RELATIVISTIC GENERAL THEORY OF REACTIONS OF THE a + b → c + d + e + . . . *TYPE*

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A relativistic theory is constructed for reactions in which there are three or more product particles. This is an extension of the theory of reactions of the type a + b → c + d in the form given by Jacob and Wick and by Chou Kuang-Chao. In this case it has been found necessary to use other variables instead of the Jacobi variables for the relative momenta of the particles. The selection rules for such reactions that follow from parity-conservation are written out.

1. There is a well known form of relativistic general theory of reactions of the type a + b → c + d (that is, there are general expressions for the angular distribution and the polarization in terms of phase shifts), which uses the description of the spin state in terms of projections along the momenta of the particles.[1-3] Besides being relativistic, this form of the theory has the great advantage that its formulas are less cumbersome as compared with the widely used formulas of phase-shift analysis (which are used, for example, in the phase-shift analysis of p-p scattering).

In the present paper the method of Chou Kuang-Chao, Jacob, and Wick, and also that of a previous paper by the writer,[1-3] are applied to construct the theory of reactions of the type a + b → c + d + e + . . . (and also a → c + d + e + . . .). The spins and rest masses of the particles are arbitrary. It is shown that in the relativistic theory it is necessary to use instead of the variables of Jacobi (cf. e.g., Fabri[4]) different momentum variables, which were first used by Dalitz for the description of τ decay.[5]

2. Let us begin with the case of three particles a + b → 1 + 2 + 3. The main propositions of the theory of reactions are presented, for example, in the papers by Jacob and Wick[2] and by the writer.[6]

First of all we must introduce instead of the momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ of the three particles the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$ and two relative momenta $\mathbf{p}$ and $\mathbf{p}'$ (whose exact definition is given later). The conservation of $\mathbf{P}$ allows us to eliminate this variable from the description of the reaction.[6] Hereafter we shall suppose that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$.

Our main problem is to find the transformation functions $\langle \mathbf{p}\mathbf{p}' m_1 m_2 m_3 | \ldots JM \rangle$ which enable us to go from the description in terms of experimentally measured quantities (the momenta of the particles) to a description in a representation that involves the conserved total angular momentum J and its projection M. The analogous function for two particles is of the form[1-3]

$$\langle \mathbf{p} m_1 m_2 | m_1' m_2' \tilde{p} j m \rangle = \sqrt{\frac{2j+1}{4\pi}} D^j_{m_1+m_2,\, m}(-\pi, \vartheta, \pi - \varphi) \times \delta_{p,\tilde{p}} \delta_{m_1, m_1'} \delta_{m_2, m_2'} q(m_1, m_2, p). \qquad (1)$$

The total angular momentum j of the two particles and its projection m are referred to the center-of-mass system (c.m.s.) of the two particles, with arbitrarily chosen axes z′, y′, x′. ϑ and φ are the spherical angles of the relative momentum $\mathbf{p}$ of the two particles in this system of axes; m_1 and m_2 are the spin projections of particles 1 and 2, referred to the system of axes $z_c \parallel \mathbf{p}$ and $y_c \parallel [\mathbf{z}' \times \mathbf{p}]$ (referred to Lorentz frames which are the rest systems of the particles), and m_1' and m_2' are quantized relative to the same axes z_c, y_c, x_c, but are referred to the c.m.s. of the two particles. The factor q appears as a result of the transformation of the spin variables from the rest systems of the particles to the c.m.s.[3]

To solve the problem with three particles we can in principle go from the variables ($\mathbf{p} m_1 m_2$) to the variables ($m_1' m_2' j m$) and then from ($\mathbf{p}' m m_3$) to ($m m_3 J M$), regarding the system of particles 1 and 2 as if it were a single particle with the spin variable j. If, however, we use the Jacobi variables (for example, $\mathbf{p}_1 = \mathbf{p}'/2 + \mathbf{p}$, $\mathbf{p}_2 = \mathbf{p}'/2 - \mathbf{p}$, and $\mathbf{p}_3 = -\mathbf{p}'$), the spin projections along $\mathbf{p}$ will not be projections along the momenta of the particles (helicities). And in the formula (1) m_1 and m_2 must be such projections.

Let $\mathbf{p}$ be the momentum of particle 1 in the Lorentz system $K_{1,2}$ in which the total momentum of particles 1 and 2 is zero.[5] Then $-\mathbf{p}$ is the momentum of particle 2. For the total momentum of the system (1, 2, 3) to be zero, the total momentum of the system (1, 2) (relative to the c.m.s. $K_{1,2,3}$ of the three particles) must be equal to $-\mathbf{p}_3$ or $+\mathbf{p}'$. The velocity β of the system $K_{1,2}$ relative to $K_{1,2,3}$ is $\mathbf{p}'/E_{1,2}$, where $E_{1,2} = (p'^2 + \kappa_{1,2}^2)^{1/2}$, where $\kappa_{1,2}$ is the "mass" of the system (1, 2) ($\kappa_{1,2}$ is defined in the usual way as the total energy of (1, 2) in the Lorentz system in which (1, 2) is at rest:

$$\varkappa_{1,2} = \sqrt{p^2 + \varkappa_1^2} + \sqrt{p^2 + \varkappa_2^2};$$

the speed of light is taken to be unity). The momenta of particles 1 and 2 in $K_{1,2,3}$ are obtained by Lorentz transformation (cf. Sec. 18 in the book by Møller[7]):

$$\mathbf{p}_1 = \mathbf{p} - \mathbf{p}'\left[\frac{\mathbf{p}'\mathbf{p}}{p'^2}\left(\frac{E_{1,2}}{\varkappa_{1,2}} - 1\right) - \frac{\sqrt{p^2+\varkappa_1^2}}{\varkappa_{1,2}}\right],$$

$$\mathbf{p}_2 = -\mathbf{p} - \mathbf{p}'\left[-\frac{\mathbf{p}'\mathbf{p}}{p'^2}\left(\frac{E_{1,2}}{\varkappa_{1,2}} - 1\right) - \frac{\sqrt{p^2+\varkappa_1^2}}{\varkappa_{1,2}}\right]. \quad (2)$$

To find $\mathbf{p}$, knowing the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, we must make the inverse transformation

$$\mathbf{p} = \mathbf{p}_1 + (\mathbf{p}_1 + \mathbf{p}_2)\left[\frac{(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{p}_1)}{(\mathbf{p}_1 + \mathbf{p}_2)^2}\left(\frac{E_{1,2}}{\varkappa_{1,2}} - 1\right) - \frac{\sqrt{p_1^2+\varkappa_1^2}}{\varkappa_{1,2}}\right],$$

$$E_{1,2} = \sqrt{p_1^2 + \varkappa_1^2} + \sqrt{p_2^2 + \varkappa_2^2},$$

$$\varkappa_{1,2} = \sqrt{E_{1,2}^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2}. \quad (3)$$

Thus if Eq. (1) is written for particles 1 and 2, then $\mathbf{p}$ must be the Dalitz variable, and m_1', m_2', j, and m must refer to the Lorentz system $K_{1,2}$. The transformation function from $(\mathbf{p}'mm_3)$ to $(m'm_3'JM)$ is also of the form of Eq. (1), and the required function is given by

$$\sum_m \langle \mathbf{p}m_1m_2 \,|\, m_1'm_2'\tilde{\mathbf{p}}jm\rangle \langle \mathbf{p}'mm_3 \,|\, m'm_3'\tilde{\mathbf{p}}'JM\rangle$$

$$= \sqrt{\frac{2j+1}{4\pi}} D^j_{m_1+m_2\, m'}(-\pi, \vartheta, \pi - \varphi)$$

$$\times \delta_{p,\tilde{p}}\, \delta_{m_1, m_1'}\, \delta_{m_2, m_2'}\, q(m_1, m_2, p)$$

$$\times \sqrt{\frac{2J+1}{4\pi}} D^J_{m'+m_3,\, M}(-\pi, \vartheta', \pi - \varphi')$$

$$\times \delta_{p', \tilde{p}'}\, \delta_{m_3, m_3'}\, q(m', m_3, p')$$

$$\equiv \langle \mathbf{p}\mathbf{p}'m_1m_2m_3 \,|\, m_1'm_2'm_3'jm'JM\tilde{\mathbf{p}}\tilde{\mathbf{p}}'\rangle. \quad (4)$$

In this formula J, M, j, m', and m_3' refer to the system $K_{1,2,3}$, M is quantized relative to some axis system z, y, x, and ϑ' and φ' are the spherical angles of $\mathbf{p}'$ relative to z, y, x. The projections m_3 and m, and also m_3' and m', are quantized relative to a system of axes with $z' \parallel \mathbf{p}'$ and $y' \parallel [\mathbf{z} \times \mathbf{p}']$; ϑ and φ are the spherical angles of $\mathbf{p}$ in this same system of axes, i.e., ϑ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. Finally, m_1, m_2, m_3 relate to the rest systems of the respective particles.

An important further virtue of the Dalitz variables is the possibility of expressing the total energy of the system of three particles as a function of $|\mathbf{p}|$ and $|\mathbf{p}'|$ only:

$$E = E_{1,2} + \sqrt{p'^2 + \varkappa_3^2} = [p'^2 + (\sqrt{p^2 + \varkappa_2^2} + \sqrt{p^2 + \varkappa_1^2})^2]^{1/2} + \sqrt{p'^2 + \varkappa_3^2}. \quad (5)$$

In the case of Jacobi variables E would also depend on the angle between the relative momenta, and the law of conservation of energy could not be written in the representation containing the absolute values of the momenta and J, M.

We can now write the following expression for an element of the S matrix (or of the matrix R = S − 1) of the reaction a + b → 1 + 2 + 3:

$$\langle \mathbf{p}\mathbf{p}'m_1m_2m_3 \,|\, S \,|\, \mathbf{p}_a m_a m_b\rangle = \frac{1}{4\pi\sqrt{4\pi}} \sum_{j,m,J,M} (2J+1)\sqrt{2j+1}$$

$$\times D^j_{m_1+m_2,m}(-\pi, \vartheta, \pi - \varphi)\, D^J_{m+m_3,M}(-\pi, \vartheta', \pi - \varphi')$$

$$\times \langle m_1m_2m_3jmp \,|\, S^{JE} \,|\, m_am_b\rangle\, D^{*J}_{m_a+m_b,M}(-\pi, \vartheta_a, \pi - \varphi_a). \quad (6)$$

The functions q are included in the notation $\langle\,|\, S^{JE} \,|\,\rangle$. More exactly, after going over to a representation containing J and M, by their aid we can return from the representation in the projections m_1', m_2', m_3', and m' [see the explanation of Eq. (4)] to the representation in m_1, m_2, m_3, and m (cf. reference 3).

We can perform[6] the summation over M and get

$$\sum_M D^J_{m+m_3,M}(-\pi, \vartheta', \pi - \varphi')\, D^{*J}_{m_a+m_b,M}(-\pi, \vartheta_a, \pi - \varphi_a)$$

$$= D^J_{m+m_3,m_a+m_b}(-\pi, \tilde{\vartheta}', \pi - \tilde{\varphi}'), \quad (7)$$

where $\tilde{\vartheta}'$ and $\tilde{\varphi}'$ are the spherical angles of $\mathbf{p}'$ in the axis system z_a, y_a, x_a relative to which m_a and m_b are quantized (in particular, $z_a \parallel \mathbf{p}_a$). Therefore we can write element (6) in the form

$$\langle m_1m_2m_3\mathbf{p}\mathbf{p}' \,|\, S(\vartheta, \varphi, \vartheta', \varphi') \,|\, m_am_b\mathbf{p}_a\rangle$$

(here and in what follows we omit the signs ~ over ϑ' and φ').

The formula for a → 1 + 2 + 3 has the analogous form

$$\langle \mathbf{p}\mathbf{p}'m_1m_2m_3 \,|\, S \,|\, JM\rangle = \frac{1}{4\pi} \sum_{j,m} \sqrt{(2j+1)(2J+1)}\, D^j_{m_1+m_2,m}$$

$$\times(-\pi, \vartheta, \pi - \varphi)$$

$$\times D^J_{m+m_3,M}(-\pi, \vartheta', \pi - \varphi')\, \langle m_1m_2m_3jmp \,|\, S^{J\varkappa} \,|\,\rangle. \quad (8)$$

Here J denotes the spin of the decaying particle, M its projection with respect to some axis system, and κ the rest mass of the decaying particle.

3. With the usual method of compounding angular momenta by means of Clebsch-Gordan coefficients we get for $a + b \rightarrow 1 + 2 + 3$ the following formula (if the rest masses of all particles are different from zero):

$$\langle \mathbf{p}\mathbf{p}' n_1 n_2 n_3 | S | \mathbf{p}_a n_a n_b \rangle = \sum Y_{l\mu}(\vartheta, \varphi) Y_{l'\mu'}(\vartheta', \varphi') \langle i_1 i_2 n_1 n_2 | i\sigma \rangle$$
$$\times \langle i l \sigma \mu | j n_j \rangle \langle j i_3 n_j n_3 | s' n' \rangle \langle s' l' n' \mu' | JM \rangle$$
$$\times \langle i l j s' l' p | S^{JE} | s l_a p_a \rangle \langle s l_a n \mu_a | JM \rangle$$
$$\times \langle i_a i_b n_a n_b | sn \rangle Y^*_{l_a \mu_a}(\vartheta_a, \varphi_a). \quad (9)$$

In Eq. (9) all of the spin projections n are quantized relative to a single system of axes z, y, x, to which all spherical angles also refer; i_1, i_2, i_3 are the spins of particles 1, 2, 3; the summation is over repeated indices. The formula (9) is also relativistic, if we understand $\mathbf{p}$ and $\mathbf{p}'$ to be the Dalitz variables and use for the description of the relativistic spin the representation of Foldy and Yu. Shirokov.[8] The formula (6) can be obtained from Eq. (9). To do so we must go over to spin projections along the directions of the momenta of the particles (m-projections) by means of the relation

$$\Big| m \rangle = \sum_{n=-i}^{+i} \Big| n \rangle D^i_{n,m}(g), \quad (10)$$

where g is the rotation that brings the axes z, y, x to coincidence with the corresponding system of axes for the m projection. On carrying out some rather cumbersome transformations (cf. e.g., Sec. 2 of reference 6), we get the formula (6), if we use the notation (cf. reference 1 and Appendix B of reference 2)

$$\langle m_1 m_2 m_3 j m p | S^{JE} | m_a m_b \rangle = \sum \left\{ \sqrt{\frac{2l+1}{2j+1}} \langle i_1 i_2 m_1 m_2 | i m_1 + m_2 \rangle \right.$$
$$\left. \times \langle i l m_1 + m_2 0 | j m_1 + m_2 \rangle \right\} \left\{ \sqrt{\frac{2l'+1}{2J+1}} \langle j i_3 m m_3 | \; s' m \right.$$
$$\left. + m_3 \rangle \langle s' l' m + m_3 0 | Jm + m_3 \rangle \right\} \langle i l j s' l' p | S^{JE} | s l_a \rangle$$
$$\times \left\{ \sqrt{\frac{2l_a+1}{2J+1}} \langle i_a i_b m_a m_b | s m_a + m_b \rangle \right.$$
$$\left. \times \langle s l m_a + m_b 0 | J m_a + m_b \rangle \right\}. \quad (11)$$

It is clear how much more cumbersome the formula (9) is in comparison with Eq. (6). In particular, in Eq. (6) there is not a single Clebsch-Gordan coefficient. Of course they will appear if for the description of the spin state of the ensemble of particles we use the polarization tensors (cf., e.g., references 1 and 6):

$$\rho(q\tau) = \sqrt{2i+1} \sum_{m,m'} (-1)^{i-m'} \langle i i m - m' | q\tau \rangle \rho_{m,m'}.$$

Here i is the spin of the particle, and $\rho_{m,m'}$ is the density matrix that describes the spin state of the ensemble of particles. The extension of the expressions for the angular distribution and the polarization tensors of the products of the reaction (cf. references 6 and 9, Sec. 1) to the case of three particles presents no difficulties.

4. No important new difficulties appear in the extension of the formulas (6) and (8) to the case of more than three product particles. Instead of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, ... one introduces Dalitz variables; the spherical angles ϑ_i are the angles between the Dalitz momenta. When the number of particles is increased by one an additional transformation function of the type of Eq. (1) appears.

5. In conclusion we write out the selection rules that follow from the invariance of the transition matrix with respect to space reflection and the existence of definite parities for all the particles. These rules are obtained in exactly the same way as in references 1, 2, and 9.

We have

$$\langle m_1 m_2 \ldots | R(\vartheta, \varphi, \vartheta', \varphi' \ldots) | m_a m_b \rangle$$
$$= \pi_1^* \pi_2^* \ldots \pi_b \pi_a (-1)^{i_1+i_2+\ldots+i_a+i_b} (-1)^{m_1+m_2+\ldots+m_a+m_b}$$
$$\times \langle -m_1, -m_2, \ldots | R(\vartheta, -\varphi; \vartheta', -\varphi', \ldots)$$
$$\times | -m_a, -m_b \rangle. \quad (12)$$

In terms of the coefficients W introduced by the writer,[9] this selection rule has the form

$$\langle q_1 \tau_1 \ldots | W(\vartheta, \varphi, \vartheta', \varphi' \ldots) | q_a \tau_a q_b \tau_b \rangle = (-1)^{q_1+\ldots+q_a+q_b}$$
$$\times (-1)^{\tau_1+\ldots+\tau_a+\tau_b} \langle q_1, -\tau_1 \ldots | W(\vartheta, -\varphi; \vartheta',$$
$$-\varphi', \ldots) | q_a, -\tau_a; q_b - \tau_b \rangle, \quad (13)$$

which in particular means for the angular distributions

$$\sigma(\vartheta, \varphi, \vartheta') = \sigma(\vartheta, -\varphi, \vartheta') \quad (14)$$

in the case of three particles,

$$\sigma(\vartheta, \varphi, \vartheta', \varphi', \vartheta'') = \sigma(\vartheta, -\varphi, \vartheta', -\varphi', \vartheta'') \quad (15)$$

in the case of four, and so on. We recall that the angle φ' is measured from the axis $x'' \parallel [\mathbf{p}_a \times \mathbf{p}''] \times \mathbf{p}_a$, and the angle φ from $x' \parallel [\mathbf{p}'' \times \mathbf{p}'] \times \mathbf{p}''$. The rule (13) (in other formulations) has been repeatedly mentioned in the literature (cf., e.g., a paper by Sona[10]).

Those azimuthal symmetries in cascade reactions involving reactions of the type $a + b \rightarrow 1 + 2 + 3 + \ldots$ that follow from parity conservation are obtained by an obvious extension of the symmetries enumerated by the writer[9] (for cascades of binary reactions, i.e., of reactions of the type $a + b \rightarrow 1 + 2$).

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