ON THE THEORY OF RELAXATION PROCESSES IN FERRODIELECTRICS WITH WEAK MAGNETIC ANISOTROPY AT LOW TEMPERATURES

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We consider the relaxation of the magnetic moment and the equalization of spin and lattice temperatures of ferrodielectrics with a weak magnetic anisotropy in weak magnetic fields. We show that the magnetic dipole interaction establishes the equilibrium of the magnetic moment, both as to its magnitude and as to its direction. The relaxation time of the absolute magnitude of the magnetic moment is in this case of the same order of magnitude as the characteristic time of rotation of the magnetic moment to its equilibrium direction. The relaxation time for the equalization of spin and lattice temperatures is also evaluated.

1. Akhiezer, Bar'yakhtar, and Peletminskiĭ[1] presented a general theory for the relaxation of the magnetic moment in ferrodielectrics; this theory was based upon the fact that there are two types of interaction between spin waves: the strong exchange interaction and the weak relativistic interactions (magnetic dipole interaction and interactions caused by the magnetic anisotropy).

The exchange interaction leads to the establishment of a Bose distribution of the spin waves with a non-equilibrium value of the magnetic moment, but the weak interactions lead to the establishment of an equilibrium value of the magnetic moment both in absolute magnitude and in direction. In the cited paper this scheme was applied to an evaluation of the relaxation time of the magnetic moment in those circumstances where the magnetic anisotropy constant β and the external magnetic field H_0 were sufficiently large. Under those circumstances one can easily check that spin waves with wave vector $\mathbf{k} = 0$ cannot split up into two spin waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$. The strongest of the "weak" interactions, which describe the processes of the combination of two spin waves into one and the splitting of one spin wave into two, can therefore not cause a change in the number of spin waves with $\mathbf{k} = 0$, which determines the component of the magnetic moment of the body perpendicular to the axis of easiest magnetization.

Because of this, one invokes the relativistic interactions, which describe processes involving a large number of spin waves one of which has a momentum $\mathbf{k} = 0$, to explain the relaxation of the transverse component of the magnetic moment.

The situation is different in crystals with a small magnetic anisotropy constant β in weak magnetic fields ($\beta + H_0/M_0 < 4\pi/3$), since under those conditions the splitting of a spin wave with $\mathbf{k} = 0$ into two spin waves now turns out to be possible. Because of this it is not necessary, when describing the relaxation of the magnetic moment in such crystals, to take the interactions describing spin wave-spin wave scattering into account, and it is sufficient to restrict ourselves to the magnetic dipole interaction which describes the splitting of one spin wave into two and the amalgamation of two spin waves into one.

We note, finally, the following fact: it is well known (see Néel[2]) that many ferrites, which at low temperatures can be considered to be dielectrics, have a complicated magnetic structure, i.e., they are described not by one but by several magnetic sublattices. This leads to the occurrence of high-frequency branches of the magnetic energy spectrum with a large activation energy, as well as a low-frequency branch (without an activation energy). The contribution of these high-frequency branches to the thermodynamic and transport properties of ferrodielectrics at low temperatures is, of course, exponentially small. One might think that the strong exchange interaction $\gamma(\mathbf{M}_1 \cdot \mathbf{M}_2)$ between uniformly magnetized sublattices could essentially change the interactions between low-frequency spin waves. This would thereby lead to an influencing of the magnetic structure of the transport and relaxation properties of a ferrodielectric at low temperatures.

A detailed analysis (see Appendix) shows that, indeed, the interaction between low-frequency spin waves, which is caused by the energy of exchange between sublattices, turns out to be of the same order of magnitude as the relativistic interaction describing spin wave-spin wave scattering. This enables us to neglect the magnetic structure of a ferrodielectric with weak anisotropy when we study its relaxation processes.

2. The Hamiltonian of a ferrodielectric with cubic symmetry can be written in the form

$$\mathcal{H} = \mathcal{H}^{(s)} + \mathcal{H}^{(p)},$$

$$\mathcal{H}^{(s)} = \int \Big[\frac{1}{2}\alpha \frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_i} + \frac{\beta}{2M_0^2}(M_x^2 M_y^2 + M_x^2 M_z^2 + M_y^2 M_z^2) + \frac{\mathbf{H}^2}{8\pi}\Big] dv,$$

$$\mathcal{H}^{(p)} = \frac{1}{2}\frac{\Theta_c a^2}{\mu M_0}\int \Big[\delta_1 \frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_k} u_{ik} + \delta_2 \frac{\partial M_l}{\partial x_i}\frac{\partial M_l}{\partial x_i} u_{kk}\Big] dv, \quad (1)$$

where $\mathbf{M}$ is the magnetic moment density, $\mathbf{H}$ the magnetic field acting in the crystal, u_{ik} the deformation tensor, α a constant connected with the exchange integral $[\alpha \equiv (\Theta_c/\mu M_0)a^2$, where Θ_c is a quantity of the same order of magnitude as the Curie temperature, a the lattice constant, M_0 the saturated magnetic moment, and μ the Bohr magneton], and δ_1 and δ_2 are the magnetostriction constants. The first term in Eq. (1) is the magnetic energy of the ferrodielectric and the second term the exchange part of the magnetostriction energy which is connected with the inhomogeneity of the magnetic moment. Kaganov and Tsukernik[3] have shown that one can neglect the energy of relativistic origin, which describes the magnetostriction effect when there is a uniform magnetization, when one considers relaxation and transport processes in the temperature range $T \gg 2\pi\mu M_0 \sim 1°K$.

If we now make the well-known transition (see Holstein and Primakoff[4] and also Kaganov and Tsukernik[5]) to the creation and annihilation operators of the spin waves, $c_{\mathbf{k}}^+$ and $c_{\mathbf{k}}$, we get

$$\mathcal{H}^{(s)} = \mathcal{H}^{(0)} + \mathcal{H}^{(3)} + \mathcal{H}^{(4)}, \quad (2)$$

$$\mathcal{H}^0 = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} c_{\mathbf{k}}^+ c_{\mathbf{k}}, \qquad \varepsilon_{\mathbf{k}} = \sqrt{A_{\mathbf{k}}^2 - |B_{\mathbf{k}}|^2}, \quad (3)$$

$$\mathcal{H}^{(3)} = \sum_{1,2,3} \Phi_{12,3} c_1^+ c_2^+ c_3 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) + \text{c.c.}, \quad (4)$$

$$\mathcal{H}^{(4)} = \sum_{1,2,3,4} \Phi_{12,34}\, c_1^+ c_2^+ c_3 c_4 \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4), \quad (5)$$

where we have used the following notation

$$A_k = \mu M_0(\alpha k^2 + \beta + H_0/M_0 + 2\pi \sin^2\theta_k), \quad (6)$$

$$B_k = 2\pi\mu M_0 \sin^2\theta_{\mathbf{k}} e^{-2i\varphi_{\mathbf{k}}},$$

$$\Phi_{12,3} = -\pi\mu(2\mu M_0)^{1/2} V^{-1/2}[\sin 2\theta_1 (e^{-i\varphi_1} u_1^* + e^{i\varphi_1} v_1^*)(u_2^* u_3 + v_2^* v_3) + \sin 2\theta_2 (e^{-i\varphi_2} u_2^* + e^{i\varphi_2} v_2^*)(u_1^* u_3 + v_1^* v_3) + \sin 2\theta_3 (e^{i\varphi_3} u_3 + e^{-i\varphi_3} v_3)(v_1^* u_2^* + v_2^* u_1^*)], \quad (7)$$

$$\Phi_{12,34} = -(\mu^2\alpha/2V)(\mathbf{k}_1\mathbf{k}_2 + \mathbf{k}_3\mathbf{k}_4), \quad (8)$$

$$u_{\mathbf{k}} = \sqrt{(A_{\mathbf{k}} + \varepsilon_{\mathbf{k}})/2\varepsilon_{\mathbf{k}}}, \qquad v_{\mathbf{k}} = -e^{2i\varphi_{\mathbf{k}}}\sqrt{(A_{\mathbf{k}} - \varepsilon_{\mathbf{k}})/2\varepsilon_{\mathbf{k}}}, \quad (9)$$

$c_j \equiv c_{\mathbf{k}_j}$, θ_j and φ_j are the polar angles of the vector $\mathbf{k}_j$. Since $\Phi_{12,34}$ is very small for small k ($\sim \alpha k^2$) the main role in the collisions caused by the operator $\mathcal{H}^{(4)}$ will at not too low temperatures ($T \gg 2\pi\mu M_0$) be played by spin waves with large wave vectors: $\epsilon_{\mathbf{k}} \sim T$, $\alpha k^2 \sim T/\mu M_0 \gg 1$. One sees easily from (6) and (9) that for such values of the wave vector $|v_{\mathbf{k}}| \ll |u_{\mathbf{k}}| \approx 1$.

The considerations given above refer also to the Hamiltonian describing the interaction between spin waves and phonons, $\mathcal{H}^{(p)}$. The Hamiltonian $\mathcal{H}^{(p)}$ expressed in terms of creation and annihilation operators for the spin waves and the phonons is of the form

$$\mathcal{H}^{(p)} = \sum_{1,2,3} \Psi_{12,3} c_1^+ c_2 b_3 \Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{f}_3) + \text{c.c.} \quad (10)$$

where

$$\Psi_{12,3} = i\frac{a^2\Theta_c}{4\sqrt{\omega_3}}\left(\frac{\hbar}{2\rho V}\right)^{1/2}[\delta_1(\mathbf{k}_1\mathbf{e}_3)(\mathbf{k}_2\mathbf{f}_3) + \delta_1(\mathbf{k}_2\mathbf{e}_3)(\mathbf{k}_1\mathbf{f}_3) + 2\delta_2(\mathbf{k}_1\mathbf{k}_2)(\mathbf{e}_3\mathbf{f}_3)], \quad (11)$$

the index 3 indicates the wave vector $\mathbf{f}_3$ and the polarization s of the phonon, $\mathbf{e}_3$ and ω_3 are the polarization unit vector and the frequency of the phonon, b_3^+ and b_3 are the creation and annihilation operators for the phonons and ρ is the density of the medium.

Using Eqs. (4), (5), and (10) for the Hamiltonians $\mathcal{H}^{(3)}$, $\mathcal{H}^{(4)}$, and $\mathcal{H}^{(p)}$, we can now write down the transport equation for the number of spin waves n_1 with wave vector $\mathbf{k}_1$

$$\dot{n}_1 = \dot{n}_1^{\text{coll}} \equiv L_{\mathbf{k}_1}^{(e)}\{n\} + L_{\mathbf{k}_1}^{(r)}\{n\} + L_{\mathbf{k}_1}^{(p)}\{n, N\}, \quad (12)$$

where

$$L_{\mathbf{k}_1}^{(e)}\{n\} = \frac{96\pi}{\hbar}\sum_{2,3,4}|\Phi_{12,34}|^2[(n_1+1)(n_2+1)n_3 n_4 - n_1 n_2(n_3+1)(n_4+1)]\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,\Delta((\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4), \quad (13)$$

$$L_{\mathbf{k}_1}^{(r)}\{n\} = \frac{8\pi}{\hbar}\sum_{2,3}\{2|\Phi_{12,3}|^2[(n_1+1)(n_2+1)n_3 - n_1 n_2(n_3+1)]\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3)\,\Delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3) + |\Phi_{23,1}|^2[(n_1+1)n_2 n_3 - n_1(n_2+1)(n_3+1)]\,\delta(\varepsilon_2+\varepsilon_3-\varepsilon_1)\,\Delta(\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_1)\}, \quad (14)$$

$$L_{\mathbf{k}_1}^{(p)}\{n, N\} = \frac{2\pi}{\hbar}\sum_{2,3}\Big\{|\Psi_{123}|^2[(n_1+1)n_2 N_3 - n_1(n_2+1)(N_3+1)]\,\delta(\varepsilon_1-\varepsilon_2-\hbar\omega_3)\,\Delta(\mathbf{k}_1-\mathbf{k}_2-\mathbf{f}_3) + |\Psi_{123}|^2[(n_1+1)n_2(N_3+1) - n_1(n_2+1)N_3]\,\delta(\varepsilon_1+\hbar\omega_3-\varepsilon_2)\,\Delta(\mathbf{k}_1+\mathbf{f}_3-\mathbf{k}_2)\Big\}, \quad (15)$$

N_3 is the number of phonons with momentum $\mathbf{f}_3$ and polarization s. The operator $L_{\mathbf{k}}^{(e)}$ describes the spin wave-spin wave scattering processes caused by the exchange interaction in the transport equation; the operator $L_{\mathbf{k}}^{(r)}$ describes the recombination of two spin waves into one and the splitting up of one spin wave into two, which are caused by the magnetic dipole interaction; the operator $L_{\mathbf{k}}^{(p)}$ describes the emission and absorption of a phonon by a spin wave. Starting from Eqs. (13) to (15) for the operators $L_{\mathbf{k}}^{(e)}$, $L_{\mathbf{k}}^{(r)}$, and $L_{\mathbf{k}}^{(p)}$ one can show by a treatment similar to the one in reference 1 that in the temperature range $\Theta_c \gg T \gg \Theta_c \times (\mu M_0/\Theta_c)^{4/7} \approx 10°K$ the main role in the transport equation is played by the operator $L_{\mathbf{k}}^{(e)}$ ($L_{\mathbf{k}}^{(e)} \gg L_{\mathbf{k}}^{(r)}, L_{\mathbf{k}}^{(p)}$). To a first approximation one can thus use the equation

$$L_{\mathbf{k}}^{(e)}\{n\} = 0$$

to determine the spin wave distribution function.

The solution of this equation is of the form

$$n_{\mathbf{k}} = \begin{cases} n_0, & \mathbf{k} = 0, \\ [\exp\{(\varepsilon_{\mathbf{k}} - \gamma)/T_s\} - 1]^{-1}, \end{cases} \tag{16}$$

where γ and n_0 are arbitrary constants which can be connected with the mean values of the absolute magnitude of the magnetic moment $\langle \mathfrak{M}^2 \rangle$ and of the dispersion of the component of the magnetic moment $\langle \mathfrak{M}_\perp^2 \rangle$ perpendicular to the axis of easiest magnetization (the z axis)

$$\langle \mathfrak{M}^2 \rangle = \left\langle \left[\int \mathbf{M}(\mathbf{r}, t)\, dv \right]^2 \right\rangle$$
$$= M_0 V \left[M_0 V - 2\mu \sum_{\mathbf{k}} |v_{\mathbf{k}}|^2 - 2 \sum_{\mathbf{k}} \mu_{\mathbf{k}} n_{\mathbf{k}} \right],$$
$$\langle \mathfrak{M}_\perp^2 \rangle = \langle \mathfrak{M}_x^2 \rangle + \langle \mathfrak{M}^2 \rangle = 2\mu n_0 M_0 V,$$
$$\mu_{\mathbf{k}} = \mu A_{\mathbf{k}}/\varepsilon_{\mathbf{k}} = -\partial \varepsilon_{\mathbf{k}}/\partial H. \tag{17}$$

The presence of weak interactions ($L_{\mathbf{k}}^{(p)}, L_{\mathbf{k}}^{(r)}$) causes the quantities γ, T_s, T_p, and n_0 to change in time, but slowly compared to the establishment of the distribution (16).

Proceeding as in reference 1, we can obtain the following equations to determine the quantities γ, n_0, and $\Delta T = T_s - T_p$

$$\Delta\dot{T} + G_1 \dot{\gamma} - \varepsilon_0 \dot{n}_0 / c = B_{\gamma\gamma}\gamma + B_{\gamma 0} n_0,$$
$$\Delta\dot{T} + G_2 \dot{\gamma} + \varepsilon_0 \dot{n}_0 / c_s = B_{TT}\Delta T, \qquad \dot{n}_0 = -n_0/\tau_\perp, \tag{18}$$

where

$$c_s = \frac{15}{32}\frac{\zeta(5/2)}{\pi^{3/2}}\frac{V}{a^3}\left(\frac{T}{\Theta_c}\right)^{3/2}, \qquad c_p = \frac{2\pi^2}{5}\frac{V}{a^3}\left(\frac{T}{\Theta_D}\right)^3$$

are the spin and phonon specific heats, ϵ_0 the energy of a spin wave with $\mathbf{k} = 0$, and

$$G_1 = -\frac{T}{c_p}\frac{\partial}{\partial T}\sum_{\mathbf{k}} n_{\mathbf{k}}^0 + \frac{c_s + c_p}{c_p}\left(\frac{\partial}{\partial H}\sum_{\mathbf{k}} n_{\mathbf{k}}^0 \Big/ \frac{\partial}{\partial T}\sum_{\mathbf{k}} \mu_{\mathbf{k}} n_{\mathbf{k}}^0\right),$$
$$G_2 = -\frac{T}{c_s}\frac{\partial}{\partial T}\sum_{\mathbf{k}} n_{\mathbf{k}}^0, \tag{19}$$

$$B_{\gamma\gamma} = A \sum_{\mathbf{k}} \left(\frac{\partial L_{\mathbf{k}}}{\partial \gamma}\right)_0 \mu_{\mathbf{k}} = -\frac{8\pi A}{\hbar T}\sum_{123} |\Phi_{12,3}|^2 (\mu_1 + \mu_2 - \mu_3)$$
$$\times n_1^0 n_2^0 (n_3^0 + 1)\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3)\, \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3),$$

$$B_{\gamma_0} = A \sum_{\mathbf{k}} \mu_{\mathbf{k}} \left(\frac{\partial L_{\mathbf{k}}}{\partial n_0}\right)_0$$
$$= \frac{16\pi A}{\hbar}\sum_{\mathbf{k}} |\Phi_{\mathbf{k},-\mathbf{k};0}|^2 (1 + 2n_{\mathbf{k}}^0)\mu_{\mathbf{k}}\, \delta(\varepsilon_0 - 2\varepsilon_{\mathbf{k}}), \tag{20}$$

$$B_{TT} = \left(\frac{1}{c_s} + \frac{1}{c_p}\right)\sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \left(\frac{\partial L_{\mathbf{k}}}{\partial \Delta T}\right)_0$$
$$= -\frac{2\pi\hbar}{T^2}\frac{c_p + c_s}{c_p c_s}\sum_{123} |\Psi_{123}|^2 \omega_3^2 (n_1^0 + 1)$$
$$\times n_2^0 N_3^0\, \delta(\varepsilon_1 - \varepsilon_2 - \hbar\omega_3)\, \Delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{f}_3), \tag{21}$$

$$\frac{1}{\tau_1} = -\left(\frac{\partial L_0}{\partial n_0}\right)_0 = \frac{8\pi}{\hbar}\sum_{\mathbf{k}} |\Phi_{\mathbf{k},-\mathbf{k};0}|^2 (1 + 2n_{\mathbf{k}}^0)\, \delta(\varepsilon_0 - 2\varepsilon_{\mathbf{k}}),$$
$$A = (1/c_s + 1/c_p)T/G_2\mu. \tag{22}$$

Assuming the quantities ΔT, γ, and n_0 to change with time as $e^{-\lambda t}$ we find the following expressions for the relaxation constants

$$\lambda_{1,2} = \frac{G_1 B_{TT} + B_{\gamma\gamma} \pm \sqrt{(G_1 B_{TT} - B_{\gamma\gamma})^2 + 4B_{TT}B_{\gamma\gamma}G_2}}{2(G_2 - G_1)},$$
$$\lambda_3 = \frac{1}{\tau_\perp}. \tag{23}$$

If $T \gg \Theta_c (\mu M_0/\Theta_c)^{4/7} \gg 2\pi\mu M_0$ we get

$$\lambda_1 \approx \begin{cases} \dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^{1/2}\dfrac{T}{\Theta_c}\ln^{-1}\dfrac{T}{\mu M_0}, & \dfrac{\mu M_0}{T} \ll \beta + \dfrac{H_0}{M_0} \ll 1 \\ \dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^{1/2}\dfrac{T}{\Theta_c}, & \beta + \dfrac{H_0}{M_0} \sim 1, \end{cases} \tag{24}$$

$$\lambda_2 = \begin{cases} \left(\dfrac{\hbar}{pa^5}\right)[\delta_1^2 + 2(\delta_1 + \delta_2)^2]\left(\dfrac{T}{\Theta_c}\right)^{1/2}, & T \gg \dfrac{\Theta_D^2}{\Theta_c} \\ \left(\dfrac{\hbar}{pa^5}\right)\delta_1^2\left(\dfrac{T}{\Theta_c}\right)^2\exp\left(-\dfrac{\Theta_t^2}{4T\Theta_c}\right), & T \ll \dfrac{\Theta_D^2}{\Theta_c}, \end{cases} \tag{25}$$

where Θ_D is the Debye temperature, $\Theta_t = s_t\hbar/a$, and s_t the velocity of a transverse sound wave.

It is clear from Eqs. (22) and (23) that the relaxation time of the quantity n_0 is determined by the process where a spin wave with wave vector $\mathbf{k} = 0$ splits into two spin waves with wave vectors $\mathbf{k}$ and $-\mathbf{k}$. This process can occur when the energy conservation law

$$\varepsilon_0 = 2\varepsilon_{\mathbf{k}},$$

where ϵ_0 is the energy connected with the uniform precession of the magnetic moment, is satisfied. Kittel[7] has shown that the uniform precession frequency ω_0 and thus $\epsilon_0 = \hbar\omega_0$ is strongly shape dependent. If we consider a ferrodielectric occupying one half of space bounded by one of the crystal planes with the field H_0 in this plane, then

$$\varepsilon_0 = \mu\sqrt{(H_0 + \beta M_0)(H_0 + \beta M_0 + 4\pi M_0)}\,. \qquad (26)$$

We can obtain the expression for ϵ_0 from Eqs. (3) and (6) by putting $\mathbf{k} = 0$ and $\theta_{\mathbf{k}} = \pi/2$ which corresponds to a quantum mechanical consideration of uniform oscillations of the magnetic moment.

In evaluating the energy spin wave $\epsilon_{\mathbf{k}}$ we neglect the influence of boundary effects* which is permissible provided the spin-wave mean free path l is much shorter than the dimensions of the specimen L

$$l = \bar{v} / W^{(e)} \approx (\Theta_c / T)^{7/2} a \ll L. \qquad (27)$$

When condition (27) is satisfied, the average spin wavelength $\bar{\lambda} \sim (\Theta_c/T)^{1/2} a$ is at the same time much shorter than the dimensions L of the body.

Using (26) one sees easily that the energy conservation law $\epsilon_0 = 2\epsilon_{\mathbf{k}}$ is satisfied, provided

$$\beta + H_0 / M_0 < 4\pi / 3. \qquad (28)$$

We do not give here the detailed calculations, but quote the final results for λ_3

$$\lambda_3 = \frac{1}{\tau_\perp} \approx \begin{cases} \dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^{1/2}\dfrac{T}{\Theta_c}, & \beta + \dfrac{H_0}{M_0} \ll 1, \\ 10^2\,\dfrac{\mu M_0}{\hbar}\left(\dfrac{\mu M_0}{\Theta_c}\right)^{1/2}\dfrac{T}{\Theta_c}\left(\dfrac{4\pi}{3} - \beta - \dfrac{H_0}{M_0}\right)^{5/2}, & \dfrac{4\pi}{3} - \beta - \dfrac{H_0}{M_0} \ll 1. \end{cases} \qquad (29)$$

The quantity $\tau_\perp$ tends in this approximation to infinity for $\beta + H_0/M_0 = 4\pi/3$, as should have been expected. This means that it is now necessary to take the relativistic interactions which describe the spin wave-spin wave scattering into account to evaluate $\tau_\perp$, as was done before.[1] Using Eq. (18), and also Eq. (19), we get the following formulae which describe the change in time of the quantities $\langle\mathfrak{M}^2\rangle$, $\langle\mathfrak{M}_\perp^2\rangle$, and ΔT

$$\frac{\Delta T}{T} = \frac{\Delta T_0}{T} e^{-\lambda_2 t} + \frac{(2\pi)^2}{3\Gamma(3/2)\zeta(3/2)}\frac{G_2}{G_1}\left(\frac{\Theta_c}{T}\right)^{3/2}\frac{\langle\mathfrak{M}_0^2\rangle - \overline{\mathfrak{M}}^2}{(M_0 V)^2}$$
$$\times (e^{-\lambda_2 t} - e^{-\lambda_1 t}) + \frac{G_2}{2G_1}\frac{\mu M_0}{\hbar(\lambda_3 - \lambda_1)(2\pi + \beta + H_0/M_0)}\left(\frac{\mu M_0}{T}\right)^{1/2}$$
$$\times \frac{\langle\mathfrak{M}_{\perp 0}^2\rangle}{(M_0 V)^2}\left[e^{-\lambda_2 t} - e^{-\lambda_1 t} + \frac{\lambda_3}{\lambda_1}(e^{-\lambda_2 t} - e^{-\lambda_3 t})\right], \qquad (30)$$

$$\frac{\langle\mathfrak{M}^2\rangle - \overline{\mathfrak{M}}^2}{(M_0 V)^2} = \frac{\langle\mathfrak{M}_0^2\rangle - \overline{\mathfrak{M}}^2}{(M_0 V)^2} e^{-\lambda_1 t} + \frac{3\Gamma(3/2)\zeta(3/2)}{2\pi^2}\left(\frac{\lambda_1}{\lambda_2}\right)^2\left(\frac{T}{\Theta_c}\right)^{3/2}$$
$$\times \frac{\Delta T_0}{T}(e^{-\lambda_2 t} - e^{-\lambda_1 t}) + \frac{3\Gamma(3/2)\zeta(3/2)}{8\pi^2}\frac{\mu M_0}{\hbar(\lambda_3 - \lambda_1)(2\pi + \beta + H_0/M_0)}$$
$$\times \left(\frac{\mu M_0}{\Theta_c}\right)^{1/2}\left(\frac{T}{\Theta_c}\right)\frac{\langle\mathfrak{M}_{\perp 0}^2\rangle}{(M_0 V)^2}(e^{-\lambda_1 t} - e^{-\lambda_3 t}), \qquad (31)$$

$$\langle\mathfrak{M}_\perp^2\rangle = \langle\mathfrak{M}_{\perp 0}^2\rangle e^{-\lambda_3 t}, \qquad (32)$$

*We are indebted to M. I. Kaganov for this remark.

where ΔT_0, $\langle\mathfrak{M}_{\perp 0}^2\rangle$, and $\langle\mathfrak{M}_0^2\rangle$ are the initial values of the temperature difference, the transverse components, and the magnitude of the magnetic moment of the body, while $\overline{\mathfrak{M}^2}$ is the average value of the absolute magnitude of the magnetic moment at the given temperature.

It is clear from the formulae given here that $2/\lambda_3$ is the relaxation time of the transverse component of the magnetic moment of the body.

To elucidate the physical meaning of the relaxation constant λ_1 we assume that $\Delta T_0 = \langle\mathfrak{M}_{\perp 0}^2\rangle = 0$. We have then

$$\langle\mathfrak{M}^2\rangle - \overline{\mathfrak{M}}^2 = [\langle\mathfrak{M}_0^2\rangle - \overline{\mathfrak{M}}^2]\, e^{-\lambda_1 t}. \qquad (33)$$

Under those initial conditions $1/\lambda_1$ has thus the meaning of the relaxation time of the mean square of the magnetic moment of the body.

If the initial data are such that $\langle\mathfrak{M}_{\perp 0}^2\rangle = 0$ and $\langle\mathfrak{M}_0^2\rangle = \overline{\mathfrak{M}^2}$ then

$$\Delta T = \Delta T_0 e^{-\lambda_2 t}, \qquad (34)$$

and we must treat the quantity $1/\lambda_2$ as the time for equalizing the spin and lattice temperatures.

Let us estimate the magnitudes of λ_1, λ_2, and λ_3. Putting

$\Theta_c \sim 10^3\,°\mathrm{K}$, $\Theta_l \sim 10^2\,°\mathrm{K}$, $\beta + H_0/M_0 \sim 1$,

$M_0 \sim 10^3$ emu, $\mu = 10^{-20}$ emu,

$\rho = 10\,\mathrm{g/cm^3}$, $a \approx 2\cdot 10^{-8}\,\mathrm{cm}$,

we get

$\lambda_1 \sim \lambda_3 \sim 10^8 (T/\Theta_c)\,\mathrm{sec}^{-1}$, $\lambda_2 \sim 10^{11}(T/\Theta_c)^{5/2}\,\mathrm{sec}^{-1}$.

Provided $\beta + H_0/M_0$ is not too close to $4\pi/3$, the relaxation times of the absolute magnitude and of the transverse component of the magnetic moment are thus of the same order of magnitude. One should describe such a relaxation of the magnetic moment phenomenologically by the Bloch equation. In this equation is contained the difference between the relaxation for the case of a small anisotropy and weak fields and the relaxation in the case of large anisotropy or strong fields, when the establishment of the equilibrium value of the magnetic moment is appreciably faster than the rotation of the moment towards its equilibrium direction.

In the temperature range $T \ll \Theta_c(\mu M_0/\Theta_c)^{4/7}$ the main role in the transport equation is played by the operator $L_{\mathbf{k}}^{(r)}\{n\}$. In ferrodielectrics with a small magnetic anisotropy the equilibrium value of the magnetic moment is thus at these temperatures established at the same time as the Bose distribution of the spin waves.

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APPENDIX

We consider a ferrodielectric with two magnetic sublattices, the Hamiltonian of which can be written in the form

$$\mathcal{H} = \int \Big\{ \frac{1}{2}\alpha_1 \frac{\partial \mathbf{M}_1}{\partial x_i}\frac{\partial \mathbf{M}_1}{\partial x_i} + \frac{1}{2}\alpha_2 \frac{\partial \mathbf{M}_2}{\partial x_i}\frac{\partial \mathbf{M}_2}{\partial x_i} + \alpha_{12}\frac{\partial \mathbf{M}_1}{\partial x_i}\frac{\partial \mathbf{M}_2}{\partial x_i} + \gamma \mathbf{M}_1\mathbf{M}_2 + \frac{\mathbf{H}^2}{8\pi}\Big\} dv, \tag{A.1}$$

where $\mathbf{M}_j$ is the magnetic moment of the j-th sublattice, $\mathbf{H}$ the magnetic field acting in the ferrodielectric, and α_1, α_2, α_{12}, and γ quantities which are connected with the exchange integrals

$$\alpha_1 \sim \alpha_2 \sim \alpha_{12} \sim \Theta_c a^2/\mu M_0, \qquad \gamma \sim \Theta_c/\mu M_0.$$

We do not write down here the anisotropy energy which is inessential for the following.

Putting

$$\mathbf{M}_j = \mathbf{M}_{0j} + \mathbf{m}_j, \qquad \mathbf{H} = \mathbf{H}_0 + \mathbf{h},$$

where $\mathbf{M}_{0j}$ is the equilibrium value of the magnetic moment of the j-th sublattice, $\mathbf{H}_0$ the constant magnetic field, in the direction of the z axis, and $\mathbf{m}$ and $\mathbf{h}$ small corrections to $\mathbf{M}_{0j}$ and $\mathbf{H}_0$, we find

$$\mathcal{H} = \int \Big\{ \frac{1}{2}\alpha_1 \frac{\partial \mathbf{m}_1}{\partial x_i}\frac{\partial \mathbf{m}_1}{\partial x_i} + \frac{1}{2}\alpha_2 \frac{\partial \mathbf{m}_2}{\partial x_i}\frac{\partial \mathbf{m}_2}{\partial x_i} + \alpha_{12}\frac{\partial \mathbf{m}_1}{\partial x_i}\frac{\partial \mathbf{m}_2}{\partial x_i} + \gamma M_{01}^z m_2^z + \gamma M_{02}^z m_1^z + \gamma \mathbf{m}_1\mathbf{m}_2 - \boldsymbol{H}_0 (m_1^z + m_2^z) + \frac{\mathbf{h}^2}{8\pi}\Big\} dv$$

Going over from $\mathbf{m}$ and $\mathbf{h}$ to their Fourier components

$$\mathbf{m}_j = \frac{1}{V^{1/2}}\sum_{\mathbf{k}} \mathbf{m}_{\mathbf{k}j} e^{i\mathbf{k}\mathbf{r}}, \qquad \mathbf{h} = \frac{1}{V^{1/2}}\sum_{\mathbf{k}} \mathbf{h}_{\mathbf{k}} e^{ik} \tag{A.2}$$

and using the equations of magnetostatics, we get

$$\mathcal{H} = \sum_{\mathbf{k}}\Big[\frac{1}{2}\alpha_1 k^2 \mathbf{m}_{\mathbf{k}1}\mathbf{m}_{-\mathbf{k}1} + \frac{1}{2}\alpha_2 k^2 \mathbf{m}_{\mathbf{k}2}\mathbf{m}_{-\mathbf{k}2} + \alpha_{12}k^2 \mathbf{m}_{\mathbf{k}1}\mathbf{m}_{-\mathbf{k}2} + \gamma V^{1/2}(M_{01}^z \mathbf{m}_{\mathbf{k}2}^z + M_{02}^z \mathbf{m}_{\mathbf{k}1}^z)\Delta(\mathbf{k}) + \gamma \mathbf{m}_{\mathbf{k}1}\mathbf{m}_{-\mathbf{k}2} - HV^{1/2}(\mathbf{m}_{\mathbf{k}1}^z + \mathbf{m}_{\mathbf{k}2}^z)\Delta(\mathbf{k}) + (2\pi/k^2)(\mathbf{m}_{\mathbf{k}1} + \mathbf{m}_{\mathbf{k}2}, \mathbf{k}) \times (\mathbf{m}_{-\mathbf{k}1} + \mathbf{m}_{-\mathbf{k}2}, \mathbf{k})\Big]. \tag{A.3}$$

We now define the operators of the magnetic moments $\mathbf{m}_1$ and $\mathbf{m}_2$ by the equations

$$m_1 = (2\mu M_1)^{1/2}(1 - \mu a_1^+ a_1/2M_1)^{1/2} a_1 \approx (2\mu M_1)^{1/2}(a_1 - \mu a_1^+ a_1 a_1/4M_1),$$

$$m_1^+ = (2\mu M_1)^{1/2} a_1^+ (1 - \mu a_1^+ a_1/2M_1)^{1/2} \approx (2\mu M_1)^{1/2}(a_1^+ - \mu a_1^+ a_1^+ a_1/4M_1),$$

$$m_1^z = M_1^z - M_1 = -\mu a_1^+ a_1,$$

$$m_2^+ = (2\mu M_2)^{1/2}(1 - \mu a_2^+ a_2/2M_2)^{1/2} a_2 \approx (2\mu M_2)^{1/2}(a_2 - \mu a_2^+ a_2 a_2/4M_2),$$

$$m_2^- = (2\mu M_2)^{1/2} a_2^+ (1 - \mu a_2^+ a_2/2M_2)^{1/2} \approx (2\mu M_2)^{1/2}(a_2^+ - \mu a_2^+ a_2^+ a_2/4M_2),$$

$$m_2^z = M_2^z + M_2 = \mu a_2^+ a_2, \qquad m_j^{\pm} = m_{xj} \pm i m_{yj},$$

$$M_1 = M_{01}^z > 0, \qquad M_2 = -M_{20}^z > 0,$$

$$[a_j(\mathbf{r}, t), a_{j'}(\mathbf{r}', t)] = \delta_{jj'}^{+}\delta(\mathbf{r} - \mathbf{r}'). \tag{A.4}$$

Using these formulae one can express the Hamiltonian $\mathcal{H}$ in terms of the variables $a_{\mathbf{k}j}$ and $a_{\mathbf{k}j}^+$

$$a_{\mathbf{k}j} = \frac{1}{V^{1/2}}\sum_{\mathbf{k}} a_j(\mathbf{r}, t)e^{-i\mathbf{k}\mathbf{r}}, \qquad a_{\mathbf{k}j}^+ = \frac{1}{V^{1/2}}\sum_{\mathbf{k}} a_j^+(\mathbf{r}, t)e^{i\mathbf{k}\mathbf{r}}. \tag{A.5}$$

We then get

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}^{(3)} + \mathcal{H}^{(4)}, \tag{A.6}$$

where

$$\mathcal{H}_0 = \sum_{\mathbf{k}}(A_{\mathbf{k}1}a_{\mathbf{k}1}^+ a_{\mathbf{k}1} + A_{\mathbf{k}2}a_{\mathbf{k}2}^+ a_{\mathbf{k}2} + B_{\mathbf{k}}a_{\mathbf{k}1}a_{-\mathbf{k}2} + B_{\mathbf{k}}^* a_{-\mathbf{k}2}^+ a_{\mathbf{k}1}^+),$$

$$A_{\mathbf{k}1} = \mu(\alpha_1 k^2 M_1 + \gamma M_2 + H), \quad A_{\mathbf{k}2} = \mu(\alpha_2 k^2 M_2 + \gamma M_1 - H),$$

$$B_{\mathbf{k}} = \mu(M_1 M_2)^{1/2}(\alpha_{12}k^2 + \gamma). \tag{A.7}$$

We have neglected here the influence of the magnetic dipole interaction on the spectrum; taking it into account does not change the result, but complicates the calculations greatly.

The Hamiltonians $\mathcal{H}^{(3)}$ and $\mathcal{H}^{(4)}$ contain, respectively, three and four operators $a_{\mathbf{k}j}$ and $a_{\mathbf{k}j}^+$.

The Hamiltonian $\mathcal{H}_0$ which is quadratic in the operators $a_{\mathbf{k}j}$ and $a_{\mathbf{k}j}^+$ can be diagonalized by a Bogolyubov canonical transformation[9]

$$a_{\mathbf{k}1} = u_{11}c_{\mathbf{k}1} + v_{12}^* c_{-\mathbf{k}2}^+, \qquad a_{\mathbf{k}2} = u_{22}c_{\mathbf{k}2} + v_{21}^* c_{-\mathbf{k}1}^+, \tag{A.8}$$

where the operators $c_{\mathbf{k}j}$ and $c_{\mathbf{k}j}^+$ satisfy the commutation relations

$$[c_{\mathbf{k}j}, c_{\mathbf{k}'j'}^+] = \delta_{jj'}\Delta(\mathbf{k} - \mathbf{k}')$$

and where u and v satisfy the usual conditions

$$|u_{11}|^2 - |v_{12}|^2 = |u_{22}|^2 - |v_{21}|^2 = 1.$$

If we use the equations of motion

$$\dot{a}_{\mathbf{k}j} = (i/\hbar)[\mathcal{H}, a_{\mathbf{k}j}],$$

we can obtain another four equations to determine u and v, and also the eigenvalues of the Hamiltonian $\mathcal{H}_0$. These equations are of the form

$$u_{11}\varepsilon_{\mathbf{k}1} = A_{\mathbf{k}1}u_{11} + B_{\mathbf{k}}v_{21}, \qquad u_{22}\varepsilon_{\mathbf{k}2} = A_{\mathbf{k}2}u_{22} + B_{\mathbf{k}}v_{12},$$

$$-v_{21}\varepsilon_{\mathbf{k}1} = A_{\mathbf{k}2}v_{21} + B_{\mathbf{k}}u_{11}, \qquad -v_{12}\varepsilon_{\mathbf{k}2} = A_{\mathbf{k}1}v_{12} + B_{\mathbf{k}}u_{22},$$

and hence

$$\varepsilon_{\mathbf{k};1,2} = [(A_{\mathbf{k}1} + A_{\mathbf{k}2})^2/4 - B_{\mathbf{k}}^2]^{1/2} \pm (A_{\mathbf{k}1} - A_{\mathbf{k}2})/2,$$

$$v_{12} = v_{21} = (A_{2\mathbf{k}} - \varepsilon_{\mathbf{k}2})/[B_{\mathbf{k}}^2 - (A_{\mathbf{k}2} - \varepsilon_{\mathbf{k}2})^2]^{1/2},$$

$$u_{11} = u_{22} = -B_{\mathbf{k}}/[B_{\mathbf{k}}^2 - (A_{\mathbf{k}2} - \varepsilon_{\mathbf{k}2})^2]^{1/2}. \tag{A.9}$$

Since $\gamma \gg 1$, we have*

*We note that the expression for ε_1 given in reference 8 is incorrect.

$$\varepsilon_1 \approx (\mu / M)(\alpha_1 k^2 M_1^2 + \alpha_2 k^2 M_2^2 - 2\alpha_{12} k^2 M_1 M_2) + \mu H,$$
$$\varepsilon_2 \approx \mu\gamma M - \mu H + (\mu M_1 M_2 / M)(\alpha_1 + \alpha_2 - 2\alpha_{12}) k^2,$$
$$v_{12} = v_{21} \approx (M_2 / M)^{1/2}, \qquad u_{11} = u_{22} \approx -(M_1 / M)^{1/2},$$
$$M = M_1 - M_2 > 0.$$

We see that one of the branches of the energy spectrum has a large activation energy caused by the exchange interaction between the sublattices. The contribution from the oscillations of the magnetic moments described by the variables $c_{\mathbf{k}2}$ and $c^+_{\mathbf{k}2}$ to the thermodynamic and transport properties of the ferrodielectric at low temperatures is thus exponentially small, and when one studies these properties of the ferrodielectric one need only take into account the low-frequency spin waves, described by the operators $c_{\mathbf{k}1}$ and $c^+_{\mathbf{k}1}$.

The interactions between the low-frequency spin waves themselves, caused by the exchange energy connected with the inhomogeneities of the magnetic moments and by the magnetic dipole energy, have the same structure as in the case of a ferrodielectric with one magnetic sublattice.

As regards the interactions caused by the exchange energy between sublattices when the magnetization is homogeneous,

$$\mathcal{H}^{\gamma}_{int} = -\frac{\gamma\mu^2}{4}\int\Big[\Big(\frac{M_2}{M_1}\Big)^{1/2}(a_1^+ a_1^+ a_1 a_2^+ + a_1^+ a_1 a_1 a_2) + \Big(\frac{M_1}{M_2}\Big)^{1/2}(a_1^+ a_2^+ a_2^+ a_2 + a_1 a_2^+ a_2 a_2) + 4a_1^+ a_1 a_2^+ a_2\Big] dv,$$

one can use (A.5), (A.8), and (A.9) to show that that part of it which describes the interactions of the low frequency spin waves with one another does not contain the large parameter γ and is of the form

$$\mathcal{H}^{\gamma}_{int} = \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \Phi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} c^+_{\mathbf{k}_1 1} c^+_{\mathbf{k}_2 1} c_{\mathbf{k}_3 1} c_{\mathbf{k}_4 1} \Delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4),$$

where

$$\Phi_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \sim \mu^2 / V,$$

i.e., $\mathcal{H}^{(\gamma)}_{int}$ is a small correction compared with $\mathcal{H}^{(3)}$. This makes it possible to neglect the magnetic structure when studying the low-temperature properties of ferrodielectrics.

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