ANALYTICAL PROPERTIES OF THE AMPLITUDE IN THE QUASIPOTENTIAL SCATTERING PROBLEM

O. I. ZAV'YALOV, M. K. POLIVANOV, and S. S. KHORUZHIĬ

Mathematics Institute, Academy of Sciences, U.S.S.R.

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It is shown that in the quasipotential method recently proposed the second Born term for the scattering amplitude does not satisfy the Mandelstam representation if the mean particle radius is independent of the energy.

1. In a series of recently published papers [1-3] a "quasi-optical" approach to the scattering of elementary particles is proposed, which permits an investigation of the nature of the bound states, development of a method for calculating the effective cross sections, etc. The method makes liberal use of dispersion relations. Consequently, a study of the analytic properties of the amplitude within the framework of the new theory is quite timely.

The principal equation of the quasi-potential approach

$$[E^2 - \mathbf{p}^2 - m^2]\sqrt{\mathbf{p}^2 + m^2}\,\Psi(\mathbf{p}) - \int V[E, (\mathbf{p}-\mathbf{p}')^2]\,\Psi(\mathbf{p}')\,d\mathbf{p}' = 0$$

makes it possible to write for the scattering amplitude $A(\mathbf{p}_1\mathbf{p}_2)$, which is connected with the S matrix by the formula

$$\hat{S} = \hat{1} + (i\pi/2E^2)\,\delta(E - E')\,\hat{A}, \tag{1}$$

an equation of the Lippmann-Schwinger type

$$A(\mathbf{p}_1\mathbf{p}_2) = V(\mathbf{p}_1 - \mathbf{p}_2) + \frac{1}{(2\pi)^3}\int d\mathbf{p} V(\mathbf{p}_1 - \mathbf{p})\,\frac{A(\mathbf{p}\mathbf{p}_2)}{\sqrt{\mathbf{p}^2 + m^2}\,(\mathbf{p}^2 - \mathbf{p}_2^2)}\,. \tag{2}$$

The first possibility consists in finding the region of analyticity of the amplitudes A as a whole on the basis of the Fredholm method, applied to Eq. (2). Such a possibility is realized in [4]. It appears useful, however, to study in greater detail the structure of the function A by using Born iterations. The calculation of the second Born term corresponding to Eq. (2) can be carried through almost to conclusion, and the resultant formulas yield directly the character and position of the singularities of the amplitude. This is the purpose of the present article.

The second term of the Born series for (2) is of the form

$$A^{(2)}(\mathbf{p}_1\mathbf{p}_2) \sim \int V(\mathbf{p}_1 - \mathbf{p})\,\frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2 + m^2}\,(\mathbf{p}^2 - s)}\,V(\mathbf{p} - \mathbf{p}_2). \tag{3}$$

We use the following notation: $s = \mathbf{p}_1^2 = \mathbf{p}_2^2$, $t = -(\mathbf{p}_1 - \mathbf{p}_2)^2 = -2s(1 - \cos\theta)$. For simplicity we choose a Yukawa potential

$$V(\mathbf{p}) \sim (\mathbf{p}^2 + \mu^2)^{-1}. \tag{4}$$

However, the results of the paper apply also to a superposition of Yukawa potentials with weight function σ that depends on the energy, and generally speaking is complex:

$$V(s, t) = \int_{\mu}^{\infty} d\nu\,\frac{\sigma(s,\nu)}{\nu - t} \tag{5}$$

(these are precisely the superpositions used in the quasi-potential description of scattering).

It will be shown below that the amplitude $A^{(2)}$, as a function of s for fixed negative t (in the physical region), is analytic everywhere in the complex s plane, with two cuts along the positive real axis and part of the negative axis. When $0 < t < 4\mu^2$, the start of the left cut shifts (an "anomalous threshold" of sorts appears). When $t > 4\mu^2$, the anomalous singularity goes into the region of complex s. For fixed real s, the amplitude $A^{(2)}$ has as a function of t only a cut along the real axis, the position of the cut depending, generally speaking, on s. Thus, the amplitude $A^{(2)}$ does not satisfy a representation of the Mandelstam type.

We note also that the structure of the second Born approximation has a direct analogy with the structure of the Feynman triangle diagram, and therefore many calculations of the singularities of the function $A^{(2)}$ are analogs of the corresponding singularities of the triangle.

2. We proceed to a direct calculation of $A^{(2)}$. Substituting (4) in (3) we get

$$A^{(2)}(\mathrm{p}_1\mathrm{p}_2) \sim \int \frac{1}{(\mathrm{p}_1-\mathrm{p})^2+\mu^2}\,\frac{dp}{\sqrt{p^2+m^2}\,(\mathrm{p}^2-s)}\,\frac{1}{(\mathrm{p}-\mathrm{p}_2)^2+\mu^2}. \qquad (6)$$

We represent the root $\sqrt{p^2+m^2}$ contained in (6) in spectral form

$$\frac{1}{\sqrt{\mathrm{p}^2+m^2}} = \frac{1}{\pi}\int_0^\infty \frac{d\xi}{\sqrt{\xi}}\,\frac{1}{\mathrm{p}^2+m^2+\xi} \qquad (7)$$

and use the partial-fraction expansion

$$\frac{1}{\mathrm{p}^2+m^2+\xi}\,\frac{1}{\mathrm{p}^2-s} = -\frac{1}{s+m^2+\xi}\left[\frac{1}{\mathrm{p}^2+m^2+\xi}-\frac{1}{\mathrm{p}^3-s}\right]. \qquad (8)$$

Substituting (7) and (8) in (6) we find that $A^{(2)}$ breaks up into two terms:

$$A^{(2)} \sim \frac{1}{\pi}\int_0^\infty \frac{\widetilde{A}(st\xi)\,d\xi}{\sqrt{\xi}\,(s+m^2+\xi)} - \frac{A^{(2)}_{\mathrm{N.R.}}(st)}{\sqrt{s+m^2}}. \qquad (9)$$

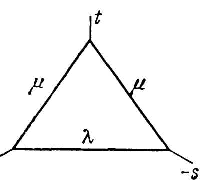
Here $A^{(2)}_{\mathrm{N.R.}}(st)$ coincides with the second Born approximation for the usual Schrodinger equation with Yukawa potential. It is well known [5,6] that $A^{(2)}_{\mathrm{N.R}}$ satisfies the Mandelstam representation, and the cut in the complex plane encloses the entire real positive axis. The corresponding term in our expression differs from $A^{(2)}_{\mathrm{N.R.}}$ only in the factor $(s+m^2)^{-1/2}$, which adds to the known singularities only the kinematic singularity $s=-m^2$. We therefore concern ourselves with the first term in the right half of (9), which reflects the specific nature of the initial equation.

Disregarding the kinematic singularities, which are connected with the zeros of the denominator in the integral with respect to ξ, we consider the function $\widetilde{A}$. It is of the form

$$\widetilde{A}(st\xi) = \int dp\,\frac{1}{(\mathrm{p}_1-\mathrm{p})^2+\mu^2}\,\frac{1}{p^2+\lambda^2}\,\frac{1}{(\mathrm{p}-\mathrm{p}_2)^2+\mu^2};$$

$$\lambda^2 = m^2+\xi. \qquad (10)$$

It is easy to see that Expression (10) is simply related to the Feynman triangle diagram. In fact, if the diagram (see the figure) is represented by the formula $\int dp_0 f(p_0, M_1^2, M_2^2, M_3^3)$, then Expression (10) is equal to $f(0, -s, -s, t)$. It is known [7] that the singular points of the triangle are determined by the singularities of the function f for



$p_0 = 0$. It is therefore natural to expect the function (10) to have the same singularities as the diagram in the figure. We note in this connection that since the integral (10) can be evaluated completely, the final formula is of additional interest as an explicit expression for the vertex part in the third order perturbation theory, for the case of three-dimensional integration over the internal momentum (we shall henceforth call a function of the type (10) a "three-dimensional" vertex).

Using the well-known α-representation and integrating with respect to p in (10) we get

$$\widetilde{A}(st) \sim \int_0^1 d\alpha_1\,d\alpha_2\,d\alpha_3\,\frac{\delta(1-\alpha_1-\alpha_2-\alpha_3)}{[Q(s,t,\alpha_1,\alpha_2,\alpha_3)]^{3/2}},$$

where Q is given by

$$Q = s(\alpha_2+\alpha_3)\,\alpha_1 - t\alpha_2\alpha_3 + \mu^2(\alpha_2+\alpha_3) + \lambda^2\alpha_1.$$

Integrating with respect to α_3 with the aid of the δ function, we arrive at

$$\widetilde{A} = \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2\,\frac{1}{[Q(s,t,\alpha_1,\alpha_2,1-\alpha_1-\alpha_2)]^{3/2}}. \qquad (11)$$

Integration with respect to α_2 in (11) leads to the tabulated integral

$$\int d\alpha_2\,(a+b\alpha_2+c\alpha_2^2)^{-3/2}$$

and yields

$$\widetilde{A}(st) = \int_0^1 d\alpha_1 \times \frac{1-\alpha_1}{\left[s\alpha_1(1-\alpha_1)+\mu^2(1-\alpha_1)+\lambda^2\alpha_1-\frac{1}{4}(1-\alpha_1)^2t\right]\sqrt{Q}\,\big|_{t=0}}. \qquad (12)$$

This integral can be readily reduced to the form

$$\widetilde{A}(st) = -\frac{1}{s+t/4}\int_0^1 d\alpha_1\,\frac{1-\alpha_1}{(\alpha_1+p)(\alpha_1+q)\sqrt{a+b\alpha_1+c\alpha_1^2}}, \qquad (13)$$

where p and q are the roots of the quadratic trinomial in the denominator of (12), taken with opposite signs, while the coefficients a, b, and c are determined from the expansion of $Q\,|_{t=0}$ in powers of α_1. Expanding (13) in partial fractions, we reduce it to a linear combination of two tabulated integrals.

Simple but cumbersome calculations lead to the following final expression:

$$\widetilde{A}(st) = \frac{\mathrm{const}}{\sqrt{Rt}}\ln\frac{K+\sqrt{Rt}}{K-\sqrt{Rt}}; \qquad (14)$$

$$K = \lambda(4\mu^2-t) + 2\mu(s+\lambda^2+\mu^2),$$
$$R = -\lambda^2(4\mu^2-t) + (s+\lambda^2+\mu^2)^2. \qquad (15)$$

Using the conventional variables for the vertex diagram

$$x = (2\mu^2 - t)/2\mu^2,\ y = z = (\lambda^2 + \mu^2 + s)/2\lambda\mu,$$

we rewrite (15) in the form

$$\widetilde{A}(st) = \frac{\text{const}}{\sqrt{x^2 + 2y^2 - 2xy^2 - 1}} \times \ln \frac{1 + x + 2y + \sqrt{x^2 + 2y^2 - 2xy^2 - 1}}{1 + x + 2y - \sqrt{x^2 + 2y^2 - 2xy^2 - 1}}. \quad (16)$$

In the general case when all the masses on the external lines of our three-dimensional vertex are different ($y \neq z$) we would have in place of (16)

$$\mathfrak{A}(xyz) = \frac{\text{const}}{\sqrt{x^2 + y^2 + z^2 - 2xyz - 1}} \times \ln \frac{1 + x + y + z + \sqrt{x^2 + y^2 + z^2 - 2xyz - 1}}{1 + x + y + z - \sqrt{x^2 + y^2 + z^2 - 2xyz - 1}},$$

which is indeed the explicit expression for the three-dimensional vertex part in the lower order of perturbation theory. It is easy to ascertain that this expression contains all the singularities of the triangular diagram and has no other singularities.

3. The specific singularities of the second Born approximation for the scattering amplitude, corresponding to (2), are contained in the function (14). The singularities of $\widetilde{A}(st)$ can be due to the zeros of $(Rt)^{1/2}$ and to the singularities of the logarithm, that is, to the zeros of the expressions $K + (Rt)^{1/2}$ and $K - (Rt)^{1/2}$. The position of these latter singularities is given by a single equation

$$K^2 - Rt = 0. \quad (17)$$

Recalling (15), we find for the roots of (17)

$$s = -(\lambda + \mu)^2 \quad (y = z = -1),$$
$$t = 4\mu^2 \quad (x = -1),$$

which corresponds to a stationary left-hand cut along the real s axis and a stationary right-hand cut along the real t axis. In terms of the triangle diagram, these cuts correspond to the "physical" thresholds connected with the violation of the stability condition at the vertices of the triangle.

We proceed to an investigation of the root singularities

$$R = [s + (\lambda^2 + \mu^2 + \lambda\sqrt{4\mu^2 - t})] \times [s + (\lambda^2 + \mu^2 - \lambda\sqrt{4\mu^2 - t})] = (s - s_-)(s - s_+) = 0.$$

We fix $0 < t < 4\mu^2$ and regard $\widetilde{A}$ as a function of s. The physical sheet is identified by the condition that A be real at large s. It is easy to verify that on the chosen sheet the point $s = s_+$ is not a singularity of $\widetilde{A}(s)$. Indeed, as s moves towards s_+ from higher positive values, the argument of the logarithm in (14) tends to unity, remaining real, and we should take in the formula $\ln 1 = 2i\pi n$ the principal value $n = 0$. Therefore the expansion of the logarithm in powers of $(Rt)^{1/2}/K$ in the vicinity of s_+ will contain only odd powers of $(Rt)^{1/2}$, and the expansion of the complete function $\widetilde{A}$ only even powers. With further variation of s from s_+ to s_-, the argument of the logarithm becomes complex, varying along the unit circle and returning to unity with a circuit around zero. It is therefore clear that the expansion of the logarithm in the vicinity of the point s_- in powers of $(Rt)^{1/2}$ will begin with the free term $\ln 1 = 2\pi i$. Because of this, the series for the function $\widetilde{A}(s)$ in the vicinity of s_- will contain, in addition to the even powers of the root, also a term of the form $2\pi/\sqrt{-Rt}$, which leads to a singularity of $\widetilde{A}$ at the point $s = s_-$.

Thus, one of the branches of the surface $R = 0$ is singular. We write

$$s_- = -\lambda^2 - \mu^2 - \lambda\sqrt{4\mu^2 - t}.$$

We see that when $4\mu^2 > t > 0$ this singularity goes onto the physical sheet (in terms of the triangle diagram—an anomalous threshold appears), and when $t = t_0 + i\varepsilon$ ($t_0 > 4\mu^2$) the singularity goes over into the complex plane

$$s_- = -\lambda^2 - \mu^2 + i\lambda\sqrt{t_0 - 4\mu^2},$$

which is also in full correspondence with the known results of the analysis of the triangle diagram.

The point

$$s_+ = -\lambda^2 - \mu^2 - i\lambda\sqrt{t_0 - 4\mu^2}$$

will not be singular in this region of t. When t is fixed on the other edge of the cut ($t = t_0 - i\varepsilon$), the singularity goes over to the lower half plane

$$s_- = -\lambda^2 - \mu^2 - i\lambda\sqrt{t_0 - 4\mu^2}.$$

It is therefore clear that the spectral density in the dispersion relation for $\widetilde{A}$ with respect to t will contain two branch points symmetrically located with respect to the real axis. (This can already be seen from the fact that the aforementioned spectral density is proportional to $R^{-1/2}$, as can be readily understood.)

Thus, the Mandelstam representation is violated for $\widetilde{A}$. This is the consequence of the well-known occurrence of anomalous singularities in the triangular Feynman diagram. In general, from the discussed analogy with the triangle it is quite clear that the singular surface for $A(st)$ is given by the equation $\theta_x + 2\theta_y = 2\pi$; $\theta_x = \cos^{-1} x$; $\theta_y = \cos^{-1} y$. Therefore no complex singularities arise in the t plane for any real s, and the singularities in t are confined to the cut along the real axis which, generally speaking, moves.

Let us return to the total amplitude $A^{(2)}$, which is obtained from $\widetilde{A}$ by integrating with respect to ξ in (9). It is easy to ascertain that all the singularities of the function $\widetilde{A}$ at $\lambda^2 = m^2$ ($\xi = 0$) remain also after integration with respect to ξ. Indeed, the equality $\lambda^2 = m^2$ ($\xi = 0$) corresponds to the end point of the contour of integration with respect to ξ. Therefore an analytic continuation of the amplitude $A^{(2)}$, say to values $s = s_-|_{\xi=0}$ by deformation of the contour is impossible. This can also be readily seen directly. Indeed, as s approaches any of the mentioned danger points, the integral (9) becomes arbitrarily large, something possible only when these points are singular. It is easy to understand also that all the foregoing arguments apply also to the case of the potential (5).

4. Let us indicate in conclusion a possibility of retaining the Mandelstam representation within the framework of the quasi-optical approach. The weight function σ of the potential (5) depends on the energy s. We are therefore justified in assuming that the lower limit of integration in (5) also varies with energy: $\mu = \mu(s)$. At the same time, as already shown, the anomalous singularities of the amplitude $A^{(2)}$ appear only when $-s \sim \mu^2$. Therefore if we choose the function $\mu(s)$ such as to satisfy, for example, the condition $-s \ll \mu^2(s)$, then the complex singularities disappear.

Thus, for example, the equality $\mu(s) = s^2/\mu_1 + \mu_2$ with suitably chosen dimensional constants μ_1 and μ_2 ensures simple analytic properties for the amplitude $A^{(2)}$. We note that with such a choice of the energy dependence of $\mu(s)$, the left cut in the s plane likewise disappears. Of course, this choice is not unique. Physically, the use of potentials of the type (5) with variable lower limit is equivalent to assuming that the average radius of the particle depends on the energy.

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[1] A. A. Logunov and A. N. Tavkhelidze, Preprint, Joint Inst. Nuc. Res. E-1145 (1962).

[2] Logunov, Tavkhelidze, and Khrustalev, Preprint, Joint Inst. Nuc. Res. D-1191 (1963).

[3] Logunov, Tavkhelidze, and Arbuzov, Preprint, Joint Inst. Nuc. Res. R-1195 (1963).

[4] M. K. Polivanov and S. S. Khoruzhiĭ, JETP (in press).

[5] R. Jost and A. Pais. Phys. Rev. **82**, 840 (1951).

[6] Blankenbecler, Goldberger, Khuri, and Treiman. Ann. of Phys. **10**, 62 (1960).

[7] L. D. Landau, JETP **37**, 62 (1959), Soviet Phys. JETP **10**, 45 (1960).

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