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NATURAL OSCILLATIONS OF A BOUND PLASMA

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The general properties of cylindrical waves in a cold plasma are examined. The results are applied to low-frequency natural oscillations of a plasma cylinder surrounded by conductive walls. The conditions of magneto-acoustic resonances that ensure effective penetration of the oscillations inside the plasma have been found. The nature of the resonance phenomena depends on the linear density of the electrons. Approximate formulas are given for the natural oscillations of a long plasma cylinder. It is shown that purely radial oscillations are not feasible in a region close to the geometric mean of the electron and ion cyclotron frequencies for even slight deviations from it sharply change the resonance frequency.

WHILE the propagation of waves in a plasma has been adequately treated in the literature (see, for example, references 1-4) natural oscillations of a bound plasma have been studied mainly in the high frequency regions,[5,6] where the ion motion can be neglected. Low-frequency (magnetohydrodynamic and magnetoacoustic) oscillations of a bound plasma have been examined only for special cases.[7-10]

In this paper we investigate oscillations of a plasma cylinder surrounded by conducting walls in a homogeneous static longitudinal magnetic field. The main problem is to find the natural oscillations, but we study first the general properties of cylindrical waves in a cold plasma. Attention is directed particularly to the case when the oscillation frequency is low compared with the plasma frequency (magnetohydrodynamic and magnetoacoustic oscillations).

The entire examination is carried out in a linear approximation by neglecting collisions and other dissipative processes (an ideal plasma) and thermal motion (cold plasma). The plasma is assumed to consist of two components (electrons and homogeneous ions) and its density is constant over the entire area of the examined cylinder.

BASIC EQUATIONS

By assuming that the electron mass is small compared with the ion mass, we can write equations for the motion of charged particles in a plasma in the form of equations for the mass-velocity:

$$\mathbf{v} = (n_i M \mathbf{V}_i + n_e m \mathbf{v}_e)/(n_i M + n_e m) \tag{1}$$

and for the current density

$$\mathbf{j} = e(Zn_i\mathbf{V}_i - n_e\mathbf{v}_e) \approx en(\mathbf{V}_i - \mathbf{v}_e). \tag{2}$$

In the linear approximation for a cold plasma without collisions these equations have the form

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{1}{\rho c}[\mathbf{j}\times\mathbf{H}_0], \tag{3}$$

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{\omega_0^2}{4\pi}\left(\mathfrak{E} + \frac{1}{c}[\mathbf{v}\times\mathbf{H}_0]\right) - \frac{e}{mc}[\mathbf{j}\times\mathbf{H}_0]. \tag{4}$$

Here

$$\rho = Mn_i + mn_e \approx Mn_i, \tag{5}$$

$\mathbf{H}_0$ is the static magnetic field, which is assumed to be homogeneous and, according to the linearity condition, is large compared with the alternating magnetic field $\mathfrak{H}$.

Setting the time dependence in the form $e^{-i\omega t}$, we can write (3) in the form of

$$\mathbf{v} = (i/\rho c\omega)[\mathbf{j}\times\mathbf{H}_0]. \tag{6}$$

Substituting in (4) and expanding the triple vector product we finally get:

$$\omega^2\mathbf{j} = i\frac{\omega_0^2\omega}{4\pi}\mathfrak{E} + \omega_i\omega_e[\mathbf{j} - \mathbf{h}(\mathbf{jh})] - i\omega\omega_e[\mathbf{j}\times\mathbf{h}], \tag{7}$$

where $\mathbf{h}$ is a unit vector along the static magnetic field $\mathbf{H}_0$, ω_0 is the electron plasma frequency, while ω_i and ω_e are the ion and electron cyclotron frequencies.

If the electron mass is not neglected compared with the ion mass, but the quasi-neutrality condition is retained, then the form of the equations can be preserved by changing only the definitions of the characteristic frequencies:

$$\omega_0^2 = \frac{4\pi ne^2}{m}\left(1 + \frac{Zm}{M}\right) \approx \frac{4\pi ne^2}{m}, \quad \omega_e = \frac{eH_0}{mc}\left(1 - \frac{Zm}{M}\right) \approx \frac{eH_0}{mc},$$

$$\omega_i = \frac{ZeH_0}{Mc(1 - Zm/M)} \approx \frac{ZeH_0}{Mc}.$$

Obviously, Zm/M can almost always be neglected compared with unity.

To analyze the oscillations, it is more convenient to rewrite Maxwell's equations in the form:

$$\nabla^2\mathfrak{E} - \text{grad div}\,\mathfrak{E} + \frac{\omega^2}{c^2}\mathfrak{E} = -i\frac{4\pi\omega}{c^2}\mathfrak{j}. \quad (9)$$

SIMPLE CYLINDRICAL WAVES

We locate two coordinates q_1 and q_2 of the locally orthogonal coordinate system in a plane perpendicular to the magnetic field, and align the coordinate q_3 with the static magnetic field $\mathbf{H}_0$. The electric field and the current density are best represented in the form

$$\mathfrak{E}_\pm = \mathfrak{E}_1 \pm i\mathfrak{E}_2 = (E_1 \pm iE_2)F_\pm(q_1, q_2, q_3; t),$$
$$\mathfrak{E}_3 = E_3F_3(q_1, q_2, q_3, t),$$
$$\mathfrak{j}_\pm = \mathfrak{j}_1 \pm i\mathfrak{j}_2 = (j_1 \pm ij_2)F_\pm(q_1, q_2, q_3, t),$$
$$\mathfrak{j}_3 = j_3F_3(q_1, q_2, q_3, t). \quad (10)$$

The Gothic letters indicate the variable quantities to distinguish them from the constant amplitudes.

Equation (7) can be written in the variables (10) as:

$$(\omega_0^2\omega/4\pi)\mathfrak{E}_\pm = i(\omega_i\omega_e - \omega^2 \pm \omega\omega_e)\mathfrak{j}_\pm, \quad (11)$$

$$\mathfrak{j}_3 = i(\omega_0^2/4\pi\omega)\mathfrak{E}_3. \quad (12)$$

We shall use henceforth a cylindrical system of coordinates: $q_1 = \mathbf{r}$; $q_2 = \varphi$; $q_3 = z$. The function F is sought in the form:

$$F = Z(k_1r)e^{i\psi}, \qquad \psi = k_3z + m\varphi - \omega t. \quad (13)$$

Here m is the azimuthal number, k_1 and k_3 are the components of the wave vector and Z designates a cylindrical function. The solution that remains finite on the cylinder axis has the form

$$Z_\pm = J_{m\pm1}(k_1r), \qquad Z_3 = J_m(k_1r), \quad (14)$$

where J is the Bessel function of the first kind. For a coaxial gap they are replaced by general cylindrical functions, i.e., linear combinations from the Bessel and Neumann functions with the same indices. Substitution of (11)–(13) into the expression for the differential operators and use of the properties of the Bessel function gives:

$$\text{div}\,\mathfrak{E} = i(k_1E_2 + k_3E_3)J_m(k_1r)e^{i\psi}, \quad (15)$$

$$\nabla^2\mathfrak{E} = -(k_1^2 + k_3^2)\mathfrak{E} = -k^2\mathfrak{E}, \quad (16)$$

$$(\text{grad div}\,\mathfrak{E})_\pm = \mp ik_1(k_1E_2 + k_3E_3)J_{m\pm1}(k_1r)e^{i\psi}, \quad (17)$$

$$(\text{grad div}\,\mathfrak{E})_3 = -k_3(k_1E_2 + k_3E_3)J_m(k_1r)e^{i\psi}. \quad (18)$$

We see from this that Eq. (9) is satisfied for the selected form of the functions. Substitution of (10)–(18) into (9) yields a characteristic system of linear homogeneous equations connecting the integration constants E and j, while the determinant of this system yields the dispersion equation.

We shall first examine the components along the magnetic field. Substituting (14) in (10) and taking (10), (16), and (18) into account, we get

$$E_3 = E_2k_1k_3c^2/(k_1^2c^2 + \omega_0^2 - \omega^2). \quad (19)$$

It is seen directly that in the two special cases $k_1 = 0$ and $k_3 = 0$ (axial and purely-radial oscillations) the equation for E_3 separates. In these cases there are two independent oscillation modes. The mode with $E_3 = 0$ constitutes for $k_3 = 0$ purely radial magnetoacoustic oscillations and for $k_1 = 0$ magnetohydrodynamic waves propagated along the field. The mode with $E_3 \neq 0$ constitutes in the second case longitudinal plasma waves and in the first a transverse electromagnetic wave. The dispersion equation for this mode is obtained by equating to zero the denominator of the left portion of (19), and is independent of the presence of the magnetic field.

In the general case of oblique propagation, the equation for E_3 does not separate and (19) gives only the connection between E_3 and E_2. Inserting (19) in (15), (16), (17), and (9), we get

$$k_1E_2 + k_3E_3 = \frac{k^2c^2 + \omega_0^2 - \omega^2}{k_1^2c^2 + \omega_0^2 - \omega^2}k_1E_2, \quad (20)$$

$$(-k^2 + \omega^2/c^2)(E_1 \pm iE_2) \pm ik_1(k_1E_2 + k_3E_3)$$
$$= -i4\pi\omega c^{-2}(j_1 \pm ij_2). \quad (21)$$

From (21) we can get the connection between the amplitudes of the electric field and current in the form

$$(k^2c^2 - \omega^2)E_1 = i4\pi\omega j_1, \qquad (k_3^2c^2q - \omega^2)E_2 = i4\pi\omega j_2, \quad (22)$$

where

$$q = (\omega_0^2 - \omega^2)/(k_1^2c^2 + \omega_0^2 - \omega^2). \quad (23)$$

From (11) we get for the same amplitudes

$$(\omega_0^2\omega/4\pi)E_1 = i(\omega_i\omega_e - \omega^2)j_1 + \omega\omega_ej_2,$$
$$(\omega_0^2\omega/4\pi)E_2 = i(\omega_i\omega_e - \omega^2)j_2 - \omega\omega_ej_1. \quad (24)$$

The system of characteristic equations (22)–(24) yields the dispersion equation

$$\left(\frac{\omega_0^2\omega^2}{k^2c^2 - \omega^2} - \omega_i\omega_e + \omega^2\right)\left(\frac{\omega_0^2\omega^2}{k_3^2c^2q - \omega^2} - \omega_i\omega_e + \omega^2\right) - \omega^2\omega_e^2 = 0. \quad (25)$$

We note that the parameter q contains $k_1^2c^2$ in the denominator, so that the dispersion equation is the second degree in k_1^2 (or k^2) if k_3^2 and ω^2 are given, or in k_3^2 if k_1^2 and ω^2 are given. The dispersion equation does not contain m and is identical with the equation for plane waves,[1] if the direction of the propagation is assumed to lie in the r, z plane.

Maxwell's equation:

$$(i\omega/c)\,\widetilde{\mathbf{H}} = \mathrm{curl}\,\mathfrak{E} \tag{26}$$

makes it possible to find the alternating magnetic field $\widetilde{\mathbf{H}}$. We can then write the complete solution with the azimuthal number m:

$$\mathfrak{E}_r = i\frac{E_2}{2}[(1-p)J_{m+1}(k_1r) - (1+p)J_{m-1}(k_1r)]\,e^{i\psi}, \tag{27}$$

$$\mathfrak{E}_\varphi = \frac{E_2}{2}[(1-p)J_{m+1}(k_1r) + (1+p)J_{m-1}(k_1r)]\,e^{i\psi}, \tag{28}$$

$$\mathfrak{E}_z = E_3J_m(k_1r)\,e^{i\psi}, \tag{29}$$

$$\mathfrak{j}_r = \frac{j_2}{2}[(1-s)J_{m+1}(k_1r) - (1+s)J_{m-1}(k_1r)]\,e^{i\psi}, \tag{30}$$

$$\mathfrak{j}_\varphi = -i\frac{j_2}{2}[(1-s)J_{m+1}(k_1r) + (1-s)J_{m-1}(k_1r)]\,e^{i\psi}. \tag{31}$$

Here

$$p = i\frac{E_1}{E_2} = \frac{w^2{}_0\dfrac{w}{w_e} - \left(\dfrac{w_i}{w} - \dfrac{w}{w_e}\right)\left(k^2{}_3\,c^2q - w^2\right)}{k^2\,c^2 - w^2},$$

$$s = \frac{k^2c^2 - \omega^2}{k_3^2c^2q - \omega^2}\,p; \tag{32}$$

j_2 is related to E_2 by formula (22). Furthermore,

$$\mathfrak{j}_z = i\frac{\omega_0^2}{4\pi\omega}E_3J_m(k_1r)e^{i\psi}, \tag{33}$$

$$\widetilde{H}_r = -\frac{E_2}{2}\frac{k_3c}{\omega}[(q-p)J_{m+1}(k_1r) + (q+p)J_{m-1}(k_1r)]\,e^{i\psi}, \tag{34}$$

$$\widetilde{H}_\varphi = i\frac{E_2}{2}\frac{k_3c}{\omega}[(q-p)J_{m+1}(k_1r) + (q-p-2)J_{m-1}(k_1r)]\,e^{i\psi}, \tag{35}$$

$$\widetilde{H}_z = ipE_2\frac{k_1c}{\omega}J_m(k_1r)\,e^{i\psi}. \tag{36}$$

Here $k_1 = k_r$ and $k_3 = k_z$ are the radial and axial wave numbers.

We note that the radial magnetic field $\widetilde{\mathbf{H}}_r$ is expressed in terms of the electric field components $\mathfrak{E}_z$ and $\mathfrak{E}_\varphi$ in the form

$$i\frac{\omega}{c}\widetilde{H}_r = i\frac{m}{r}\mathfrak{E}_z - ik_z\mathfrak{E}_\varphi. \tag{37}$$

We shall call the complex solutions of type (27)–(36) simple cylindrical waves. The real part of such a solution directly gives directly a helical traveling wave. Because of the gyrotropic plasma properties, the natural oscillations have the form of standing waves along the axis but are traveling waves in azimuth.

Natural oscillation with given k_3^2 can be expressed as the sum or difference of the two simple cylindrical waves with axial wave numbers $+k_3$ and $-k_3$. It is convenient to set the origin at a node of the standing wave and to express the natural oscillation as a difference of simple cylindrical waves. In this case, since the first power of k_3 enters only into formulas (19) (35) and (36), we have

$$\mathfrak{E}_r,\ \mathfrak{E}_\varphi,\ \mathfrak{j}_r,\ \mathfrak{j}_\varphi,\ \widetilde{H}_z \backsim i\sin k_3z; \quad \mathfrak{E}_z,\ \mathfrak{j}_z,\ \widetilde{H}_r,\ \widetilde{H}_\varphi \backsim \cos k_3z. \tag{38}$$

From such formulas (27)–(36) it is apparent that the radial distributions of the amplitudes are different in oscillations with azimuthal numbers +m and −m. This discloses the gyrotropic properties of the plasma; this is precisely why the natural oscillations of the plasma should be represented by azimuthal traveling waves.

In deriving all the previous formulas we have neglected only the non-linearity, thermal motion and collisions. Otherwise, the formulas are general and applicable for arbitrary frequencies. In what follows we shall examine a number of very simple cases, with special attention being directed to low frequencies, where the ion motion is significant, i.e., to magnetoacoustic and magnetohydrodynamic oscillations.

All the results cited can be also obtained from the general theory of propagation of electromagnetic waves in gyrotropic media[8-11] by substituting the dielectric tensor of the plasma. However, it is necessary in this case to express not the field in terms of the current, but the current in terms of the field, which results in a rather cumbersome derivation. For the cases of specific interest to us, the method given above is the simplest and clearest.

EXACT BOUNDARY CONDITIONS

The natural oscillations of a bounded plasma are described by the obtained solutions of the equations if they satisfy the boundary conditions. The electromagnetic fields as well as the plasma motion are relevant in this respect and, therefore, the boundary conditions may generally be both electrodynamic and hydrodynamic.

The hydrodynamic boundary condition must be imposed when the plasma comes into direct contact with the solid walls. Then the normal component of the mass velocity must vanish at the wall surface. In a cold plasma, according to formula (6), the velocity along the magnetic field is also equal to zero. Therefore, the hydrodynamic boundary condition is imposed only on the lateral surface of the plasma cylinder. It appears from (6) that if this surface comes into direct contact with the solid walls, then the following condition must be in effect on it:

$$j_\varphi(k_1R_0) = 0, \tag{39}$$

where R_0 is the radius of the plasma cylinder.

The electrodynamic boundary conditions are that the normal component of the alternating magnetic and the tangential components of the electric field at the plasma surface have at all times the same values as in the surrounding medium. At the ends of the cylinder the conditions are imposed on $\tilde{H}_z$, $\mathfrak{E}_r$ and $\mathfrak{E}_\varphi$. For solutions of type (38), values of k_3 satisfying such conditions can always be found. In particular, if the plasma cylinder of length L is bounded on the ends by ideally conducting walls, then the boundary conditions at the ends give

$$k_3 = l\pi/L, \tag{40}$$

where l is an integer.

It is apparent from (38) that the solution with $l = 0$ contains only $\mathfrak{E}_z$, j_z, $\tilde{H}_r$ and $\tilde{H}_\varphi$. This special solution, with $k_3 = 0$ and $E_3 \neq 0$ constitutes, as we have already pointed out, a transverse electromagnetic wave polarized along the magnetic field and propagated in the same manner as without a field. For the oscillations of interest to us, which depend on the magnetic field, and especially for all the low-frequency oscillations, the eigenvalues k_3 correspond to values of l from 1 to ∞ in (40).

Considerably more complex are the electrodynamic boundary conditions on the lateral surface of the cylinder. Here, they are imposed on $\tilde{H}_r$, $\mathfrak{E}_z$, and $\mathfrak{E}_\varphi$. The connection (37) between these values makes it possible in the simplest cases to reduce the three conditions to two.

It is seen from (27) and (28) that the radial dependences of $\mathfrak{E}_z$ and $\mathfrak{E}_\varphi$ are expressed by various combinations of Bessel functions. Therefore, one simple cylindrical wave, generally speaking, cannot satisfy the boundary conditions; a linear combination of two simple cylindrical waves with different k_1 is required for this.

Since the dispersion equation is of the second degree in k_1^2, two such cylindrical waves can always be constructed for given ω^2 and k_3^2. If k_1 is imaginary for one of these solutions, their linear combination can still satisfy the boundary conditions for the natural oscillations. In a region where both values of k_1 are imaginary, however, only forced oscillations are possible.

APPROXIMATE BOUNDARY CONDITIONS

The problem of natural oscillations makes sense if the plasma is located inside a closed cavity with ideally conducting walls, for example, coaxially inside a metallic cylinder. Then an exact condition for the lateral surface will require matching to the oscillations of a coaxial dielectric gap.

The radial functions for a coaxial gap are combinations of Bessel and Neumann functions of the argument k_er. Here k_e is the external radial wave number, determined from the relationship

$$k_e^2 = \lambda_0^{-2} - \lambda_3^{-2}, \tag{41}$$

$\lambda_0 = c/\omega$ is the vacuum wavelength and $\lambda_3 = 1/k_3$ is the longitudinal wavelength. If the thickness of the gap is small compared with λ_0 and λ_3, the variation of the fields in it may be neglected. Then the boundary condition on the plasma surface can be assumed given in the following form:

$$\mathfrak{E}_\varphi(k_1R_0) = 0, \qquad \mathfrak{E}_z(k_1R_0) = 0, \tag{42}$$

where R_0 is the radius of the plasma cylinder. It is apparent from (37) that the boundary condition for $\tilde{H}_r$ is thus satisfied automatically. On the other hand, for small k_3 and low frequencies, $\mathfrak{E}_z$ may, according to (19), be neglected. Then the conditions at the lateral surface of the plasma cylinder will have the form

$$\mathfrak{E}_\varphi(k_1R_0) = 0, \qquad j_\varphi(k_1R_0) = 0. \tag{43}$$

The second condition pertains only to cases when the plasma surface is in direct contact with the solid walls. In case of a free plasma surface the approximate boundary condition reduces to

$$k_1R_0 = \alpha_n, \tag{44}$$

where α_n are the roots of the right half of (28). One can speak of oscillations of a plasma confined by a free surface only to the extent that the periods of the considered oscillations are small compared with the skin times.

MAGNETOACOUSTIC REGION

For frequencies which are low compared with ionic cyclotron ones, the gyrotropic properties of the plasma do not affect the oscillations. We shall call this frequency region the magnetoacoustic region. From Eqs. (26)—(30) two independent oscillation branches are obtained for it. For the first:

$$E_2 = 0, \quad k^2c^2/\omega^2 \approx \omega_0^2/\omega_i\omega_e; \tag{45}$$

for the second:

$$E_1 = 0, \quad k_3^2c^2/\omega^2 \approx \omega_0^2/\omega_i\omega_e. \tag{46}$$

This approximation is valid when

$$k_1^2c^2 \ll \omega_0^2, \tag{47}$$

$$k_3^2c^2 \ll \omega_0^2. \tag{48}$$

Newcomb[12] calls the first branch the TE mode, and the second the TEM mode.

LINEAR DENSITY OF THE ELECTRONS

For small k_3, the magnetoacoustic region is determined by inequality (47). We substitute the plasma frequency and express k_1 from the boundary condition, written in the form of (44). Then (47) will have the form

$$\omega_0^2/k_1^2c^2 = 4\alpha_n^{-2}(e^2/mc^2)\,\pi R_0^2 n_e \gg 1. \tag{49}$$

The quantity

$$(e/mc^2)\,\pi R_0^2 n_e = \Pi \tag{50}$$

contained in (50) has a simple meaning. This is the total number of electrons along the length of a cylinder equal to the classical electron radius. At the suggestion of S. É. Braginskiĭ, we call Π the linear density of the electrons.

We shall call the following quantity,

$$\omega_0^2/k_1^2c^2 = 4\alpha_n^{-2}\Pi = \Pi^* \tag{51}$$

which depends on the boundary conditions, the effective linear density of the electrons.

Now inequality (49) acquires the following meaning: the magnetoacoustic region is realized when the effective linear density of the electrons is large.

DISPERSION EQUATION IN DIMENSIONLESS VARIABLES

In going over to dimensionless quantities, the dispersion equation (25) can be given in a form containing two dimensionless parameters:

$$A = \omega_0^2/\omega_i\omega_e = c^2/u_A^2, \qquad B = \omega_e/\omega_i$$

and the dimensionless variables

$$x = (k_1^2c^2 + \omega_0^2 + \omega^2)/\omega_0^2, \qquad y = k_3^2c^2/\omega_0^2, \qquad \Omega = \omega^2/\omega_i\omega_e.$$

The parameter A depends only on the velocity u_A; it is equal to the square of the index of refraction of the plasma in the magnetoacoustic region. The parameter B depends only on the nature of the gas, and is always large ($B \geq M/m$).

Expanding the left side of the dispersion equation (25) in powers of X and Y, we can set it in the symmetric form:

$$a_{20}x^2 + a_{10}x + a_{11}xy + a_{01}y + a_{02}y^2 = 0,$$

$$a_{20} = (1-\Omega)(A+1-\Omega) - B\Omega, \quad a_{10} = B\Omega - A - 1 + \Omega,$$

$$a_{11} = A(1-\Omega) + (A-2\Omega)[B\Omega - (1-\Omega)^2]/\Omega,$$

$$a_{01} = (A-\Omega)(1-\Omega-B\Omega)/\Omega,$$

$$a_{02} = (A-\Omega)[B\Omega - (1-\Omega)^2]/\Omega. \tag{52}$$

The index of refraction of the plasma tends to infinity when the highest-order coefficients a_{20} or a_{02} vanish. The coefficient a_{02} vanishes at frequencies close to the electron and ion cyclotron frequencies. a_{20} vanishes at two frequencies which, after Hurwitz,[8] are called hybrid. The higher one is close to the plasma frequency and the lower one is determined by the relationship:

$$\omega_h^2 = \omega_i\omega_e/[1 + (\omega_e/\omega_0)^2]. \tag{53}$$

LOW-FREQUENCY OSCILLATIONS OF A LONG CYLINDER

We shall examine a case when an approximate expression for the lower natural frequencies can be obtained in explicit form. We set in (25)

$$\omega_0^2\omega^2/(k_3^2c^2q - \omega^2) \gg \omega_i\omega_e - \omega^2. \tag{54}$$

Obviously, this approximation is suitable for not too high k_3; we shall therefore call it the long-cylinder approximation. A special case where approximation (54) is inapplicable is the TEM mode in a magnetoacoustic region. Otherwise, its applicability region is rather extensive: in particular, it is always applicable when $k_3^2c^2q$ is close to ω^2 or when ω^2 is close to $\omega_i\omega_e$. Moreover, by setting:

$$\omega^2 \ll k^2c^2, \qquad \omega^2 \ll \omega_0^2, \tag{55}$$

we get from (25)

$$\omega^2 \approx \omega_i\omega_e \frac{1 + (\omega_e/\omega_i)\,k_3^2c^2/(\omega_0^2 + k_1^2c^2)}{1 + (\omega_0/kc)^2 + (\omega_e/\omega_0)^2}. \tag{56}$$

If $k_3 \ll k_1$, we can introduce the effective linear density of the electrons Π according to (51) and write (56) in the form

$$\omega^2 \approx \omega_i\omega_e \frac{1 + (\Pi^* + 1)^{-1}(\omega_e/\omega_i)(k_3/k_1)^2}{\Pi^* + 1 + \omega_e^2/\omega_0^2}. \tag{57}$$

In the limit of a very large linear electron density, (57) goes into expression (45) for the magnetoacoustic region. But this limit is attained only when

$$\Pi^* \gg (\omega_e/\omega_i)(k_3/k_1)^2, \tag{58}$$

which, for not too small k_3, is a very stringent requirement. If the less stringent condition is imposed:

$$\Pi^* \gg 1, \quad \Pi^* \gg (\omega_e/\omega_0)^2,$$

then (57) will give:

$$\omega^2 \approx (k_1^2 + k_3^2\omega_e/\Pi^*\omega_i)\, u_A^2. \quad (59)$$

Here the natural frequency is almost as if the propagation had an Alfvén velocity across the field and a velocity $u_e/\sqrt{\Pi^*}$ along the field, where u_e is the "electron Alfvén velocity" which is $\sqrt{M/Zm}$ times larger than the Alfvén velocity.

$$u_e^2 = \omega_e^2 c^2/\omega_0^2 = H_0^2/4\pi n_e m. \quad (60)$$

When $k_1^2/k_3^2\, \Pi^* \ll \omega_e/\omega_i$ the natural frequency no longer depends on the ion mass and becomes inversely proportional to the length of the cylinder. This region can be called pseudo-magnetohydrodynamic.

EXCITATION OF THE OSCILLATIONS

The forced plasma-cylinder oscillations are best analyzed by expansion in a series of natural oscillations. When the forcing frequency approximates one of the natural frequencies of the cylinder, the corresponding term of the expansion sharply increases and resonance occurs.

We consider the simplest case of excitation. Let the plasma be surrounded by an ideally-conducting metallic cylinder, the lateral surface of which has a dielectric cut along the generatrix. A sinusoidal external voltage of given frequency ω is applied to this cut. By assuming the cut to be infinitesimally thin we may write the boundary condition in the form:

$$\mathfrak{E}_\varphi = (V/R_1)\,\delta(\varphi), \quad \text{for } r = R_1, \quad (61)$$

where V is the voltage on the cut. We assume

$$V = V_0 e^{-i\omega t} \quad (62)$$

and expand the δ function in a Fourier series

$$\delta(\varphi) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{im\varphi}. \quad (63)$$

Because of the gyrotropic properties of the plasma, waves with positive as well as negative azimuthal numbers must be independently present in the expansion.

The solution for the forced oscillation is found in the form:

$$\mathfrak{E}_\varphi = \sum_{-\infty}^{\infty} C_m Z_m(k_1 r)\, e^{i\psi}, \quad (64)$$

where Z_m is a function of the form (28) or a linear combination of two such functions; k_1 satisfies the dispersion equation for given ω and k_3, but does not satisfy the boundary conditions for $\mathfrak{E}_\varphi$. For low-frequency oscillations, an approximate boundary condition can be used by considering that (61) is given not at the inner radius R_1 of the metallic housing, but at the radius R_0 of the plasma cylinder. Then

$$C_m = V/2\pi R_1 Z_m(k_1 R_0). \quad (65)$$

When k_1R_0 approaches one of the roots of the right side of (28), one of the terms of the series in (73) sharply increases. Under these conditions resonances should be observed, accompanied by an effective penetration of the alternating field into the plasma.

Other excitation schemes are considerably more difficult to calculate, since it is necessary to expand not only in azimuthal, but in radial functions. The qualitative conclusion that resonance phenomena are present is general.

MAGNETOACOUSTIC RESONANCE IN A PLASMA

The dispersion equation (25) or (52) for a cold plasma is of the fifth power with respect to the square of the frequency. Therefore, generally speaking, up to five different resonance frequencies correspond to given values of k_1, k_3 and m, but these frequencies differ in character.

For a cold unbounded plasma there are also five characteristic frequencies: 2 cyclotron, 2 hybrid, and 1 plasma. Often these frequencies are called natural or resonant. In this case the term resonance is not always unambiguously defined.

Near cyclotron and hybrid frequencies, one of the indices of refraction of the plasma tends to become infinite on passing from positive to negative values. This phenomenon is similar to anomalous dispersion in optics.

The tendency for the index of refraction to become infinite indicates that the phase velocity vanishes. Near this point, thermal motion can no longer be neglected. If the phase velocity is small and there is even a slight thermal motion in the direction of propagation, there will always be particles for which the thermal velocity component in this direction will be close to the phase velocity. These particles move in phase with the wave and irreversibly draw energy from it. In other words, it can be said that these particles are in resonance with the wave. Examples of such a single-particle resonance are ion and electron cyclotron resonances.

Single-particle resonance is associated with the conversion of oscillation energy into energy of other degrees of freedom of the plasma motion (for example, cyclotron rotation). Therefore, single-particle resonance results in a unique energy absorption not associated with collisions and oscillation damping. It can be stated that if account is taken of even the slightest thermal motion, then the imaginary part of the index of refraction must tend to infinity together with the real part.

For a bounded plasma, a phenomenon of a completely different nature takes place — collective resonance at the natural frequencies of the plasma volume. These natural frequencies have been examined above; they depend on the plasma concentration and on the boundary conditions. Resonance at the natural frequencies of a bounded plasma results, generally speaking, in the penetration of alternating fields inside the plasma.

However, if the geometric dimensions of the plasma volume are small compared with the vacuum wavelength, then high indices of refraction are required for resonance at the natural frequencies. Therefore, certain of the natural frequencies of a plasma volume are in many cases close to the anomalous dispersion frequencies, i.e., the collective resonance practically coincides with the single-particle resonance.[6]

If the natural frequency of a plasma volume coincides with the single-particle resonance, then the plasma cannot be made to oscillate at this frequency, because of specific absorption; i.e., it is impossible to produce in the plasma large alternating field amplitudes. We call such resonances absorption resonances; they include the ion and electron cyclotron resonances. They result only in surface heating of the plasma, i.e., the latter is opaque to the corresponding frequencies.

Anomalous dispersion type resonances constitute a larger group and include all the resonances close to frequencies where the index of refraction tends to infinity, i.e., resonances at hybrid as well as at cyclotron frequencies. All these resonances may be called trivial, since their frequencies are close to the natural frequencies of an unbounded plasma.

Natural-frequency resonances that depend on the boundary conditions, i.e., on the geometry and the plasma volume dimensions, are characteristic of a bounded plasma. These resonances are not due to specific absorption; the phase velocity is sufficiently large and assures effective penetration of the alternating fields inside the plasma. We shall call such resonances build-up resonances. The most important of them is the magnetoacoustic resonance, which is produced when the linear density of the electrons is large.

The use of build-up resonances and, particularly, of magnetoacoustic resonances, in contrast to the ordinary surface effect of high frequency fields (skin-effect), makes possible a deep penetration of the alternating field inside the plasma.

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