SUPERCONDUCTOR OF SMALL DIMENSIONS IN A STRONG MAGNETIC FIELD

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Spherical and cylindrical superconductors, whose dimensions are smaller than the penetration depth, are considered. The field at which the gap in the excitation spectrum vanishes and the field at which pairing vanishes are determined. The dependence of the magnetic moment on the field strength is determined. The dependence of the Knight shift on the size of the particles and on the field strength is obtained for the case when the field is not small in comparison with the critical field.

1. INTRODUCTION

A superconductor of small dimensions has the same properties as a bulk sample, so long as it is not located in a magnetic field. The behavior of small samples in a strong magnetic field has a number of special features. The gap in the one-particle excitation spectrum vanishes at a certain value of the field; however, the other properties of the superconducting state (for example, the anomalous diamagnetism) are retained. With further increase of the field, a second-order phase transition into the normal state occurs. The magnitude of the critical field depends on the dimensions of the sample and on the concentration of impurities in it.

The spin susceptibility also depends on the field strength. At zero temperature and without any spin-orbit interaction, it vanishes in that region where there is a gap in the spectrum (the Knight shift is absent). With further increase of the field strength, the spin susceptibility gradually increases up to its value in the normal metal. If a spin-orbit interaction with impurities exists, then the spin susceptibility is nonzero throughout and increases with increasing field strength.

In this article we consider superconductors whose dimensions are small in comparison with the size of a Cooper pair and with the magnetic field penetration depth, and whose shape has an axis of symmetry directed along the field (small sphere or cylinder). With specular boundary conditions and in the absence of impurities, a superconductor of such shape possesses a number of special features. They arise because in such a system the component of the angular momentum in the direction of the field is conserved. For a sufficiently large concentration of impurities, when the mean free path is smaller than the dimensions of the system, the results must be insensitive to the form of the boundary conditions. One should think that pure superconductors with diffuse boundary conditions or irregular shapes are described well by the formulas for contaminated superconductors with a mean free path the order of the dimensions of the system.

2. BASIC EQUATIONS

The Gor'kov equations[1] describing a superconductor in an external field have the form

$$(i\omega_n - \hat{H})G(\mathbf{r},\mathbf{r}') + \Delta^*(\mathbf{r})F(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'),$$
$$(i\omega_n + \hat{H}^*)F(\mathbf{r},\mathbf{r}') + \Delta(\mathbf{r})G(\mathbf{r},\mathbf{r}') = 0,$$
$$\Delta(\mathbf{r}) = gF(\mathbf{r},\mathbf{r}). \qquad (1)$$

In these equations $\omega_n = (2n+1)\pi T$, $\hat{H}$ is the electron Hamiltonian measured from the chemical potential, which includes their interaction with the boundaries, with impurities, and with the magnetic field. $\hat{H}$ differs from $\hat{H}^*$ by the sign of the magnetic field.

If the magnetic field is equal to zero, then it can be verified that Δ does not depend on the size of the superconductor. For this purpose we expand the equation in eigenfunctions of the Hamiltonian:

$$\hat{H}\psi_\lambda = \xi_\lambda\psi_\lambda. \qquad (2)$$

Assuming Δ to be constant, we obtain

$$\Delta(\mathbf{r}) = |g|T\sum_{\omega,\lambda}|\psi_\lambda(\mathbf{r})|^2\frac{\Delta}{\omega_n^2+\xi^2+\Delta^2}. \qquad (3)$$

If the dimensions of the system are large in comparison with interatomic distances, so that Δ is much larger than the distance between levels, then the summation in (3) takes place over a large number of states. The rapidly oscillating parts in the factor $|\psi_\lambda(\mathbf{r})|^2$ then cancel out, and we can replace it by its average value V^{-1}.

Therefore the assumption made earlier, that Δ does not depend on **r**, is valid. In addition, we can replace the summation over λ by an integral. As a result we obtain

$$1 = |g|\rho_0\pi T \sum_{\omega} (\omega_n^2 + \Delta^2)^{-1/2}. \quad (4)$$

Thus, the properties of a superconductor in the absence of a magnetic field are determined by the density of one-particle states, $\rho_0 = mp_F/2\pi^2$ and by the interaction constant g, and do not depend on either the dimensions of the system or the form of the boundary conditions at the surface or on the impurity concentration. This conclusion is incorrect for a superconductor in a magnetic field, since the operators $\hat{H}$ and $\hat{H}^*$ have different eigenvalues and eigenfunctions.

3. PURE SUPERCONDUCTOR

In pure superconductors, having spherical or cylindrical shapes with specular boundary conditions, the angular momentum component along the direction of the magnetic field is conserved. Therefore the operators $\hat{H}$ and $\hat{H}^*$ have the same eigenfunctions, but different eigenvalues:

$$\hat{H}\psi_\lambda = (\xi_\lambda - \mu H)\psi_\lambda, \qquad \hat{H}^*\psi_\lambda = (\xi_\lambda + \mu H)\psi_\lambda, \quad (5)$$

where $\mu = \mu_0 m$, H is the magnetic field intensity, μ and m are the components of the magnetic and mechanical moments in the direction of the field. Expanding the system (1) in these functions, we obtain

$$G = -\frac{i\omega_n + \xi + \mu H}{(\omega_n - i\mu H)^2 + \xi^2 + \Delta^2},$$

$$F = \frac{\Delta}{(\omega_n - i\mu H)^2 + \xi^2 + \Delta^2}. \quad (6)$$

After substitution of F from (6) and integration with respect to ξ, the last equation in system (1) takes the form

$$1 = |g|\rho_0\pi T \left\langle \sum_{\omega} [(\omega_n - i\mu H)^2 + \Delta^2]^{-1/2} \right\rangle. \quad (7)$$

Here and below the brackets ⟨...⟩ denote averaging over all states with different values of μ lying on the Fermi surface. Because of the large dimensions of the system, for the averaging one can use quasiclassical formulas for the level density and for the angular momentum. Thus

$$\rho_0 \langle f(\mu H)\rangle = \int \frac{d^3p\, d^3r}{(2\pi)^3} f\left(\frac{e\mathbf{H}}{2mc}[\mathbf{pr}]\right) \delta\left(\frac{p^2}{2m} - \frac{p_0^2}{2m}\right). \quad (8)^*$$

*$[\mathbf{pr}] = \mathbf{p} \times \mathbf{r}$.

At zero temperature it is necessary to replace the sum over frequencies in (7) by an integral.

If the field is smaller than

$$H_1 = 2mc\Delta_0 / ep_0R, \quad (9)$$

where R is the radius of the sphere or cylinder, then Δ coincides with its value Δ_0 in the absence of a field. This happens because $\mu H < \Delta$ even for the states with maximum angular momentum components, and in the integral with respect to ω one can displace the contour by $i\mu H$; then the integral determining Δ ceases to depend on the field.

Δ decreases for large values of the field; from (7) and (8) we obtain the following equation for a cylinder

$$\ln\frac{\Delta_0}{\Delta} = \left(1 + \frac{1}{2x^2}\right)\ln\left(x + \sqrt{x^2 - 1}\right) - \frac{3}{2}\sqrt{1 - x^{-2}}, \quad (10)$$

$x = eHp_0R/2mc\Delta$. It follows from this equation that Δ goes to zero and the pairing vanishes at the field

$$H_2 = {}^1/_2 e^{3/2} H_1 = 2.24 H_1. \quad (11)$$

For a sphere

$$H_2 = {}^1/_4 e^{7/3} H_1 = 2.57 H_1.$$

In the case considered the quantity Δ does not coincide with the gap in the one-particle excitation spectrum. The density of one-particle states ρ is determined by the imaginary part of the Green's function.[2] Using expression (6), in which we must make the substitution $\omega_n \to -i\omega$, for the Green's function, we obtain

$$\rho = \frac{1}{\pi}\,\mathrm{Im}\sum_\lambda G = \rho_0\,\mathrm{Im}\left\langle \frac{\omega + \mu H}{[\Delta^2 - (\omega + \mu H)^2]^{1/2}} \right\rangle. \quad (12)$$

For $H < H_1$ the value of the gap in the spectrum is determined by the state with the largest value of $m = p_0R$ and is given by

$$\omega_0 = \Delta_0(1 - H/H_1). \quad (13)$$

Decrease of the gap with increasing field has a simple physical interpretation. The Bose condensate is made up of pairs with zero angular momentum components. The interaction energy of such a pair with the magnetic field is equal to zero. When such a pair breaks a binding energy $2\Delta_0$ is lost, but the interaction energy of the electrons with the field is gained. Hence the gap decreases.

The gap vanishes when $H = H_1$. With further increase of the field strength, the density of levels at zero energy increases, and when $H = H_2$ it reaches its value for the normal metal. Broken pairs, which consist of electrons with large angular momentum components, are now found in the ground state in this region of field strengths. A superconducting state without a gap in its spectrum is of some interest in principle; therefore, we shall present a description of such a state using various methods.

In the Bardeen-Cooper-Schrieffer method[3] the wave function of the system has the form

$$\Psi = \prod_{|\mu H|<E} (u + v a_m^+ a_{-m}^+) \prod_{\mu H > E} a_m^+,$$

$$E^2 = \xi^2 + \Delta^2, \quad u^2 = {}^1/_2(1 + \xi/E), \quad v^2 = {}^1/_2(1 - \xi/E). \tag{14}$$

In the method of Bogolyubov[4] the operator for the creation of quasiparticles is

$$\alpha_m^+ = \begin{cases} u a_m^+ - v a_{-m}, & \mu H < E, \\ u a_m - v a_{-m}^+ & \mu H > E. \end{cases} \tag{15}$$

In the method of Gor'kov[1] the Green's function has the form (6) at finite temperatures, but when T = 0

$$G = \frac{u^2}{\omega + \mu H - E + i\delta \operatorname{sign} \omega} + \frac{v^2}{\omega + \mu H + E + i\delta \operatorname{sign} \omega}. \tag{16}$$

In states with $|\mu H| < E$ the electrons are paired, the occupation numbers are equal to v^2, and are the same for states with opposite signs of the components. The states with $\mu H < -E$ are empty, but those with $\mu H > E$ are completely occupied. Electrons in these states give a contribution to the density of levels at low energies. Pairing of electrons in the states with $|\mu H| < E$ leads to the result that certain properties of the superconducting state are retained. For example, the diamagnetic susceptibility turns out to be anomalously large.

The magnetic moment is determined from the formula:

$$\mathbf{M} = \frac{e}{2mc} \sum_{\omega} \left[\mathbf{r}, \mathbf{p} - \frac{e}{c}\mathbf{A} \right] G \frac{d^3r d^3p}{(2\pi)^3}. \tag{17}$$

Substituting G from (6), we obtain

$$\mathbf{M} = \rho_0 \int d\xi T \sum_{\omega} \left\langle \mu \frac{i\omega_n + \mu H + \xi}{(\omega_n - i\mu H)^2 + \xi^2 + \Delta^2} \right\rangle - \frac{e^2 N}{2mc} \langle \mathbf{r}_\perp^2 \rangle \mathbf{H}. \tag{18}$$

At zero temperature, it is necessary to replace the summation over ω by an integral, and the first term vanishes so long as $H < H_1$. The expression for the magnetic moment has then the same form as in the limiting London case, although the dimensions of the superconductor are small in comparison with the size of a pair.

Such a result becomes understandable if the fact that the first term is proportional to the mechanical moment is taken into consideration. Its vanishing corresponds to the well known quantum mechanical result that a body cannot rotate as a whole about an axis of symmetry. At finite temperatures, or for $H > H_1$, individual excitations may rotate. In these cases the diamagnetic susceptibility decreases with increasing field strength and vanishes when $H = H_2$.

4. SUPERCONDUCTOR CONTAINING IMPURITIES

When there are impurities in a superconductor, then it is impossible to expand the system (1) in eigenfunctions of the operators $\hat{H}$ and $\hat{H}^*$, since these functions do not coincide and are very complicated. However, one can average Eqs. (1) with respect to the positions of the impurities. Averaging is carried out in the same way as in the case of an infinite medium.[1] The corrections which arise are of order $(p_0 R)^{-1}$ and are small when the dimensions of the samples are large in comparison with atomic distances. As a result of averaging, Eqs. (1) take the form

$$(i\omega_n + i\overline{G} - \hat{H}) G(\mathbf{r}, \mathbf{r}') + (\Delta^* + \overline{F}^*) F(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

$$(i\omega_n + i\overline{G} + H^*) F(\mathbf{r}, \mathbf{r}') + (\Delta + \overline{F}) G(\mathbf{r}, \mathbf{r}') = 0, \tag{19}$$

where $\hat{H}$ is the Hamiltonian of the electrons in a pure superconductor,

$$i\overline{G} = \pi n \sigma m^{-2} G(\mathbf{r}, \mathbf{r}'), \quad \overline{F} = \pi n \sigma m^{-2} F(\mathbf{r}, \mathbf{r}'), \tag{20}$$

n is the impurity concentration and σ is the scattering cross section which, for simplicity, is assumed isotropic. The considerations discussed following formula (3) enable us to verify that $\overline{G}$, $\overline{F}$, and Δ do not depend on r.

Let us expand the system (19) in eigenfunctions of the Hamiltonian $\hat{H}$ which satisfy Eqs. (5). The system is then easily solved. Substituting the expressions for G and F into (20), we obtain equations for $\overline{G}$ and $\overline{F}$:

$$2\tau\overline{G} = \left\langle \frac{\omega_n + \overline{G} - i\mu H}{[(\omega_n + \overline{G} - i\mu H)^2 + (\Delta + \overline{F})^2]^{1/2}} \right\rangle,$$

$$2\tau\overline{F} = \left\langle \frac{\Delta + \overline{F}}{[(\omega_n + \overline{G} - i\mu H)^2 + (\Delta + \overline{F})^2]^{1/2}} \right\rangle, \tag{21}$$

where $\tau = l/v$ and l is the mean free path.

In the case considered the mean free path is less than or of the order of the dimensions of the sample, and in any case it is much less than the pair dimension v/Δ. As will be clear from the solution, the important fields are of order $\mu H \sim (\Delta/\tau)^{1/2}$, and $\overline{G} \sim \tau^{-1}$. Therefore one can expand (21) in powers of μH. Here it is convenient to introduce the parameter

$$\alpha = 2\tau\langle \mu^2\rangle H^2;$$

$$\alpha = \begin{cases} {}^1/_3\tau(eHp_0R/2mc)^2 & \text{for a cylinder,} \\ {}^4/_{15}\tau(eHp_0R/2mc)^2 & \text{for a sphere.} \end{cases} \qquad (22)$$

It is convenient to write the solution of Eq. (21) in parametric form:

$$2\tau\overline{G} = \sin\varphi, \quad 2\tau\overline{F} = \cos\varphi,$$

$$\omega_n = \Delta\,\mathrm{tg}\,\varphi - \alpha\sin\varphi. \qquad (23)^*$$

From the last equation of system (1) we obtain an equation for the determination of the quantity Δ:

$$\Delta = |g|\rho_0\pi T\sum_\omega \cos\varphi, \qquad (24)$$

where φ is expressed in terms of ω_n by formula (23).

For small Δ

$$\cos\varphi = \Delta / |\omega + \alpha|$$

and from (24) we obtain an equation for the dependence of the critical temperature on the field strength:

$$\ln\frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \frac{\alpha}{2\pi T}\right) - \psi\left(\frac{1}{2}\right). \qquad (25)$$

At zero temperature it is possible to change in (24) from an integral with respect to ω to an integral with respect to φ. We obtain

$$\ln\frac{\Delta_0}{\Delta} = \begin{cases} \pi\alpha/4\Delta, & \alpha > \Delta, \\ \ln\dfrac{\alpha + (\alpha^2 - \Delta^2)^{1/2}}{\Delta} - \dfrac{(\alpha^2 - \Delta^2)^{1/2}}{2\alpha} + \dfrac{\alpha}{2\Delta}\arcsin\dfrac{\Delta}{\alpha}, & \alpha < \Delta, \end{cases} \qquad (26)$$

where Δ_0 is the value of Δ in the absence of the field. The dependence of Δ on the field strength is shown in the figure (curve 1). It follows from (26) or from (25) that Δ goes to zero when $\alpha = \Delta_0/2$. For large field strengths, pairing is not present even at zero temperature.

As in the case of a pure superconductor considered above, Δ does not coincide with the gap in the excitation spectrum, and the gap vanishes at a field smaller than the critical. In analogy to formula (12), we obtain

$$\rho = \rho_0\,\mathrm{Im}\,2\tau i\overline{G}. \qquad (27)$$

In this formula, it is necessary to substitute $\overline{G}$ from formula (23), in which we make the substitutions $\omega_n \to -i\omega$, $\varphi \to -i\varphi$. As a result

$$\rho = \rho_0\,\mathrm{Im}\,\mathrm{sh}\,\varphi, \quad \omega = \Delta\,\mathrm{th}\,\varphi - \alpha\,\mathrm{sh}\,\varphi. \qquad (28)^\dagger$$

The gap in the spectrum is equal to the smallest value of ω for which φ becomes complex:

$$\omega_0 = (\Delta^{2/3} - \alpha^{2/3})^{3/2}. \qquad (29)$$

The gap vanishes for $\alpha = \Delta$. The dependence of ω_0 on the field is represented in the figure by curve 2.

Using (26) and (22), we obtain the relation between the field H_1 at which the gap vanishes and the critical field H_2 at which pairing vanishes:

$$H_1^2 = 2e^{-\pi/4}H_2^2 = 0.91H_2^2. \qquad (30)$$

The physical meaning of a superconducting state without a gap is less clear than in the case of a pure superconductor. And in this case a fraction of the pairs are broken when $H > H_1$. However, the electron states in the presence of impurities are rather complicated, and it is impossible to determine which pairs are broken.

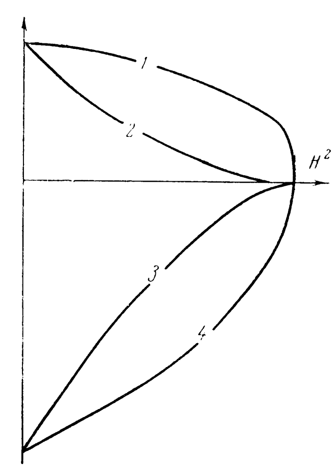
As follows from (28) and (29), the density of states near ω_0 is given by

$$\rho = \rho_0\left(\frac{\Delta^2}{\alpha\omega_0}\right)^{1/6}\left(\frac{\omega - \omega_0}{\alpha}\right)^{1/2}. \qquad (31)$$

In the interval $H_1 < H < H_2$, the density of states at zero energy is

$$\rho = \rho_0(1 - \Delta^2/\alpha^2)^{1/2}. \qquad (32)$$

In order to determine the correction to the thermodynamic potential we use the formula[1]



*tg = tan.

†sh = sinh, th = tanh.

$$\Omega_s - \Omega_n = \int_0^g \Delta^2 \frac{dg}{g^2}. \qquad (33)$$

Expressing g in terms of Δ, we obtain with the aid of (24) and (23)

$$\Omega_s - \Omega_n = \frac{1}{2} V\rho_0 \pi T \sum_\omega \omega_n \left(2 - \sin\varphi - \frac{1}{\sin\varphi} \right), \qquad (34)$$

where φ is expressed in terms of Δ, α, and ω_n by formula (23). At zero temperature the sum is replaced by an integral, and for the energy of the ground state we obtain

$$E_s - E_n = -{}^1/_2 V\rho_0 \begin{cases} \Delta^2 - \frac{\pi}{2}\Delta\alpha + \frac{2}{3}\alpha^2, & \Delta > \alpha, \\ \Delta^2 - \Delta\alpha \arcsin\frac{\Delta}{\alpha} + \frac{2}{3}\alpha^2 - \frac{1}{3\alpha}(\Delta^2 + 2\alpha^2)\sqrt{\alpha^2 - \Delta^2}, & \Delta < \alpha. \end{cases} \qquad (35)$$

The dependence of the energy on the magnetic field is represented in the figure by curve 3.

The magnetic moment of the sample is equal to the derivative of Ω_s with respect to the magnetic field:

$$M = \frac{\partial\Omega}{\partial H} = \frac{\partial\Omega}{\partial\alpha}\frac{2\alpha}{H}.$$

Differentiating (34) with account of (23) and (24), we obtain

$$M = 2V\rho_0 \frac{\alpha}{H} \pi T \sum_\omega \cos^2\varphi. \qquad (36)$$

At zero temperature, we can replace the sum by an integral and obtain for the magnetic susceptibility

$$\chi = \frac{M}{H} = \frac{\chi_0}{\Delta_0} \begin{cases} \Delta - \frac{4\alpha}{3\pi} & \Delta > \alpha, \\ \frac{2}{\pi}\left[\Delta \arcsin\frac{\Delta}{\alpha} - \frac{2}{3}\alpha + \frac{1}{3}\left(2 + \frac{\Delta^2}{\alpha^2}\right)\sqrt{\alpha^2 - \Delta^2} \right], & \Delta < \alpha. \end{cases} \qquad (37)$$

This function is represented in the figure by curve 4. Here χ_0 is the susceptibility in a weak field; from (36) we obtain

$$\chi_0 = \pi V\rho_0 \frac{\alpha\Delta_0}{H^2} \operatorname{th}\frac{\Delta_0}{2T}. \qquad (38)$$

It is of interest to compare this expression with the corresponding formula for the magnetic susceptibility of a pure superconducting sphere with diffuse boundary conditions:[5]

$$\chi_0 = \frac{\pi^2}{12}\frac{e^2N}{mc}R^6\frac{\Delta}{v}\operatorname{th}\frac{\Delta}{2T}.$$

Substituting α from (22) into (37), we find that diffuse boundary conditions are equivalent to an impurity concentration leading to a mean free path

$$l = 5\pi R/6.$$

It can be expected that such an assumption leads to small errors in the description of diffuse boundary conditions in the remaining cases as well.

Formulas similar to (25)—(30) are obtained for a superconductor containing paramagnetic impurities.[6] The same formulas were obtained by Maki[7], who investigated a superconductor in a constant magnetic field.

5. SPIN SUSCEPTIBILITY (KNIGHT SHIFT)

Neither the Landau diamagnetism nor the spin paramagnetism was considered above. When the size of the sample is large in comparison with atomic distances, these effects are small in comparison with the considered anomalous diamagnetism and do not give an appreciable contribution to the magnetic moment. However, in nuclear magnetic resonance experiments the spin susceptibility is measured directly. For the case of a weak field, it was determined by Abrikosov and Gor'kov.[8] In some experiments the field is not small in comparison with the critical field; therefore it is of interest to determine the susceptibility for this case as well. Here the interaction of the electron spins with the field is small, but it is convenient to make the expansion after completion of certain exact calculations.

The influence of a magnetic field on the electron spins can be taken into account by adding the term $\mu_0\sigma \cdot \mathbf{H}$ to $\hat{H}$ in the initial equations (1), and subtracting the same term from $\hat{H}^*$ (μ_0 is the Bohr magneton). Thus, it is necessary to make the following substitution in the initial equations:

$$\omega_n \to \omega_n - i\mu_0\sigma\mathbf{H}. \qquad (39)$$

The average spin is determined in terms of the Green's function by the formula

$$S = \rho_0 \int d\xi T\pi \sum_{\omega} \left\langle \mathrm{Sp}\, \frac{1}{2}\, \sigma G \right\rangle. \quad (40)$$

For a pure superconductor, we substitute G from formula (6), taking (39) into account. At zero temperature, the sum over ω is replaced by an integral. If $H < H_1$ we can displace the integration contour and verify that the integral does not depend on the spin. Thus, the average spin is equal to zero in this region of field strengths. For field strengths greater than H_1, the spin susceptibility is proportional to the density of one-particle states. Such a result is natural; it means that only unpaired electrons contribute to the spin susceptibility.

A similar result is obtained in a superconductor containing light elements as impurities, when the spin-orbit interaction is small. In this case the spin does not change during scattering, and one can make the substitution (39) in the final answer (23). Substituting (23) into (40), we obtain

$$S = \rho\, \mathrm{Sp}\, \frac{\pi}{2} \boldsymbol{\sigma} T \sum_{\omega} \sin\varphi,$$

$$\omega_n - i\mu_0 \boldsymbol{\sigma}\mathbf{H} = \Delta\, \mathrm{tg}\, \varphi - \alpha \sin\varphi. \quad (41)$$

One should take into consideration that the integration with respect to ξ must be performed after the summation over ω; therefore formula (41) gives the difference between the spins in the superconducting and normal states. With this in mind, expanding (41) with respect to $\mu_0 \sigma \cdot H$, we obtain the following expression for the spin susceptibility

$$\chi_s = \chi_n \left(1 - \pi T \sum_{\omega} \frac{\cos^3\varphi}{\Delta - \alpha \cos^3\varphi} \right), \quad (42)$$

where φ is expressed in terms of ω according to formula (23).

At zero temperature, the integral appearing in place of the sum can be evaluated. In accordance with its physical interpretation, the spin susceptibility turns out to be proportional to the density of one-particle states at zero energy. When $H < H_1$, all electrons give the following contribution to the susceptibility

$$\chi_s = \chi_n (1 - \Delta^2 / \alpha^2)^{1/2}. \quad (43)$$

A different result is obtained if the spin-orbit interaction of the electrons with the impurities is taken into consideration. In this case the spin of the electrons does not have a definite value, and a spin susceptibility arises even when all electrons are paired. The interaction of the electrons with impurities is described by the amplitude

$$f = a + ibp_0^{-2}([\mathbf{p}_1\mathbf{p}']\boldsymbol{\sigma}). \quad (44)$$

The presence of the second term does not have any effect on results which do not depend on the spins. The total lifetime τ, appearing in formulas (22) and (23), is expressed in terms of the constants a and b in the following manner:

$$\frac{1}{\tau} = \frac{1}{\tau_0} + \frac{1}{\tau_1}; \quad \frac{1}{\tau_0} = \frac{1}{2} n\rho_0 \int d\Omega |a|^2,$$

$$\frac{1}{\tau_1} = \frac{1}{2} n\rho_0 \int d\Omega |b|^2 \sin^2\theta. \quad (45)$$

However, the averaging (with respect to the position of the impurities) of quantities which are proportional to $\boldsymbol{\sigma}$ leads to different expressions. Therefore it is impossible to make the substitution (39) in the final expressions.

In the course of averaging of the quantity proportional to $\boldsymbol{\sigma} \cdot \mathbf{H}$, an expression $f_{\alpha\beta}\sigma_{\beta\gamma} \cdot \mathbf{H} f_{\gamma\delta}$ appears which contains not only terms proportional to $\boldsymbol{\sigma} \cdot \mathbf{H}$, but terms proportional to $(\boldsymbol{\sigma} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{H})$ as well. It is convenient to describe the result of averaging expressions proportional to $\boldsymbol{\sigma} \cdot \mathbf{H}$ and $(\boldsymbol{\sigma} \cdot \mathbf{p})(\mathbf{p} \cdot \mathbf{H})$ by the matrix

$$L = \frac{1}{2} n\rho_0 \int d\Omega \begin{pmatrix} |a|^2 & -|b|^2 \sin^2\theta \\ \frac{1}{2}\sin^2\theta\, (|a|^2 - |b|^2 \sin^2\theta) & \frac{1}{2}(3\cos^2\theta - 1)(|a|^2 - |b|^2\sin^2\theta) \end{pmatrix}. \quad (46)$$

This matrix is equivalent to the factor $(2\tau)^{-1}$ which appears in connection with the averaging of expressions that do not contain σ.

The corrections proportional to $\sigma \cdot H$ in the Green's functions integrated with respect to ξ will be denoted by G' and F'. Allowing for the field acting on the electron spin, Eqs. (21) take the form

$$2\tau\bar{G} + G' = \left\langle \frac{\omega_n + \bar{G} - i\mu H + LG' - i\mu_0\boldsymbol{\sigma}\mathbf{H}}{\{(\omega_n + \bar{G} - i\mu H + LG' - i\mu_0\boldsymbol{\sigma}\mathbf{H})^2 + (\Delta + \bar{F} + LF')^2\}^{1/2}} \right\rangle,$$

$$2\tau\bar{F} + F' = \left\langle \frac{\Delta + \bar{F} + LF'}{\{(\omega_n + \bar{G} - i\mu H + LG' - i\mu_0\boldsymbol{\sigma}\mathbf{H})^2 + (\Delta + \bar{F} + LF')^2\}^{1/2}} \right\rangle. \quad (47)$$

Expanding these equations with respect to the quantities proportional to $\sigma \cdot \mathbf{H}$, and substituting for $\overline{G}$ and $\overline{F}$ their expressions from (23), we obtain

$$(\tau^{-1} - L + \Delta / \cos\varphi - \alpha\cos^2\varphi) G' = i\mu_0\sigma\mathbf{H}\cos^2\varphi. \quad (48)$$

Let us substitute L from (46), and assume that the spin-orbit interaction is smaller than the ordinary interaction, $b \ll a$. Then

$$G' = \frac{\cos^2\varphi}{\Delta/\cos\varphi - \alpha\cos^2\varphi - 2/3\tau_1} i\mu_0\sigma\mathbf{H}, \quad (49)$$

where τ_1 is defined by (45) and denotes the lifetime with respect to rotation of the spin. Substituting (49) into (40) and taking the remark following (41) into consideration, we obtain the following expression for the spin susceptibility

$$\chi_s = \chi_n\left(1 - \pi T\sum_\omega \frac{\cos^2\varphi}{\Delta/\cos\varphi - \alpha\cos^2\varphi - {}^2/_3\tau_1}\right). \quad (50)$$

For large values of τ_1, this expression goes over into (42), and the result of Abrikosov and Gor'kov[8] is obtained in the case of a weak magnetic field ($\alpha = 0$). For small values of τ_1, the spin susceptibilities of the normal and superconducting states are nearly equal. The difference is proportional to the anomalous diamagnetic susceptibility [(37) and (38)] and is represented in the figure by curve 4.

The magnetic field influences the spin susceptibility through the parameter α which enters directly into Eq. (50) and, in addition, has an effect on the relation (23) between φ and ω_n, and on the value of Δ given by Eq. (24). It should be noted that, according to (22), α depends on the size of the samples. Therefore, in a field which is not very small in comparison with the critical field, the magnitude of the Knight shift must depend on the size of the samples, and for a large degree of dispersion of the particles, the nuclear magnetic resonance line must be broad.

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