $f_H^\alpha(v_\perp)$, so that

$$\tau = \frac{1}{16}\left(\frac{e_\alpha}{m_\alpha}\right)^2 \frac{\sin^2\vartheta\,\varepsilon_{\mathbf{k}}(0)}{|v_\parallel|\gamma}(1 - e^{-2\gamma t}).$$

We now consider the relaxation of unstable longitudinal oscillations in a magnetic field, in the case when the spectrum of the oscillations is in general not one-dimensional. If the oscillation energy ϵ_k in the final state differs from zero, then it follows from (28) that $f_\infty = \text{const}$. The region of $v_\parallel$ with specified n, in which a "plateau" is formed, can be determined from the equality $\omega_k = k_\parallel v_\parallel + n\omega_\alpha$, by assigning to the vector $\mathbf{k}$ all possible values for which $\epsilon_k(\infty) \neq 0$. The region of $v_\perp$ is limited by the inequalities $a < v_\perp < b$, where the limits a and b must be determined from the particle-number conservation condition

$$\frac{\partial}{\partial t}\int_a^b v_\perp\,dv_\perp \int f_0^\alpha\,dv_\parallel = 0.$$

If several regions $\Omega_V^{(n)}$ with different n overlap, then, integrating (23) over the aggregate of these regions between the limits $a < v_\perp < b$, we find that

$$\frac{\partial}{\partial t}\int f_0^\alpha\,d\mathbf{v} = 2\pi^2\left(\frac{e_\alpha}{m_\alpha}\right)^2\int dv_\parallel \sum_{n=-\infty}^{\infty} n \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2}$$
$$\times\,\delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha)$$
$$\times\left[J_n^2(\lambda)\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)\right]_{v_\perp = a}^{v_\perp = b}. \qquad (36)$$

It follows from (36) that the number of particles in this region is conserved only when a = 0 and b = ∞. On the other hand, if the region $\Omega_V^{(n)}$ does not overlap the other regions, then integrating (23) with respect to $v_\parallel$ in this region and in the region $a < v_\parallel < b$, we find

$$\frac{\partial}{\partial t}\int f_0^\alpha\,d\mathbf{v} = 2\pi^2\omega_\alpha\left(\frac{e_\alpha}{m_\alpha}\right)^2\int dv_\parallel\, n \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}}{k^2}$$
$$\times\,\delta(\omega_{\mathbf{k}} - k_\parallel v_\parallel - n\omega_\alpha)$$
$$\times\left[J_n^2(\lambda)\left(k_\parallel \frac{\partial f_0^\alpha}{\partial v_\parallel} + \frac{n\omega_\alpha}{v_\perp}\frac{\partial f_0^\alpha}{\partial v_\perp}\right)\right]_{v_\perp = a}^{v_\perp = b}. \qquad (37)$$

It follows from (37) that in the case when $\Delta k_\perp \sim k_\perp$ the number of particles is conserved when a = 0 and b = ∞. But if $\Delta k_\perp \ll k_\perp$, that is, $k_\perp = \text{const}$, then the number of particles is conserved in the region

$$a = \omega_\alpha \eta_\nu / k_\perp, \qquad b = \omega_\alpha \eta_{\nu+1} / k_\perp.$$

Knowing the limits of 'plateau' formation regions, we easily obtain the height of the plateau:

$$f_\infty^\alpha = \int f_H{}^\alpha\,d\mathbf{v} \Big/ \int d\mathbf{v}, \qquad \int d\mathbf{v} = 2\pi\int_a^b v_\perp dv_\perp \int dv_\parallel.$$

In the case when a = 0, b = ∞ and $\int dv_\parallel \neq 0$, the size of the region $\Omega_V(n)$ is infinite and $f_\infty^\alpha = 0$. Since the number of the resonant particles is conserved, this means that the particles have acquired infinite energy. It follows therefore that in the final state $\int dv_\parallel = 0$, that is

$$v_\parallel = \frac{\omega_{\mathbf{k}} - n\omega_\alpha}{k_\parallel} = \text{const}.$$

Thus, when $t \to \infty$ the spectrum of the oscillations reaches a stationary state $k_\parallel = (k_\parallel)_\infty$, $k_\perp = (k_\perp)_\infty$.

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