STABILITY OF PLANE ELECTRON BEAMS IN BOUNDED SYSTEMS WITH A DECELERATING ELECTRIC FIELD

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The stability of stationary charged particle beams in bounded systems with a decelerating electric field is studied. The problem reduces to a Fredholm integral equation of the second kind with a kernel that depends on a complex parameter. The stationary solution is stable if the equation has no eigenvalues in the right half-plane and unstable otherwise. This criterion is used to establish the stability of the stationary solutions for several cases of injection of a beam into a system. In particular, the stationary solution is stable if, first, the particles are not reflected by the decelerating field and, second, all the particles are reflected by the external field and the distribution function of the injected beam is a monotonically decreasing function of the velocity.

A system of charged particles of not too high a density is usually described by a kinetic equation in Vlasov's form and by Maxwell's equations. In the one-dimensional case, in the presence of particles of one sort only (electrons), this system of equations has the form

$$\frac{\partial F}{\partial t}+v\frac{\partial F}{\partial x}-\frac{e}{m}\mathscr{E}\frac{\partial F}{\partial v}=0, \qquad (1)$$

$$\frac{\partial \mathscr{E}}{\partial x}=4\pi\rho=-4\pi e\int_{-\infty}^{\infty}F\,dv. \qquad (2)$$

Here $F = F(t, x, v)$ is the electron distribution function, and $\mathscr{E}_x = \mathscr{E}_x(t, x)$ is the electric field. The sign of the electron charge is already taken into account.

The system of equations (1) and (2) is rather complicated and as a rule cannot be solved analytically in explicit form. In the stationary case, however, this system can be integrated relatively simply. Stationary solutions of equations such as (1) and (2) were investigated by many workers[1-4], but the stability properties of the stationary solutions in the presence of external electric fields have been little studied. The stability of the stationary solutions for an electron plasma, i.e., for a system with two sorts of particles—electrons and stationary ions—is treated by Montgomery[5] and by Frieman and Pytte[6]. These papers deal with the stability of a "weakly inhomogeneous" plasma, i.e., it is assumed that the potentials of the external field are small compared with the average kinetic energy of the electrons. The stability of the inhomogeneous plasma was investigated also in a recent paper by Fowler[7].

The present work is devoted to the stability of the stationary solutions for systems bounded in space; the potentials of the external field are not assumed to be small. We study a system consisting of particles of one sort (electrons), but the results can be easily extended to include the corresponding problems for the case of a plasma. It is shown that stability of the stationary solution is equivalent to the existence of eigenvalues of a certain integral Fredholm equation with a kernel that depends on a complex parameter. The formulated integral equation is used to test the stability of the stationary solution in several particular cases.

1. STATIONARY SOLUTIONS

Let us consider Eqs. (1) and (2) in the stationary case

$$v\,\frac{\partial F}{\partial x}+\frac{e}{m}\,\varphi'(x)\,\frac{\partial F}{\partial v}=0, \qquad (3)$$

$$\varphi''(x)=-4\pi\rho(x)=4\pi e\int_{-\infty}^{\infty}F(x,v)\,dv. \qquad (4)$$

Here $\varphi(x)$ is the potential of the electric field: $\mathscr{E}(x) = -\varphi'(x)$. We seek a solution of the system (3) and (4) in the region $0 \le x \le l$ under the following supplementary conditions:

$$F(0,v)=Nf_0(v)>0 \quad (0\leqslant v<\infty), \qquad \int_0^{\infty} f_0(v)\,dv=1, \qquad (5)$$

$$F(l,v)=0 \qquad (-\infty<v\leqslant 0), \qquad (6)$$

$$\varphi(0)=0, \qquad \varphi(l)=\varphi_1\leqslant 0. \qquad (7)$$

Condition (5) defines the injection of electrons in the system, while condition (6) denotes the absorption of all the particles that reach the right end of the system. Finally, condition (7) shows that an external decelerating electric field is applied to the system. A problem of this type, with the function $f_0(x)$ specially chosen, was considered by Myakishev[2,3]. The solution of the problem is carried out in the general case quite analogously, and we therefore do not dwell on the details of the solution, but give only the final formulas. Since $\varphi'' = -4\pi\rho \geq 0$, there are only two types of possible solutions: Either the potential $\varphi(x)$ is a monotonically decreasing function of x in the entire region $0 \leq x \leq l$, or else the potential $\varphi(x)$ has at a certain point x_0 of this region ($0 < x_0 < l$) a minimum $\varphi_0 = \varphi(x_0) < \varphi(x)$ ($x \neq x_0$). In this case the potential $\varphi(x)$ decreases in the segment $0 \leq x \leq x_0$ and increases from $x_0 \leq x \leq l$. No other potential distributions are possible in this system.

The solution of Eq. (3), satisfying the boundary conditions (5) and (6), has the form

$$F(x, v) = \begin{cases} Nf_0(\sqrt{v^2 - 2e\varphi(x)/m}) & -\sqrt{2e(\varphi(x) - \varphi_1)/m} \leqslant v < \infty, \\ 0 & -\infty < v < -\sqrt{2e(\varphi(x) - \varphi_1)/m} \end{cases}, \tag{8}$$

if the potential $\varphi(x)$ is a monotonic function. If $\varphi(x)$ has a minimum, then

$$F(x, v) = \begin{cases} Nf_0(\sqrt{v^2 - 2e\varphi(x)/m}) & -\sqrt{2e(\varphi(x) - \varphi_0)/m} \leqslant v < \infty, \\ 0 & -\infty < v < -\sqrt{2e(\varphi(x) - \varphi_0)/m} \end{cases} \tag{9}$$

when $0 \leq x \leq x_0$ and

$$F(x, v) = \begin{cases} Nf_0(\sqrt{v^2 - 2e\varphi(x)/m}) & \sqrt{2e(\varphi(x) - \varphi_0)/m} \leqslant v < \infty, \\ 0 & -\infty < v < -\sqrt{2e(\varphi(x) - \varphi_0)/m} \end{cases} \tag{10}$$

when $x_0 \leq x \leq l$.

Substituting (8) or (9) and (10) in the Poisson equation (4), we can express the function $\varphi(x)$ in quadratures and ascertain the conditions under which solutions of the first and second types exist.

2. STABILITY OF THE STATIONARY SOLUTION. FORMULATION OF THE PROBLEM. REDUCTION OF THE PROBLEM TO AN INTEGRAL EQUATION

Let us proceed to investigate the stability of the stationary solutions. To this end, we consider in the linear approximation the time dependence of a solution which differs little from the stationary one at the inital instant. We represent the electron distribution function F(t, x, z) and the electric field $\mathscr{E}(t, x)$ in the form

$$F(t, x, v) = F_0(x, v) + f(t, x, v),$$
$$\mathscr{E}(t, x) = -\varphi'(x) + E(t, x), \tag{11}$$

where $F_0(x, v)$ and $-\varphi'(x)$ are the distribution function and the electric field in the stationary state, and $f(t, v)$ and $E(t, x)$ are small perturbations. Substituting (11) in (1) and leaving out the nonlinear term, we obtain the linearized kinetic equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\varphi'(x)\frac{\partial f}{\partial v} - \frac{e}{m}E\frac{\partial F_0}{\partial v} = 0. \tag{12}$$

Equation (12) must be solved together with the equation for the field. In this case it is more convenient for us to use not the Poisson equation, but the equation for the total current

$$\frac{\partial E}{\partial t} + 4\pi j = \frac{\partial E}{\partial t} - 4\pi e\int_{-\infty}^{\infty} f(t, x, v)\,v\,dv = 0. \tag{13}$$

The initial and boundary conditions are chosen in the form

$$f(0, x, v) = g(x, v),$$
$$E(0, x) = E^0 - 4\pi e\int_0^x dx\int_{-\infty}^{\infty} g(x, v)\,dv \equiv E_0(x),$$
$$f(t, 0, v) = 0, \quad (v \geqslant 0), \qquad f(t, l, v) = 0 \quad (v \leqslant 0). \tag{14}$$

We proceed to an analysis of the formulated problem. Taking the Laplace transform with respect to time, we obtain the system

$$pf + v\frac{\partial f}{\partial x} + \frac{e}{m}\varphi'(x)\frac{\partial f}{\partial v} = \frac{e}{m}E\frac{\partial F_0}{\partial v} + g, \tag{15}$$

$$E - \frac{4\pi e}{p}\int_{-\infty}^{\infty} f(p, x, v)\,v\,dv = \frac{1}{p}E_0(x), \tag{16}$$

$$f(p, 0, v) = 0 \quad (v \geqslant 0), \qquad f(p, l, v) = 0 \quad (v \leqslant 0). \tag{17}$$

We regard (15) as an inhomogeneous equation with respect to the function $f(p, x, v)$. It is easy to construct for this equation a solution satisfying the zero boundary conditions (17). In view of the linearity, the solution contains two terms, one dependent on the electric field $E(p, x)$, and the other determined by the form of the initial function $g(x, v)$: $f(p, x, v) = f_1\{E\} + f_2\{g\}$. The expression for the current is also represented by a sum of two terms

$$j(p, x) = j_1(p, x) + j_2(p, x) = -e\int_{-\infty}^{\infty} f_1 v\,dv - e\int_{-\infty}^{\infty} f_2 v\,dv. \quad (18)$$

We consider first the case when the potential of the stationary solution $\varphi(x)$ is a monotonic function. Substituting the expression for $f_1(p, x, v)$ in formula (18) for $j_1(p, x)$, we get

$$-4\pi j_1(p, x)/p = \int_0^l K(p, x, \xi) E(p, \xi)\,d\xi, \quad (19)$$

$$K(p, x, \xi) = H_1(v_1, p, x, \xi) + H_2(v_1, p, x, \xi) + H_3(v_1, p, x, \xi). \quad (20)$$

Here

$$H_1(v_1, p, x, \xi) = \frac{\omega_0^2}{p}\begin{cases} \int_{v(x)}^{\infty} \exp\left(\int_x^{\xi} \frac{pds}{U(u, s)}\right) f_0'(u)\,du & (0 \leqslant \xi \leqslant x \leqslant l), \\ -f_0(v_1)\exp\left(\int_{\xi}^{x} \frac{pds}{U(v_1, s)}\right) + \\ + \int_{v(\xi)}^{v_1} \exp\left(\int_{\xi}^{x} \frac{pds}{U(u, s)}\right) f_0'(u)\,du & (0 \leqslant x \leqslant \xi \leqslant l), \end{cases} \quad (21)$$

$$H_2(v_1, p, x, \xi) = -\frac{\omega_0^2}{p}\int_{v(x)}^{v_1} \exp\left(\int_{x(u)}^{\xi} \frac{pds}{U(u, s)} + \int_{x(u)}^{x} \frac{pds}{U(u, s)}\right) f_0'(u)\,du, \quad (22)$$

$$H_3(v_1, p, x, \xi) = \frac{\omega_0^2}{p} f_0(v_1)\exp\left(\int_l^{\xi} \frac{pds}{U(v_1, s)} + \int_l^{x} \frac{pds}{U(v_1, s)}\right);$$

$$U(u, s) = \sqrt{u^2 - \frac{2e}{m}\varphi(s)}, \quad v(x) = \sqrt{-\frac{2e}{m}\varphi(x)},$$
$$v_1 = \sqrt{-\frac{2e}{m}\varphi_1}, \quad x(u) = \psi\left(-\frac{mu^2}{2e}\right), \quad (23)$$

$x = \psi(\varphi)$ is the inverse of $\varphi(x)$, and $\omega_0 = \sqrt{4\pi e^2 N/m}$ is the plasma frequency.

Calculating the current $j_2(p, x)$, which depends on the initial distribution function, and substituting it together with (19) in (16), we obtain an integral equation with respect to the field $E(p, x)$:

$$E(p, x) = \int_0^l K(p, x, \xi) E(p, \xi)\,d\xi + h(p, x), \quad (24)$$

where

$$h(p, x) = E_0(x)/p - 4\pi j_2(p, x)/p. \quad (25)$$

The kernel $K(p, x, \xi)$ and the function $h(p, x)$ are analytic in p in the right half-plane. Therefore the only singular points of the function $E(p, x)$ in the right half-plane are those values of p, at which the corresponding homogeneous equation

$$E(p, x) = \int_0^l K(p, x, \xi) E(p, \xi)\,d\xi \quad (26)$$

has nontrivial solutions. Consequently, in the case when (26) has no eigenvalues in the right half-plane, the solution of (24) is analytic in the entire right half-plane, i.e., the stationary solution considered here is stable. On the other hand, if this condition is not satisfied, then the solution of (24) has singular points in the right half-plane and the stationary solution is unstable.

We now consider the stationary solution for which the potential has a minimum. In the region $0 \leq x \leq x_0$ the current $j_1(p, x)$ is determined by formulas (19)—(23), in which l and φ_1 are replaced by x_0 and φ_0. Inasmuch as the function

$$U(v_0, s) = \sqrt{v_0^2 - v^2(s)} \qquad \left(v_0 = \sqrt{-2e\varphi_0/m}\right)$$

has at $s = x_0$ a zero of first order, the integrals

$$\int_{x_0}^{\xi} \frac{ds}{U(v_0, s)} \text{ and } \int_{x_0}^{x} \frac{ds}{U(v_0, s)}$$

diverge, and the kernel $H_3(v_0, p, x, \xi)$ is identically equal to zero when $\operatorname{Re} p > 0$. Substituting, as before, the calculated kernels into (16) we obtain an integral equation for the electric field in the region $0 \leq x \leq x_0$:

$$E(p, x) = \int_0^{x_0} K_1(p, x, \xi) E(p, \xi)\,d\xi + h(p, x),$$
$$K_1(p, x, \xi) = H_1(v_0, p, x, \xi) + H_2(v_0, p, x, \xi). \quad (27)$$

It is easy to show that when $x_0 \leq x \leq l$

$$E(p, x) = \int_0^x K_2(p, x, \xi) E(p, \xi)\,d\xi + h(p, x),$$

$$K_2(p, x, \xi) = \frac{\omega_0^2}{p}\int_{v_0}^{\infty} \exp\left(\int_x^{\xi} \frac{pds}{U(u, s)}\right) f_0'(u)\,du. \quad (28)$$

The function $E(p, x)$ is thus determined in the region $0 \leq x \leq x_0$ by the integral equation (27) and is then continued into the region $x_0 \leq x \leq l$ with the aid of the Volterra equation (28). The kernels $K_1(p, x, \xi)$ and $K_2(p, x, \xi)$ and the function $h(p, x)$ are analytic in p in the right half-plane. Therefore the only singular points of the function $E(p, x)$ in the right half-plane are those values of p, for which the homogeneous equation

$$E(p, x) = \int_0^{x_0} K_1(p, x, \xi) E(p, \xi) d\xi \qquad (29)$$

has nontrivial solutions. Consequently, as in the first case, our stationary solution is stable if (29) has no eigenvalues in the right half-plane, and unstable otherwise.

3. STABILITY OF CERTAIN STATIONARY SOLUTIONS

In the present section we consider several cases for which we can prove, under rather general assumptions concerning the properties of the function f_0, that the homogeneous equations (26) and (29) have no eigenvalues in the right half-plane, and the corresponding stationary solutions are therefore stable.

1. Sufficiently low particle density. If the potential of the stationary solution has no minimum, then it is quite obvious from the mathematical point of view that the kernel $K(p, x, \xi)$ [(20)—(23)] has no eigenvalues on the right half-plane if the particle density is sufficiently small, for this kernel is uniformly bounded in all its arguments when $\operatorname{Re} p \geq 0$:

$$|K(p, x, \xi)| \leqslant \omega_0^2 A, \qquad (30)$$

where

$$A = 2\int_{v_1}^{\infty} |f_0'(u)|\, du \int_0^l \frac{ds}{\sqrt{v_1^2 - v^2(s)}} + 2\int_0^{v_1} |f_0'(u)|\, du \int_0^{x(u)} \frac{ds}{\sqrt{u^2 - v^2(s)}}. \qquad (31)$$

It follows from the inequality (30) that when

$$\omega_0^2 A l < 1 \qquad (32)$$

Eq. (26) has no eigenvalues in the right half-plane.

Let us discuss the physical meaning of condition (32). If the potential energy $-e\varphi_1$ of the electrons on the right end is small compared with the average kinetic energy, then the first term in the expression for A is the principal one. In this case

$$A \approx 2f_0(v_1)\, t_1 \approx (N_1/N)\, 2t_1/v_1$$

and condition (32) assumes the form

$$(N_1/N)(\omega_0 2t_1)^2 < 1, \qquad (33)$$

where N_1/N is the fraction of the electrons deflected by the external electric field, $v_1 = \sqrt{-2e\varphi_1/m}$ is the velocity of those electrons at the point $x = 0$ which arrive at the point $x = l$ with zero velocity, and $2t_1$ is the transit time of these electrons from $x = 0$ to $x = l$ and back.

In the opposite case, when the potential energy $-e\varphi_1$ is large compared with the average kinetic energy $mv^2/2$ of the electrons, it is the second term that predominates in (31). In this case $A \approx 4l/v_1^2$ and the stability condition (32) is written in the form

$$4\omega_0^2 l^2/v_1^2 < 1. \qquad (34)$$

We note that condition (34) is less stringent than the requirement that the dimensions of the system be small compared with the Debye radius

$$l^2/D^2 = 2\omega_0^2 l^2/\overline{v^2} < 1,$$

for in the case considered $v_1^2 \gg \overline{v^2}$.

When the potential of the electric field has a minimum, the kernel $K_1(p, x, \xi)$ is no longer uniformly bounded, but it is easy to show that when the particle density is sufficiently low Eq. (29) has likewise no eigenvalues in the right half-plane. The criteria for the stability will be analogous to (33) and (34).

2. There are no reflected particles. Let $f_0(v) = 0$ when $0 \leq v \leq V$, and if the potential of the stationary distribution is monotonic, let $V > v_1$; if the potential has a minimum, let $V > v_0$. In this case all the electrons injected into the system at the plane $x = 0$ have sufficient energy to reach the opposite wall $x = l$. There are no reflected particles. If the distribution function satisfies these conditions, then Eqs. (26) and (29) turn into the Volterra equation

$$E(p, x) = \int_0^x K(p, x, \xi) E(p, \xi) d\xi,$$

$$K(p, x, \xi) = \frac{\omega_0^2}{p} \int_V^{\infty} \exp\left(\int_x^{\xi} \frac{p\,ds}{U(us)}\right) f_0'(u)\, du,$$

which has no eigenvalues.

We note that no limitations are imposed on the function $f_0(v)$ when $v > V$. In particular, the function $f_0(v)$ can have several maxima. In real problems this would lead to a convective instability for an unbounded region. For a bounded region, the concept of convective instability has no meaning, and therefore on going over from convectively unstable unbounded systems to bounded systems the instability either disappears completely (as is the case in the present example), or becomes absolute. The second possibility is realized when sufficiently strong "feedback" exists in the system. It can be shown, for example, that an electron plasma and a beam which comprise a system that is convectively unstable in an unbounded region, can become absolutely unstable in a bounded region because the plasma contains electrons with negative velocities.

3. There are no transmitted particles, and the function $f_0(v)$ is monotonic. Let $f_0(v) = 0$ when $v > V$, with $V < v_1$. In this case the energy of the electrons is insufficient to overcome the decelerating electric field. As a result all the particles are reflected and leave the system at the point of injection in the plane $x = 0$. It is easy to show that in this case the potential of the stationary field should be a monotonic function of x.

Let us consider the integral equation (26). Under the assumption made, $H_3(v_1, p, x, \xi) \equiv 0$. We introduce a new function

$$Y(p,x,u) = \frac{\omega_0^2}{p}\left\{\int_0^x\left[\exp\left(\int_x^\xi \frac{pds}{U(u,s)}\right) - \exp\left(\int_{x(u)}^\xi \frac{pds}{U(u,s)} + \int_{x(u)}^x \frac{pds}{U(u,s)}\right)\right]E(p,\xi)\,d\xi + \int_x^{x(u)}\left[\exp\left(\int_\xi^x \frac{pds}{U(u,s)}\right) - \exp\left(\int_{x(u)}^\xi \frac{pds}{U(u,s)} + \int_{x(u)}^x \frac{pds}{U(u,s)}\right)\right]E(p,\xi)\,d\xi\right\}.$$

This function is defined in the region $0 \le u \le V$, $0 \le x \le x(u) \equiv \psi(-mu^2/2e)$, and by virtue of (26) we have

$$E(p,x) = \int_{v(x)}^V Y(p,x,u)\,f_0'(u)\,du \qquad (0 \leqslant x \leqslant l). \tag{35}$$

The function $Y(p, x, u)$ satisfies the ordinary differential equation

$$\frac{d}{dx}\left[U(u,x)\frac{dY}{dx}\right] - \frac{p^2}{U(u,x)}Y + 2\omega_0^2 E(p,x) = 0 \tag{36}$$

under the following boundary conditions

$$\frac{dY}{dx}(p,0,u) = \frac{p}{U(u,0)}Y(p,0,u), \quad Y(p,x(u),u) = 0. \tag{37}$$

Applying Green's formula to the solution of the boundary problem (36) and (37), we obtain

$$p^2\int_0^{x(u)} \frac{1}{U(u,x)}|Y|^2\,dx + p\,|Y(p,0,u)|^2 + \int_0^{x(u)} U(u,x)\left|\frac{dY}{dx}\right|^2 dx - 2\omega_0^2\int_0^{x(u)} Y^* E(p,x)\,dx = 0. \tag{38}$$

Let the function $f_0(v)$ be non-increasing in the region $0 \le v \le V$, so that $f_0'(v) \le 0$. We multiply (38) by $-f_0'(u)$ and integrate with respect to u from 0 to V. As a result we have

$$p^2A(p) + pB(p) + C(p) = 0, \tag{39}$$

where A, B, and C are real non-negative functions of the complex argument p. When $\operatorname{Re} p > 0$, the equality (39) can be satisfied only if $A = B = C = 0$, i.e., when $E(p, x) \equiv 0$. Thus, in our case, Eq. (26) again can have no eigenvalues in the right half-plane.

It is easy to understand the physical meaning of the result obtained. In the case under consideration, the electron distribution function (8) at any cross section $x = \text{const}$ is a monotonically decreasing function of $|v|$. Unbounded homogeneous systems with distribution functions possessing this property are stable. The proof given shows that a bounded system in an external field is likewise stable.

In conclusion we note that the results obtained are valid also for an electron plasma with a charge that is partially neutralized by a stationary ion background. It is merely important that the stationary distribution of the potential $\varphi(x)$ be of the type considered in Sec. 1, i.e., either a monotonically decreasing function, or a function which has one minimum but no maxima.

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28