THE VACUUM POLARIZATION OF A CHARGED VECTOR FIELD

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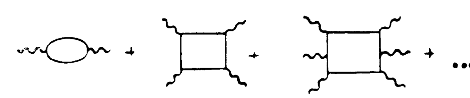
The nonlinear additions to the Lagrangian of a constant electromagnetic field, caused by the vacuum polarization of a charged vector field, are calculated in the special case in which the gyromagnetic ratio of the vector boson is equal to 2. The result is exact for an arbritrarily strong electromagnetic field, but does not take into account radiative corrections, which can play an important part in the unrenormalized electrodynamics of a vector boson. The anomalous character of the charge renormalization is pointed out.

1. INTRODUCTION

In recent times there have been frequent discussions in the literature on the properties of the charged vector boson, which is a possible carrier of the weak interactions. At present all that is known is that if such a boson exists its mass must be larger than 1.5 BeV. The theory of the electromagnetic interactions of such a particle encounters serious difficulties in connection with renormalization. Without touching on this difficult problem, we shall consider a problem, in our opinion not a trivial one, in which the nonrenormalizable character of the electrodynamics of the vector boson makes no difference. We are concerned with the calculation of the nonlinear corrections to the Lagrange function of a constant electromagnetic field interacting with the vacuum of charged vector bosons with gyromagnetic ratio 2. As is well known, the analogous problem for the case of polarization of the vacuum of spinor and scalar particles has been solved by a number of authors.[1-3]

It must be pointed out at once that the physical aspect of the statement of this problem in the electrodynamics of the vector boson is not as indisputable as in ordinary electrodynamics. The nonlinear corrections to the Lagrange function describe nonlinear effects of the type of scattering of light by light, i.e., a set of processes which correspond to the series of diagrams shown in the figure.

In ordinary electrodynamics a solid line corresponds to a vacuum electron. The vertex parts of such diagrams are proportional to the amplitude of the strong field, and the contribution from



virtual photons gives only small corrections to the solution. If we are dealing with a vector particle, then we come into the domain of nonrenormalizable theory and are not able to estimate in any reasonable way the contribution of the virtual photons to the processes represented in the figure. Although this is a very important point, all we can do here is to express the hope that in cases in which processes of this kind occur at small energies of the external field the radiative corrections will be small quantities. Moreover, because of the assumed large mass of the vector boson the processes shown in the figure will begin to be important much later than the radiative corrections to the corresponding solution in the electrodynamics of electrons.

Of course, there is always the purely mathematical aspect of the problem. If the problem admits of exact solution it is interesting to obtain this solution and study its special features.

2. GENERAL THEORY

We take the equation for the vector field in the form proposed earlier[4,5]

$$(P_k^2 g_{mn} + 2\alpha P_m P_n + ie\gamma F_{mn} - m^2 g_{mn}) B^n = 0;$$
$$P_k = i\partial_k - eA_k, \qquad [P_m, P_n] = ieF_{m,n}. \qquad (1)$$

No supplementary condition is imposed on the vector field, and in the free case it is a mixture of physical quanta with the mass m and spin 1 and nonphysical quanta with mass $m/(1 + 2\alpha)^{1/2}$

and spin 0. The parameter γ is the gyromagnetic ratio for the quanta of spin 1.[1)]

If the Green's function of Eq. (1) is represented by an integral over a parameter proportional to the proper time

$$G_{mn}(x, y) = -i \int_C ds e^{-im^2 s} U_{mn}(s;\ x, y), \tag{2}$$

then the Lagrangian with nonlinearity taken into account can be written in the following form:

$$L'(x) = -i \int_C \frac{ds}{s} e^{-im^2 s} g^{mn} U_{mn}(s;\ x, x). \tag{3}$$

The expression (3), whose derivation we have placed in the Appendix, is entirely analogous to those that occur in the cases of interaction with the vacuum of scalar and spinor fields.[3] The contour C in the complex s plane begins at zero and goes to infinity in a direction which assures the convergence of the integrals (2) and (3).

Strictly speaking the matrix $U_{mn}(s;\ x,\ y)$ must be found as the solution of a system of differential equations in five variables

$$-i\partial_s U_{mn} = (P_l^2 g_{mk} + 2\alpha P_m P_k + ie\gamma F_{mk}) U_n{}^k \tag{4}$$

with the initial condition

$$U_{mn}(0; x, y) = g_{mn}\delta(x - y). \tag{5}$$

It will suffice, however, for the special case we are to consider here (and for an understanding of the difficulties that arise in the more general case) to represent U_{mn} formally as an exponential of a differential operator acting on the δ function:

$$U = \exp\{is(P^2 + 2\alpha PP + ie\gamma F)\}\delta(x - y). \tag{6}$$

To simplify the writing we have omitted the matrix indices. All three terms in the exponent fail to commute with each other in the general case. For F = const, P^2 and F commute, but PP commutes only with the sum $P^2 + 2ieF$, so that in this case the expression (6) can be rewritten in the form

$$U = \exp\{is(2\alpha PP + ie(\gamma - 2)F)\} \times \exp\{is(P^2 + 2ieF)\}\delta(x - y). \tag{7}$$

The argument of the first exponential still contains noncommuting matrices, and this causes the difficulty of obtaining a closed solution for $\gamma \neq 2$.

In the special case of gyromagnetic ratio equal to 2, the expression (7) is greatly simplified. In this connection we point out the operator identity

$$e^{is2\alpha PP} = 1 + P\frac{e^{is2\alpha P^2} - 1}{P^2}P \tag{8}$$

and the relations

$$Pe^{is(P^2+2ieF)} = e^{isP^2}P, \qquad e^{is(P^2+2ieF)}P = Pe^{isP^2}, \tag{9}$$

which hold for F = const and follow from

$$P(P^2 + 2ieF) = P^2P, \qquad (P^2 + 2ieF)P = PP^2. \tag{10}$$

Setting $\gamma = 2$ in (7) and using (8) and (9), we get the expression

$$U(s;\ x, y) = \left(e^{is(P^2+2ieF)} + P\frac{e^{isP^2\xi} - e^{isP^2}}{P^2}P\right)\delta(x - y) \qquad (\xi = 1 + 2\alpha), \tag{11}$$

which can be transformed into

$$U(s;\ x, y) = e^{-2eFs}e^{isP^2}\delta(x - y) - iPP \times \int_{s\xi}^{s} ds'\, e^{-2eFs'}\, e^{is'P^2}\,\delta(x - y). \tag{12}$$

The problem has now been reduced to that of finding one scalar function

$$V(s; x, y) = e^{isP^2}\delta(x - y), \tag{13}$$

and we can use the result already obtained by Fock[7] and by Schwinger[3]

$$V(s; x, y) = \frac{1}{16\pi^2}\frac{1}{is^2}\exp\Big\{-ie\int_y^x dz[A(z) + \frac{1}{2}F\cdot(z - y)] - \frac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh} eFs}{eFs} - \frac{i}{4}(x - y)eF\operatorname{cth}(eFs)\cdot(x - y)\Big\}. \tag{14*}$$

*sh = sinh.

1)In the Green's function (16) and Lagrangian (24) derived later on the basis of Eq. (1) it is possible to take the limit $1 + 2\alpha \to 0$ and eliminate the contribution of the nonphysical quanta. The immediate reason for this is the fact that in the special case we are considering (F = const, $\gamma = 2$) there is a separation of the equations for the physical and nonphysical parts of the field analogous to that which occurs for the free field. All physical results contained in this paper have also been derived by us on the basis of the equations for the vector field with α set equal to $-\frac{1}{2}$ from the beginning – in this way we avoid mentioning the nonphysical quanta at all. In that case, however, the derivation of basic formulas of the type of our Eqs. (2) and (3) is more complicated: not all components of the vector field B_n are independent, and the necessity of taking account of the supplementary condition that follows from (1) for $\alpha = -\frac{1}{2}$ is rather burdensome. Therefore we have stayed with a form of exposition in which the parameter $1 + 2\alpha = \xi$ appears explicitly during the calculations and is made to go to zero only in the final results; but it must be clear to the reader that the present work has in essence no points of contact with the ξ-limit formalism of Lee and Yang.[5,6]

It is easy to verify that the function V(s; x, y) actually satisfies the differential equation $-i\partial_s V = P^2 V$, and the initial condition V(0; x, y) = δ(x − y) is satisfied owing to the fact that

$$\lim_{s\to+0}\frac{1}{16\pi^2}\frac{1}{is^2}e^{-i(x-y)^2/4s}=\lim_{s\to+0}\frac{1}{(2\pi)^4}$$

$$\times\int d^4k e^{ik^2s+ik(x-y)}=\delta(x-y). \quad (15)$$

Substituting the expressions (12), (13), (14) in Eq. (2) and reducing the resulting multiple integral to a simple integral (by changing the order of the integrations), we get the following propagation function of the vector field in the constant electromagnetic field:

$$G(x,y)=-\frac{1}{16\pi^2}\int_C\frac{ds}{s^2}\exp\left\{-\frac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh} eFs}{eFs}\right\}$$

$$\times\left[e^{-im^2s}-(e^{-im^2s}-e^{-im^2s/\xi})\frac{PP}{m^2}\right]\exp\left\{-ie\right.$$

$$\times\int_y^x dz\left[A(z)+\frac{1}{2}F\cdot(z-y)\right]$$

$$\left.-\frac{i}{4}(x-y)eF\operatorname{cth}(eFs)\cdot(x-y)-2eFs\right\}. \quad (16)^*$$

This function describes the simultaneous and independent (independent only for $\gamma = 2$) propagation of the physical and nonphysical components of the field, and therefore the contribution of the latter component can be eliminated by going to the limit $\xi \to 0$. When we do this and also perform the differentiation in (16), we get for the Green's function the representation

$$G_{mn}(x,y)=-\frac{1}{16\pi^2}\exp\left\{-ie\int_y^x dz\left[A(z)+\frac{1}{2}F\cdot(z-y)\right]\right\}$$

$$\times\int_C\frac{ds}{s^2}\exp\left\{-im^2s-\frac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh}eFs}{eFs}-\frac{i}{4}(x-y)\right.$$

$$\left.\times eF\operatorname{cth}(eFs)\cdot(x-y)\right\}\left\{\left[1-m^{-2}\left(\frac{i}{2}eF\right.\right.\right.$$

$$\times(1+\operatorname{cth}eFs)-\frac{1}{4}eF(1-\operatorname{cth}eFs)(x-y)$$

$$\left.\left.\left.\times(x-y)(1+\operatorname{cth}eFs)eF\right)\right]e^{-2eFs}\right\}_{mn}. \quad (17)$$

In the analogous expressions for the propagation functions of scalar and spinor fields, and also in the limiting case of the free vector field, the path of integration can be taken along the positive real axis. In our case this cannot be done because of the exponential increase of the integrand along the real axis. It will be more convenient for us to find the actual restrictions on the position of the contour C after deriving the Lagrangian.

*cth = coth.

3. THE LAGRANGIAN FUNCTION

For the calculation of the Lagrangian we must substitute in the expression (3) the $U_{mn}(s; x, y)$ given by Eqs. (12), (13), and (14). The same result can be obtained more rapidly by using instead of U(s; x, y) the simpler expression

$$\tilde{U}(s;\ x,y)=\left[e^{is(P^2+2ieF)}+\frac{1}{4}(e^{isP^2\xi}-e^{isP^2})\right]\delta(x-y). \quad (18)$$

The legitimacy of this replacement in the Lagrangian is proved in the Appendix. We get the Lagrangian in the form

$$L'=\frac{1}{16\pi^2}\int_C\frac{ds}{s}e^{-im^2s}\left[\exp\left\{-\frac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh}eFs}{eF}\right\}\right.$$

$$\left.\times(1-\operatorname{Sp}e^{-2eFs})-\exp\left\{-\frac{1}{2}\operatorname{Sp}\ln\frac{\operatorname{sh}eFs\xi}{eF}\right\}\right]. \quad (19)$$

For the calculation of the traces in (19) we must note that the eigenvalues of the matrix F_{mn} are $\pm F_1, \pm iF_2$, where

$$F_1=\left[\left(\frac{(\mathbf{E}^2-\mathbf{H}^2)^2}{4}+(\mathbf{EH})^2\right)^{1/2}+\frac{\mathbf{E}^2-\mathbf{H}^2}{2}\right]^{1/2},$$

$$F_2=\left[\left(\frac{(\mathbf{E}^2-\mathbf{H}^2)^2}{4}+(\mathbf{EH})^2\right)^{1/2}-\frac{\mathbf{E}^2-\mathbf{H}^2}{2}\right]^{1/2}. \quad (20)$$

This gives

$$L'=\frac{1}{16\pi^2}\int_C\frac{ds}{s}e^{-im^2s}\left\{\frac{e^2F_1F_2}{\operatorname{sh}eF_1s\cdot\sin eF_2s}\right.$$

$$\times(1-2\operatorname{ch}2eF_1s-2\cos 2eF_2s)-\frac{e^2F_1F_2}{\operatorname{sh}eF_1s\xi\cdot\sin eF_2s\xi}$$

$$\left.+\frac{1}{s^2}\left(3+\frac{1}{\xi^2}\right)+\frac{20}{3}\frac{\mathbf{E}^2-\mathbf{H}^2}{2}e^2s^2\right\}. \quad (21)^*$$

In writing the expression (21) we have subtracted an additive constant independent of the field and a term

$$-\frac{5e^2}{12\pi^2}\int_C\frac{ds}{s}e^{-im^2s}\frac{\mathbf{E}^2-\mathbf{H}^2}{2}=(Z^{-1}-1)\frac{\mathbf{E}^2-\mathbf{H}^2}{2}. \quad (22)$$

This is added to the free Lagrangian and leads to a renormalization of electromagnetic field strength and of charge:

$$E=E_0Z^{-1/2},\qquad H=H_0Z^{-1/2},\qquad e^2=e_0^2Z, \quad (23)$$

*ch = cosh.

where the index zero labels the "bare" values of charge and field strength which we have been using up to Eq. (21). By means of trigonometric transformations (21) can be put in the form

$$L' = L_1'(m^2) + 3L_0'(m^2) + L_0'(m^2/\xi), \tag{24}$$

where

$$L_1'(m^2) = \frac{e^2}{4\pi^2}\int_C \frac{ds}{s} e^{-im^2s}\left\{F_2\left(F_1\frac{\sin eF_2s}{\operatorname{sh} eF_1s} - F_2\right)\right.$$

$$\left. - F_1\left(F_2\frac{\operatorname{sh} eF_1s}{\sin eF_2s} - F_1\right)\right\}, \tag{25}$$

$$L_0'(m^2) = \frac{1}{16\pi^2}\int_C \frac{ds}{s} e^{-im^2s}$$

$$\times\left(-\frac{e^2F_1F_2}{\operatorname{sh} eF_1s\cdot\sin eF_2s} + \frac{1}{s^2} - e^2\frac{\mathbf{E}^2-\mathbf{H}^2}{6}\right). \tag{26}$$

In the region $3\pi/2 < \arg s < 2\pi$ and $|e(F_1 + iF_2)s| \gg 1$ the integrand in L_0' goes to zero exponentially, and that in L_1' behaves like $s^{-1}e^{-im^2s}\cosh e(F_1 - iF_2)s$. Therefore for convergence of the integrals it is enough to take as the contour the straight line $\arg s = \arg(F_2 - iF_1)$. It is obvious that this choice of contour assures the convergence of the integral over s in the representation (17) for the Green's function.

In the integral containing the first term in curly brackets in (25) we can rotate the contour C and place it along the positive real semiaxis. In the integral of the second term, and also in (26), the path can be placed along the negative imaginary semiaxis. After this we get the final formula for the Lagrangian:

$$L' = \frac{e^2}{4\pi^2}\int_0^\infty \frac{dt}{t}\left[e^{-im^2t}F_2\left(F_1\frac{\sin eF_2t}{\operatorname{sh} eF_1t} - F_2\right) - e^{-m^2t}F_1\right.$$

$$\left.\times\left(F_2\frac{\sin eF_1t}{\operatorname{sh} eF_2t} - F_1\right)\right] + \frac{3}{16\pi^2}\int_0^\infty \frac{dt}{t} e^{-m^2t}$$

$$\times\left(\frac{e^2F_1F_2}{\sin eF_1t\cdot\operatorname{sh} eF_2t} - \frac{1}{t^2} - e^2\frac{\mathbf{E}^2-\mathbf{H}^2}{6}\right). \tag{27}$$

Here we have dropped the contribution $L_1'(m^2/\xi)$ from the nonphysical quanta; it goes to zero for $\xi \to 0$.

The poles that arise in (27) at the zeroes of $\sin eF_1t$ must be evaded by passing above them. It can be seen immediately that $\operatorname{Im} L' = 0$ only for $F_1 = 0$. In this case a coordinate system can be chosen in which $\mathbf{E} = 0$ and $\mathbf{H} \neq 0$. For $F_1 > 0$ the Lagrangian has an imaginary part which determines the probability of pair production per unit time and unit volume.[3] In a purely electric field ($F_1 = E$, $F_2 = 0$) this probability is equal to three times the probability of production of pairs of scalar particles:

$$W = 3\operatorname{Im} L_0' = \frac{3e^2E^2}{8\pi^3}\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2}{eE}\right)$$

$$= \frac{3e^2E^2}{8\pi^3}\int_0^{\pi m^2/eE}\frac{dt}{t}\ln(1+t). \tag{28}$$

The quantity W cannot be derived in perturbation theory, since it describes the production of pairs by an infinite number of photons of infinitesimal energy.

In conclusion we give the lowest terms of the expansion of L' in powers of the field:

$$L' = \left(\frac{e^2}{4\pi}\right)^2\frac{1}{10m^4}\left[\frac{29}{4}(\mathbf{E}^2-\mathbf{H}^2)^2 + 27(\mathbf{EH})^2\right] + \ldots \tag{29}$$

This is an asymptotic series, as is also the case in ordinary electrodynamics,[8,9] and the function $L'(e)$ has an essential singularity at $e = 0$ [this singularity can be seen particularly clearly in Eq. (28)].

The asymptotic behavior of L' for $eE/m^2 \to \infty$ is given by the expression

$$L_a' = \frac{e^2}{4\pi}\frac{7E^2}{8\pi}\ln\frac{eE}{m^2}. \tag{30}$$

For $eH/m^2 \to \infty$ we get

$$L_a' = -\frac{e^2}{4\pi}\frac{7H^2}{8\pi}\ln\frac{eH}{m^2}. \tag{31}$$

Thus in our case also (just as in the electrodynamics of scalar and spinor particles) the nonlinear effects increase only logarithmically, and they are small even at quite fantastic fields.

4. CONCLUDING REMARKS

The foregoing discussion has shown that the Lagrange function of a constant and uniform field that is induced by the interaction of the field with the vacuum of a charged vector field with gyromagnetic ratio $\gamma = 2$ gives in general a reasonable and noncontradictory description of the nonlinear effects that arise in this interaction. In any case, the function L' expressed in terms of renormalized quantities does not differ in its characteristic features from the corresponding function in ordinary electrodynamics.

There is, however, a point on which we have been silent up to now and which may characterize an internal lack of complete consistency of the approach used. The renormalized charge and field strength which we have defined in Eq. (22) are connected with the corresponding "bare" quantities through a divergent integral. If we formally cut off the integral at the lower limit with a large mass Λ, so that $1/\Lambda^2 < s < \infty$, we get from (22) and (23)

$$e^2 = e_0^2\left(1 - \frac{5e_0^2}{12\pi^2}\ln\frac{\Lambda^2}{m^2}\right)^{-1} \text{with } \ln\frac{\Lambda^2}{m^2} \gg 1 \gg \frac{m^2}{\Lambda^2}. \quad (32)$$

Unlike ordinary electrodynamics, where one has

$$e^2 = e_0^2\left(1 + \frac{e_0^2}{12\pi^2}\ln\frac{\Lambda^2}{m^2}\right)^{-1}$$

our case gives $e^2 > e_0^2$; besides this it is in principle possible that $e^2 < 0$, which is a quite absurd result.

As is well known, in electrodynamics the restriction $0 \le Z \le 1$ on the Z-factor (charge renormalization) follows from the most general principles—the quantization rules and the Källen-Lehmann representation. In our case it is impossible, generally speaking, to get the restriction $0 \le Z \le 1$ by starting from these principles, since now

$$[A_m(\mathbf{x}, t), A_n(\mathbf{y}, t)] \neq -ig_{mn}\delta(\mathbf{x}-\mathbf{y})$$

and it is not known whether the Källen-Lehmann representation exists in the renormalized theory. Nevertheless, the inequality $Z > 1$ seems extremely undesirable, since it contradicts the accepted physical interpretation of the Z-factor.[2)]

To determine to what extent the result obtained is due to the condition $\gamma = 2$ and whether or not it is a reflection of shortcomings of the technique of calculation employed, the magnitude of the electric charge renormalization has been calculated independently on the basis of the Feynman technique. The expression obtained is

$$Z^{-1} = 1 + \frac{e^2}{48\pi^2}\left(4 - 6\gamma - 3\gamma^2 + \frac{3\gamma(2-\gamma)}{\xi}\right)\ln\frac{\Lambda^2}{m^2}, \quad (33)$$

which agrees with (32) for $\gamma = 2$, but for $\gamma < 2$, though indeed positive, is scarcely meaningful, since it contains two infinite parameters Λ^2 and ξ^{-1}. All of this evidently means that in a nonrenormalizable theory with virtual photons not taken into account it is impossible to count on getting a reasonable value for the charge renormalization constant.

In conclusion we must emphasize again that there are no rigorous reasons to think that the radiative corrections do not decidedly change the entire expression for the Lagrange function. By the way, the same can be said about any result of an unrenormalizable theory.

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APPENDIX

A Lagrangian for the vector field which leads to the equation (1) can be written in the form

$$L = -(P^k B_m)^* \rho_{kl}{}^{mn}(P^l B_n) + m^2 B_n^* B^n, \quad (A.1)$$

where

$$\rho_{kl}{}^{mn} = g^{mn}g_{kl} + (2\alpha + \gamma)g_k{}^m g_l{}^n - \gamma g_l{}^m g_k{}^n. \quad (A.2)$$

From this we get the symmetrized expression for the current:

$$J_k = \frac{e}{2}(\{(P^l B_n), B_m^*\} + \{B_m, (P^l B_n)^*\})\rho_{kl}{}^{mn}. \quad (A.3)$$

The equal-time commutation relations are of the form

$$[B_m(x), \pi_n(y)]_{x_0=y_0} = [B_m^*(x), \pi_n^*(y)]_{x_0=y_0}$$
$$= -ig_{mn}\delta(\mathbf{x}-\mathbf{y}), \quad [B_m(x), B_n^*(y)]_{x_0=y_0} = 0, \quad (A.4)$$

where the canonical momentum is

$$\pi^n = -i(P^k B_m)^* \rho_{k0}{}^{mn}. \quad (A.5)$$

By Eqs. (A.4) and (A.5) the vacuum average of the chronological product of field operators

$$G_{mn}(x, y) = i\langle 0|{}^1/_2\{B_m(x), B_n^*(y)\}$$
$$+ {}^1/_2\varepsilon(x-y)[B_m(x), B_n^*(y)]|0\rangle, \quad (A.6)$$

is the Green's function of Eq. (1):

$$(P_l^2 g^{mk} + 2\alpha P^m P^k + ie\gamma F^{mk} - m^2 g^{mk})G_{kn}(x, y)$$
$$\equiv (P^s \rho_{sl}{}^{mk} P^l - m^2 g^{mk})G_{kn}(x, y) = g_n{}^m \delta(x-y). \quad (A.7)$$

This causal Green's function can be connected with the vacuum value of the current (A.3)

$$i\langle 0|J_k(x)|0\rangle = e\lim_{x\to y}(P_x{}^l G_{nm}(x, y)$$
$$+ P_y^{*l} G_{mn}(x, y))\rho_{kl}{}^{mn}, \quad (A.8)$$

if we understand the limit in (A.8) as the arithmetic mean of the two expressions obtained by letting x approach y from the past and future directions.

After averaging over the vacuum of the vector field, the action operator of the interacting electromagnetic and vector fields is still a functional of the electromagnetic field. In this functional we keep only the dependence on a strong external field, neglecting effects of the quantized proper field, assuming that these effects are of the order of the fine-structure constant. Then the action S

[2)] $\dot{A}_n$ is not the canonical momentum. This is due to the structure of the interaction of the electromagnetic field with the charged vector field [see Eq. (A.1) in the Appendix].

is determined from the functional equation

$$\delta S = \int d^4x \langle 0 | J_k(x) | 0 \rangle \delta A^k(x), \qquad (A.9)$$

where the vector-field vacuum average of the current is connected with the propagation function of the vector boson by the relation (A.8).

From this point on it is convenient to introduce matrix symbolism and regard the Green's function with which we are concerned as a matrix element of the inverse of the operator of Eq. (1):

$$G_{mn}(x, y) = \langle m, x | G | n, y \rangle, \qquad (A.10)$$

$$G = (P^2 + 2\alpha PP + ie\gamma F - m^2)^{-1} \equiv (P\rho P - m^2)^{-1}. \qquad (A.11)$$

Let us consider the integral

$$S = -i \int_C \frac{ds}{s} e^{-im^2 s} \operatorname{Sp} e^{isP\rho P}, \qquad (A.12)$$

where the symbol Sp denotes the diagonal sum over spin and coordinate indices. This integral is the desired action function satisfying the relation (A.9), if the contour is chosen in such a way that the expression

$$G_{mn}(x, y) = -i \int_C ds\, e^{-im^2 s} \langle m, x | e^{isP\rho P} | n, y \rangle \qquad (A.13)$$

is the causal Green's function. In the variation of (A.12) one must take into account the possibility of a cyclic permutation of the operators under the sign of the trace.[3)]

Using the definition of the Lagrange function, $S = \int d^4x L'(x)$, we arrive at the formula

$$L'(x) = -i \int_C \frac{ds}{s} e^{-im^2 s} g^{mn} \langle m, x | e^{isP\rho P} | n, x \rangle. \qquad (A.14)$$

[3)]Strictly speaking, the possibility of such a permutation requires additional justification, since it involves the necessity of changing the order of integrations in an improper integral. In any given case the legitimacy of this operation can always be assured by a suitable choice of the class of functions over which the potential is varied.

In Eq. (3) of the main text the matrix element $\langle m, x | e^{isP\rho P} | n, x \rangle$ was denoted by $U_{mn}(s; x, x)$.

For F = const and $\gamma = 2$ we have

$$e^{isP\rho P} = e^{is(P^2+2ieF)} + P \frac{e^{isP^2\xi} - e^{isP^2}}{P^2} P, \qquad (A.15)$$

as was shown in the passage from (6) to (11). It is obvious that the traces of the operators (A.15) and

$$\widetilde{U} = e^{is(P^2+2ieF)} + {}^1/_4 (e^{isP^2\xi} - e^{isP^2}) \qquad (A.16)$$

are equal if it is possible to use a cyclic interchange of the operators in the second term in (A.15). Since such an interchange calls for some caution (see the third footnote), we have also checked the equality in question by a direct calculation. Therefore in deriving the Lagrangian we can use instead of the matrix elements $U_{mn}(s; x, y)$ of the operator (A.15) the matrix elements $\widetilde{U}_{mn}(s; x, y)$ of the simpler operator (A.16), and this is what we have done in the main text.

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