SPECTRAL REPRESENTATIONS OF MATRIX ELEMENTS

R. V. TEVIKYAN

Physics Institute, Academy of Sciences, Armenian S.S.R.

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Spectral representations are obtained for the matrix elements of the product of n scalar Heisenberg operators.

This paper presents a generalization of the integral representation of Dyson,\(^3\) based on the methods of Schwinger\(^1\) and Gribov,\(^2\) in which anomalous regions of integration do not arise.

1. Consider the mean in vacuum of the product of three scalar operators

\[ F_{123}^{(c)}(x_{12}, x_{23}) = \langle 0 | q_1(x_1) q_2(x_2) q_3(x_3) | 0 \rangle, \]

where \( x_{ik} = x_i - x_k \). The function (1) contains only positive frequencies and consequently is analytic relative to time coordinates in the region

\[ x_{12}^0 \to x_{12}^0 - i\epsilon_1, \quad \epsilon_1 > 0, \]
\[ x_{23}^0 \to x_{23}^0 - i\epsilon_2, \quad \epsilon_2 > 0, \]

where the \( \epsilon \) are arbitrary positive constants, which we consider to be infinitesimally small.

According to (2), this function will have a spectral representation with a factor in the integrand if

\[ \int _{\epsilon_3 > 0} \frac{d\epsilon_3}{2\pi} \int \frac{d\epsilon_2}{2\pi} \int \frac{dk}{2\pi} e^{ikx - i\epsilon_3}, \]

in the integrand if

\[ \alpha_1 x_{12}^0 + \alpha_2 x_{23}^0 > 0, \quad \beta_1 x_{12}^0 + \beta_2 x_{23}^0 > 0, \]

i.e.,

\[ F_{123}^{(c)}(x_{12}, x_{23}) = \int \exp \left(-i\alpha_1 x_{12}^0 - i\alpha_2 x_{23}^0 - i\alpha_3 x_{13}^0 \right) \theta(x_{12}^0) \theta(x_{23}^0) \]
\[ + \int \exp \left(-i\beta_1 x_{12}^0 - i\beta_2 x_{23}^0 \right) \theta(x_{12}^0) \theta(x_{23}^0) \psi_{123}(x_1, x_2, x_3) \psi_{123}(x_1, x_2, x_3) \]
\[ \times d\alpha_1 d\beta_1 d\alpha_2 d\beta_2 d\alpha_3 d\beta_3. \]

The spectral representation of the T-product can be obtained by assuming \( x_{12}^0 > 0 \) and \( x_{23}^0 > 0 \) in (4). In this case, it follows from (4) and the symmetry properties of the T-product that

\[ F_{123}^{(c)}(x_{12}, x_{23}) = \int \exp \left(-i\alpha x_{12}^0 - i\beta x_{23}^0 \right) \psi_{123}(x_1, x_2, x_3) \psi_{123}(x_1, x_2, x_3) \]
\[ \times d\alpha d\beta d\gamma, \]

Introducing the Fourier transform of the function

\[ \psi_{123}(1/4\alpha_1, 1/4\alpha_2, 1/4\alpha_3), \]
\[ \psi_{123}(x_1, x_2, x_3) = (2\pi)^3 \int \exp \left\{ -x_{12}^2 \frac{\alpha_1^2}{4\epsilon_1} - x_{13}^2 \frac{\alpha_2^2}{4\epsilon_2} - x_{23}^2 \frac{\alpha_3^2}{4\epsilon_3} \right\} \]
\[ \times I_{123}(x_{12}^2, x_{13}^2, x_{23}^2) d\alpha_1 d\alpha_2 d\alpha_3, \]

we obtain

\[ F_{123}^{(c)}(x_{12}, x_{23}) \]
\[ = (2\pi)^6 \int_0^\infty D^{(c)}(x_{12}, x_{12}) D^{(c)}(x_{13}, x_{13}) D^{(c)}(x_{23}, x_{23}) \]
\[ \times I_{123}(x_{12}^2, x_{13}^2, x_{23}^2) d\alpha_1 d\alpha_2 d\alpha_3; \]

\[ D^{(c)}(x, m) = \frac{1}{(2\pi)^3} \int e^{ikx - \frac{1}{2}m^2 - \frac{1}{2}k^2 - i\epsilon} dk; \]

the parameters \( k^2 \) take on only positive values, since they characterize the mass spectra.

From considerations of relativistic invariance, it follows that the conditions (3) in (4) can be replaced by the requirement

\[ \alpha_1 x_{12}^0 > 0, \quad \alpha_2 x_{23}^0 > 0, \quad \alpha_3 x_{13}^0 > 0 \]

and one can write

\[ F_{123}^{(c)}(x_{12}, x_{23}) = (2\pi)^6 \int_0^\infty D^{(c)}(x_{12}, x_{12}) D^{(c)}(x_{13}, x_{13}) D^{(c)}(x_{23}, x_{23}) \]
\[ \times I_{123}(x_{12}^2, x_{13}^2, x_{23}^2) d\alpha_1 d\alpha_2 d\alpha_3; \]

\[ D^{(c)}(x, m) = \frac{i}{(2\pi)^3} \int e^{ikx} \left( -k^2 - \frac{1}{2}m^2 \right) \delta (k^2 - m^2) dk. \]

2. Let us turn to a consideration of the matrix element of the product of three operators

\[ F_{123}(x_{12}, x_{23}) = \langle P | q_1(x_1 - \bar{x}) q_2(x_2 - \bar{x}) q_3(x_3 - x) | Q \rangle, \]

where \( \bar{x} = (x_1 + x_2 + x_3)/3 \), the prime indicates calculation only of connected diagrams, and \( P \) and \( Q \) are the total momenta of the arbitrary states \( | P \rangle \) and \( | Q \rangle \). From the spectral condition it follows that

\[ F_{123}(x_{12}, x_{23}) = \int e^{-i\alpha x_{12}^0 - i\beta x_{23}^0} \theta(2P + Q + p_1) \theta(2P + Q + p_2) \]
\[ \times \hat{F}_{123}(p_1, p_2) dp_1 dp_2, \]

where

\[ \hat{F}_{123}(p_1, p_2) \neq 0 \]

for

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\[
\left(\frac{1}{2}(2P + Q) + p_3\right)^2 \geq (m_{123}^2)^2, \quad \frac{1}{2}(2P + Q) + p_1 \in L^*; \hfill (\frac{1}{2})(2P + Q) + p_2 \geq (m_{12}^2)^2, \quad \frac{1}{2}(2P + Q) + p_2 \in L^*; \hfill
\]

where the \( m \) are the minimum masses of the intermediate states, and \( \theta(p) = \theta(p^0) \theta(p^3) \) is an invariant discontinuous function.

Function (6) can be written in the form

\[
F_{123}(x_{12}, x_{23}) = \int \exp \left( -ix_{12}p_{12} - ix_{13}(p_{13} - p_{12}) - ix_{23}p_{23} - ix_{24}p_{24} \right) dt_{12}, dt_{13}, dt_{23}, dt_{24},
\]

from which we obtain

\[
\tilde{F}_{123}(\rho_1, \rho_2) = \int \delta(\rho_1 - k_{12} - k_{13} - u_1) \delta(\rho_2 - k_{13} - k_{23} - u_2) \times \tilde{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2) dt_{12}, dt_{13}, dt_{23}, dt_{24},
\]

(10)

The occurrence of spectral representations of type (7) means that we can set

\[
\tilde{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2) = \theta(k_{12}) \theta(k_{13}) \theta(k_{23}) \left( \frac{1}{2}(2P + Q) + u_1 \right) \times \theta(\frac{1}{2}(2P + Q) + u_2) \times \tilde{F}_{123}(k_{12}, k_{13}, k_{23}, u_1, u_2),
\]

(11)

Using (11), we obtain

\[
F_{123}(x_{12}, x_{23}) = \int_{0}^{\infty} D^{(c)}(x_{12}, x_{123}) \times D^{(c)}(x_{23}, x_{123}) I_{123}(x_{12}, x_{23}, x_{123}) \times D^{(c)}(x_{12}, x_{123}) \times D^{(c)}(x_{23}, x_{123}),
\]

(12)

where \( I_{123} \) is the Fourier transform of \( \tilde{T}_{123} \) in the variables \( u_1 \) and \( u_2 \). From the causality condition it follows that the functions

\[
I_{123}(x_{12}, x_{23}, x_{123}); \quad \delta(\rho_1 + \rho_2 + p_3) \tilde{T}_{123}(x_{12}, x_{23}, x_{123}) = m_{123}^2 \quad \text{are symmetrical relative to the indices (1, 2, 3),}
\]

(13)

From Eqs. (9)—(11) and the symmetry properties of the function \( I_{123} \) it follows that

\[
x_{12} + x_{23} \geq \max \left\{ 0, \max (m_{12}^2, m_{23}^2) \right\}
\]

\[
\left( \frac{1}{2}(2P + Q) + u_1 \right)^2 \geq (m_{12}^2)^2, \quad \frac{1}{2}(2P + Q) + u_1 \in L^*; \hfill (\frac{1}{2})(2P + Q) + u_2 \geq (m_{23}^2)^2, \quad \frac{1}{2}(2P + Q) + u_2 \in L^*;
\]

\[
\left( \frac{1}{2}(2P + Q) + u_1 \right)^2 \geq (m_{12}^2)^2, \quad \frac{1}{2}(2P + Q) + u_1 \in L^*; \hfill (\frac{1}{2})(2P + Q) + u_2 \geq (m_{23}^2)^2, \quad \frac{1}{2}(2P + Q) + u_2 \in L^*;
\]

3. A matrix element of general form

\[
F_{12 \ldots n}(x_{12}, x_{23}, \ldots, x_{n-1,n}) = \langle P | q_1(x_1 - x) q_2(x_2 - x) \ldots q_n(x_n - x) | Q \rangle,
\]

(14)

where \( F_{12 \ldots n} \) is symmetric with respect to the indices \((1, 2, \ldots, n)\) [the number of its arguments \( \kappa^n \) equals \( n(n-1)/2 \)] and has a Fourier transform

\[
\tilde{T}_{12 \ldots n}(x_{12}^2, x_{23}^2, \ldots, x_{n-1,n}^2, u_1, u_2, \ldots, u_{n-1,n}) = 0
\]

(15)

for

\[
x_{12} + x_{13} + \ldots + x_{n-1,n} \geq 0
\]

\[
\left( \frac{1}{2}(2P + Q) + u_1 \right)^2 \geq (m_{12}^2)^2, \quad \frac{1}{2}(2P + Q) + u_1 \in L^*; \hfill (\frac{1}{2})(2P + Q) + u_2 \geq (m_{23}^2)^2, \quad \frac{1}{2}(2P + Q) + u_2 \in L^*;
\]

\[
\left( \frac{1}{2}(2P + Q) + u_1 \right)^2 \geq (m_{12}^2)^2, \quad \frac{1}{2}(2P + Q) + u_1 \in L^*; \hfill (\frac{1}{2})(2P + Q) + u_2 \geq (m_{23}^2)^2, \quad \frac{1}{2}(2P + Q) + u_2 \in L^*;
\]

Consideration of the symmetry properties of \( F_{12 \ldots n} \) brings in additional limitations on the variables \( \kappa \). In particular, when \( P \) and \( Q \) correspond to vacuum states, then the functions \( F_{12 \ldots n} \) and \( I_{12 \ldots n} \) become identical with each other and depend only on the variables \( \kappa^2 \).


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