ON THE THEORY OF VECTOR PARTICLES WITH ORIENTED SPINS

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The theory of vector particles with nonzero rest mass is considered. The wave function is obtained by taking spin states into account, and elastic scattering is investigated for various types of interaction (scalar, vector, tensor, etc.).

1. THE WAVE FUNCTION

PROBLEMS connected with the development of a theory for boson particles with nonzero rest mass m_0 have recently gained in importance. Interest is attached, in particular, to polarization effects of such particles. In this connection, we develop in the present article a method for the construction of the wave functions of bosons with an account of all the possible spin states, and investigate the behavior of the polarization vector in scattering processes.

To solve the formulated problem, it is advantageous to use a matrix formulation of the theory of the vector particles, in which the wave function ψ is a ten-component column and is determined from a wave equation in the form ($\hbar = c = 1$)

$$\beta_\nu \partial\psi/\partial x_\nu + k_0\psi = 0, \qquad k_0 = m_0, \qquad \nu = 1, 2, 3, 4, \quad (1)$$

where the β_ν are ten-row Duffin-Kemmer matrices which obey the following commutation rules:[1,2]

$$\beta_\rho\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\rho = \delta_{\mu\nu}\beta_\rho + \delta_{\mu\rho}\beta_\nu. \quad (2)$$

To obtain the wave function ψ with an account of all the possible spin states, we use the supplementary condition

$$(\hat{\mathbf{S}}\hat{\mathbf{P}}/P - s)\,\psi = 0, \quad (3)$$

which is compatible with (1), since the operator $\hat{\mathbf{S}}\cdot\hat{\mathbf{P}}/P$ of the projection of the spin vector $\hat{\mathbf{S}}$[3] ($S_n = -i(\beta_k\beta_m - \beta_m\beta_k)$, m, n, k = 1, 2, 3) on the direction of the momentum **P** commutes with the Hamiltonian of free motion

$$\hat{H} = -i\alpha_k\partial/\partial x_k + m_0\beta_4, \qquad \alpha_k = -i(\beta_k\beta_4 - \beta_4\beta_k). \quad (4)$$

The quantity s in (3) is an eigenvalue of the operator $\hat{\mathbf{S}}\cdot\hat{\mathbf{P}}/P$ and determines the spin projection on the direction of **P**.

We seek a simultaneous solution of (1) and (3) in the form of a plane-wave expansion

$$\psi = L^{-3/2}\sum_{s,\varepsilon,\mathbf{k}} C(s,\varepsilon,\mathbf{k})\, u(s,\varepsilon,\mathbf{k}) \exp(-i\varepsilon Kt + i\mathbf{kr}), \quad (5)$$

where the energy is $K = (k^2 + k_0^2)^{1/2}$, $\mathbf{k} = \mathbf{P}$, and $|C(s, \varepsilon, \mathbf{k})|^2$ is the probability of finding the particle in the state $(s, \varepsilon, \mathbf{k})$. The value of ε determines, as is well known [2], the sign of the charge.

From the compatibility of (1) and (2) we can readily obtain the spectrum of the possible values of ε and s. The only allowed values are

$$\varepsilon = \pm 1, \qquad s = 0, \pm 1. \quad (6)$$

Recognizing that the conservation of the boson charge leads to a normalization of the form

$$\int\bar{\psi}\beta_4\psi d^3x = \varepsilon, \qquad \bar{\psi} = \psi^+\eta_4, \qquad \eta_4 = 2\beta_4^2 - 1, \quad (7)$$

we obtain the following components for the matrix u:

for $s = 0$

$$u_1 = -i\varepsilon\sqrt{k_0/2K}\sin\theta\cos\varphi, \quad u_2 = -i\varepsilon\sqrt{k_0/2K}\sin\theta\sin\varphi,$$
$$u_3 = -i\varepsilon\sqrt{k_0/2K}\cos\theta, \quad u_4 = 0, \quad u_5 = 0, \quad u_6 = 0,$$
$$u_7 = \sqrt{K/2k_0}\sin\theta\cos\varphi, \quad u_8 = \sqrt{K/2k_0}\sin\theta\sin\varphi,$$
$$u_9 = \sqrt{K/2k_0}\cos\theta, \quad u_{10} = \varepsilon(k/K)\sqrt{K/2k_0};$$

for $s = \pm 1$

$$u_1 = -{}^1/_2 i\varepsilon\sqrt{K/k_0}(\cos\theta\cos\varphi - is\sin\varphi),$$
$$u_2 = -{}^1/_2 i\varepsilon\sqrt{K/k_0}(\cos\theta\sin\varphi + is\cos\varphi),$$
$$u_3 = {}^1/_2 i\varepsilon\sqrt{K/k_0}\sin\theta,$$
$$u_4 = -{}^1/_2 s(k/K)\sqrt{K/k_0}(\cos\theta\cos\varphi - is\sin\varphi),$$
$$u_5 = -{}^1/_2 s(k/K)\sqrt{K/k_0}(\cos\theta\sin\varphi + is\cos\varphi),$$
$$u_6 = {}^1/_2 s(k/K)\sqrt{K/k_0}\sin\theta,$$
$$u_7 = {}^1/_2\sqrt{k_0/K}(\cos\theta\cos\varphi - is\sin\varphi),$$
$$u_8 = {}^1/_2\sqrt{k_0/K}(\cos\theta\sin\varphi + is\cos\varphi),$$
$$u_9 = -{}^1/_2\sqrt{k_0/K}\sin\theta, \quad u_{10} = 0, \quad (8)$$

where θ and φ are the angular coordinates of the vector **k**. It is easy to verify that the u_i satisfy the equality

$$\sum_{\nu=1}^{10} u_\nu^+(s', \varepsilon', \mathbf{k})\, \beta_4 u_\nu(s, \varepsilon, \mathbf{k}) = \delta_{ss'}\delta_{\varepsilon\varepsilon'}. \qquad (9)$$

We shall henceforth consider particles with positive charge ($\varepsilon = 1$).

2. TRANSFORMATION PROPERTIES OF THE SPIN VECTOR

By representing the wave function in the form of an expansion in the spin states we can obtain rather simple expressions for the mean values of the spin-vector components and investigate polarization effects in various processes in which oriented bosons participate. In particular, we can consider the behavior of the spin in elastic scattering, and also establish the laws that govern the changes in its components under Lorentz rotations.

Assuming, in accordance with the general averaging rule in the Duffin-Kemmer theory, that the mean values of the spin vector components are

$$S_n = [\int \bar{\psi}\beta_4 \hat{S}_n \psi d^3x]\, S_0^{-1}, \qquad (10)$$

we obtain with the aid of (10) for the transverse components $S_{1,2}$ (perpendicular to the momentum vector) and the longitudinal component S_3 (parallel to the momentum):

$$S_1 = (f/2\sqrt{2}\, S_0)\{(C_{-1}^+ - C_1^+)\, C_0 + C_0^+\, (C_{-1} - C_1)\},$$
$$S_2 = (if/2\sqrt{2} S_0)\{(C_1^+ + C_{-1}^+)\, C_0 - C_0^+\, (C_{-1} + C_1)\},$$
$$S_3 = (C_1^+ C_1 - C_{-1}^+ C_{-1})/S_0, \quad S_0 = C_1^+ C_1 + C_{-1}^+ C_{-1} + C_0^+ C_0,$$
$$f = k_0/K + K/k_0. \qquad (11)$$

For a complete description of the quantities S_n we must ascertain their transformation properties. For this purpose we use the law for the transformation of the wave function ψ

$$\psi' = S\psi \qquad (12)$$

under a Lorentz rotation of the coordinate system, with the matrix S for Lorentz rotations defined as in [3,4]. Without loss of generality, we can assume the particle momentum **P** to lie in the xz plane, and the primed system of coordinates to move relative to the unprimed (old) system along the z axis.

Expressing with the aid of (12) the amplitudes $C'(s')$ in the new coordinate system in terms of the amplitudes $C(s)$ in the old system, and using formulas (11) in which all the unprimed quantities are replaced by primed ones, we obtain (see [3]) the following formulas for the transformed components S'_n:

$$S'_1 = \{S_1(\beta_1 - \beta\cos\theta) - S_3\beta(1 - \tfrac{1}{2}\beta_1^2)\sin\theta\}\,\gamma\Gamma,$$
$$S'_2 = S_2\gamma,$$
$$S'_3 = \left\{S_3(\beta_1 - \beta\cos\theta) + 2S_1\beta\frac{1-\beta_1^2}{2-\beta_1^2}\sin\theta\right\}\Gamma;$$
$$\Gamma = [(1 - \beta\beta_1\cos\theta)^2 - (1-\beta^2)(1-\beta_1^2)]^{-1/2},$$
$$\gamma = [(1 - \beta\beta_1\cos\theta)^2 + (1-\beta^2)(1-\beta_1^2)]/(1-\beta\beta_1\cos\theta)(2-\beta_1^2)(1-\beta^2)^{1/2}, \qquad (13)$$

$\beta_1 = k/K$, β is the coefficient of translation along the z axis, and θ is the angle between the vector **k** and the z axis in the old coordinate system.

3. ELASTIC SCATTERING

As already noted, the formalism developed above can be used to investigate various processes in which oriented vector particles participate.

By way of an application, let us consider the elastic scattering of polarized particles on a scattering center for different interaction variants: vector (V), pseudo-vector (A), tensor (T), pseudo-scalar, etc., where the pseudo-scalar and pseudo-vector matrices have been determined earlier [4].

We carry out the analysis in first-order perturbation-theory approximation, where the effective scattering cross section has the form

$$d\sigma_{ss'} = (K/2\pi)^2 C_{s'}^{'+} C'_{s'}\, d\Omega,$$
$$C'_{s'} = \sum_{s,\nu} [u^+(s', \mathbf{k}')\, \eta_4\gamma_\nu\, u(s, \mathbf{k})] V_{\mathbf{kk}'}^{(\nu)} C_s,$$
$$V_{\mathbf{kk}'}^{(\nu)} = \int \exp\{i\mathbf{r}(\mathbf{k} - \mathbf{k}')\}\, V^{(\nu)}(\mathbf{r})\, d^3x. \qquad (14)$$

The matrices γ_ν describe here the interaction variant

$$W = \sum_\nu \gamma_\nu V^{(\nu)}(\mathbf{r}).$$

The mean values of the spin components S'_n after the scattering are again expressed by formulas (11), in which the unprimed C_i must be replaced by the primed C'_i, defined in accordance with (14).

The relative orientation of the spin and momentum of the scattered particle will be characterized by an angle α such that

$$\operatorname{tg}\alpha = \sqrt{S_1'^2 + S_2'^2}\,/\,S'_3. \qquad (15)^*$$

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*tg α = tan α.

A. Scattering of longitudinally-polarized particles. Let us consider the scattering of particles with spin either parallel ($S_3 = 1$) or antiparallel to the momentum ($S_3 = -1$). Then, for an interaction determined by the spatial part of the four-vector $V = i\beta_\mu$ (such an interaction takes place, for example, in the scattering of a charged particle by a magnetic moment), we obtain

$$\operatorname{tg}\alpha = \frac{1}{2}\left(1 + \frac{K^2}{k_0^2}\right)\operatorname{tg}\frac{\theta}{2}, \quad (16)$$

$$S_0' = \left(\frac{1}{2}\frac{k^2}{k_0^2}\sin^2\frac{\theta}{2} + \frac{k^2}{K^2}\cos^2\frac{\theta}{2}\right) \times \left|\left(V_{kk'}^{(1)} + iV_{kk'}^{(2)}\right)\sin\frac{\theta}{2} + V_{kk'}^{(3)}\cos\frac{\theta}{2}\right|^2, \quad (17)$$

where θ is the scattering angle (the quantity S_0' determines the scattering cross section).

In a case when the interaction is characterized by the temporal component of the four-vector ($V_4 = i\beta_4$, Coulomb scattering) we have

$$\operatorname{tg}\alpha = \frac{1}{4}f^2\operatorname{tg}\theta; \qquad S_0' = \frac{1}{2}\left[(1+\cos^2\theta) + \frac{1}{4}f^2\sin^2\theta\right]|V_{kk'}|^2. \quad (18)$$

For a scalar interaction we have

$$\operatorname{tg}\alpha = \frac{(1 + k_0^2/K^2)\sin\theta}{(1 + k_0^2/K^2)\cos\theta - (1 - k_0^2/K^2)},$$

$$S_0' = \frac{1}{2}\left[\sin^2\theta + \frac{k_0^2}{2K^2}(1+\cos\theta)^2 + \frac{K^2}{2k_0^2}(1-\cos\theta)^2\right]|V_{kk'}|^2. \quad (19)$$

It is seen from (16) and (19) that in the nonrelativistic approximation, as in the case of Dirac particles [5,6], we have $\tan\alpha = \tan\theta$, i.e., the spin retains its initial direction.

B. Scattering of transversely-polarized particles. The effective scattering cross section of transversely-polarized particles depends in all the interaction variants (with the exception of the scalar one) on the azimuthal angle φ, i.e., azimuthal asymmetry obtains. For example¹⁾ in the case of Coulomb scattering we have

$$S_0' = \frac{1}{2}\left\{1 + \cos^2\theta + \frac{1}{4}f^2\sin^2\theta + \frac{k^4}{4k_0^2K^2}\sin^2\theta\sin^2\varphi\right\}|V_{kk'}|^2,$$

and we have for the spatial part of the vector interaction

$$S_0' = \frac{1}{2}\left|\left(V_{kk'}^{(1)}\sin\frac{\theta}{2} + V_{kk'}^{(3)}\cos\frac{\theta}{2}\right)\left(\frac{k}{k_0}\sin\frac{\theta}{2}\sin\varphi - i\frac{k}{K}\cos\frac{\theta}{2}\right) - V_{kk'}^{(2)}\frac{k}{k_0}\sin^2\frac{\theta}{2}\cos\varphi\right|^2 + \frac{i}{2}\left[\left(V_{kk'}^{+(1)}V_{kk'}^{(2)} - V_{kk'}^{(1)}V_{kk'}^{+(2)}\right)\sin^2\frac{\theta}{2} + \left(V_{kk'}^{+(3)}V_{kk'}^{(2)} - V_{kk'}^{(3)}V_{kk'}^{+(2)}\right)\sin\theta\right]\frac{k^2}{k_0K}\sin\theta\cos\varphi + \frac{1}{2}\left[\left(V_{kk'}^{+(1)}V_{kk'}^{(1)} + V_{kk'}^{+(2)}V_{kk'}^{(2)}\right)\sin^2\frac{\theta}{2} + V_{kk'}^{+(3)}V_{kk'}^{(3)}\cos^2\frac{\theta}{2} + \frac{1}{2}\left(V_{kk'}^{+(1)}V_{kk'}^{(3)} + V_{kk'}^{(1)}V_{kk'}^{+(3)}\right)\sin\theta\right]\left(\frac{k^2}{k_0^2}\sin^2\frac{\theta}{2} + \frac{k^2}{K^2}\cos^2\frac{\theta}{2}\right).$$

¹⁾As in the case of item A, the other results are too cumbersome to be given here (see [4]).

The dependence of the cross sections on the angle φ is evidence that the scattering system is an "analyzer" for the polarization. We note also that in the V, S, and T interactions the particles acquire longitudinal polarization as a result of the scattering. An exception for V and S interactions is the case when the spin is oriented prior to scattering perpendicular to the scattering plane ($\varphi = \pi/2$) and also the case $\theta = 0$. For these values of θ and φ, the initial (transverse) polarization is conserved.

C. Scattering in combined interaction. In the case when combined interaction takes place, the matrix γ, which determines the character of this interaction, is a linear combination of the matrices considered above. Such an interaction is realized, for example, when particles with charge e, which have also a magnetic moment μ, are scattered by a stationary center which carries a charge e' and a magnetic moment μ'. In the particular case when the scattering is on a Coulomb center²⁾ ($\mu' = 0$), the effective cross sections turn out to be*

$$d\sigma = \frac{K^2}{k_0^2}\frac{e'^2 d\Omega}{4k^4}\left\{e^2k_0^2\operatorname{ctg}^4\frac{\theta}{2} + (ek_0 - 2k^2\mu)^2 + \frac{1}{2K^2}\operatorname{ctg}^2\frac{\theta}{2}\left[e(k_0^2 + K^2) - 2\mu k^2k_0\right]^2\right\}$$

in the scattering of longitudinally polarized particles, and

$$d\sigma = \frac{K^2e'^2d\Omega}{8k^4\sin^4(\theta/2)}\left\{\left[\left(e\cos\theta + 2\frac{\mu k^2}{k_0}\sin^2\frac{\theta}{2}\right)^2 + \frac{1}{4}\sin^2\theta\left(ef - 2\frac{\mu k^2}{K}\right)^2\right] \times (1 + \sin^2\varphi) + \left(e - 2\frac{\mu k^2}{k_0}\sin^2\frac{\theta}{2}\right)^2\cos^2\varphi\right\}$$

in the scattering of transversely-polarized particles.

It follows from these expressions that at low energies ($k_0/K \sim 1$), the principal role will be played by Coulomb interaction. With increasing energy ($\mu k \gg e$), the dominant role is assumed by the tensor forces which arise when the charge interacts with the magnetic field.

The proposed formalism and some of the results obtained on its basis may be useful in the analysis of possible processes involving particles with unity spin.

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*ctg = cot.

²⁾Scattering by a magnetic moment has been considered in [4].

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