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ON THE ENERGY LEVELS OF μ MESIC ATOMS

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An approximate method is proposed for the calculation of the energy levels of μ mesic atoms from radial Dirac equations with a potential which simultaneously accounts for both nuclear volume and screening effects.

1. For the calculation of the energy levels and the wave functions of the meson in mesic atoms it is necessary to solve the quantum mechanical problem of the motion of a meson in the electrostatic field of the nucleus with the most reasonable (from the point of view of nuclear physics) distribution of protons. In the case of μ mesons one usually solves the relativistic Dirac equation with the potential

$$V(r) = Ze(3R^2 - r^2)/2R^3 \qquad (r < R),$$
$$V(r) = Ze/r \qquad (r > R), \qquad (1)$$

which corresponds to a proton distribution whose density is constant inside, and zero outside the nucleus. The screening effect is neglected by assuming that the electrons, which move at relatively large distances from the nucleus, do not have any appreciable effect on the μ meson which moves close to, or inside, the nucleus.

These considerations, however, are not the only reason for neglecting the screening effect in the calculation of the energy levels. Another reason which made the inclusion of the screening effect impossible was the absence of reliable approximate methods for integrating the Dirac equation with a combined potential of more complicated form than the potential (1). We note that even the integration of the radial Dirac equations with a potential of such a simple structure as potential (1) cannot be done exactly. The most exact solution of this problem is obtained by joining the wave functions at the boundary of the nucleus.[1]

In the present paper we propose the use of the quasi-classical method of solving the radial Dirac problem (see reference 2) for the calculation of the energy levels of μ mesic atoms including nuclear volume and screening effects.

2. We propose to take account simultaneously of both the nuclear volume and screening effects in the calculation of the energy levels of μ mesic atoms with the help of a combined potential which inside the nucleus is a Thompson oscillator potential and outside, a nonrelativistic statistical Thomas-Fermi potential, corrected at small distances from the nucleus:

$$U(r) = Ze(3R^2 - r^2)/2R^3 \qquad (r < R),$$
$$U(r) = Ze[1 - B(r - R)]/r \qquad (R < r < a_0/Z),$$
$$U(r) = e/r + (Z-1)e\varphi_0(r/\mu)/r \qquad (r > a_0/Z), \qquad (2)$$

where R is the radius of the nucleus, $\varphi_0(r/\mu)$ is the universal function of the statistical model of the atom, $\mu = 0.8853\,a_0/(Z-1)^{1/3}$, and a_0 is the Bohr radius of the hydrogen atom. The constant B, as determined from the continuity condition on the potential (2) at $r = a_0/Z$, is equal to

$$B = (Z-1)[1 - \varphi_0(a_0/\mu Z)]/(a_0 - RZ). \qquad (3)$$

The qualitative agreement of potential (2) with potential (1) allows us to solve the radial Dirac problem with the potential (2) for the μ mesic atom with the help of the quasi-classical method developed by the author,[2] by constructing an approximate quasi-classical solution of the radial Dirac problem with the potential (2) from the more exact solution of the analogous problem with the potential (1), obtained by the method of joining the wave functions at the nuclear boundary.[1]

As was shown earlier,[2] the approximate eigenvalues of the radial Dirac problem are, in the above-mentioned quasi-classical method, determined by the equations (14a) and (14b), which in our case have the form

$$\int_{r_1}^{r_2} K(\rho)\,d\rho = \int_{s_1}^{s_2} L(\sigma)\,d\sigma \quad \text{(for states with } \varkappa \neq -1), \qquad (4)$$

$$\int_{0}^{r_1} K(\rho)\,d\rho = \int_{0}^{s_1} L(\sigma)\,d\sigma \qquad \text{(for S states; } \varkappa = -1), \qquad (5)$$

where κ is the quantum number of the radial Dirac problem, and r_1 and r_2 are the zeroes of the function under the integral sign, $K(\rho)$; s_1 and s_2 are the zeroes of $L(\sigma)$. The functions $K(\rho)$ and $L(\sigma)$ have the form

$$K^2(\rho) = \alpha^2\left[E + \frac{Z}{2R^3}(3R^2 - \rho^2)\right]^2 + 2\left[E + \frac{Z}{2R^3}(3R^2 - \rho^2)\right] - \frac{\varkappa(\varkappa+1)}{\rho^2} + \frac{Z\varkappa}{(E + 2/\alpha^2 + 3Z/2R)R^3 - Z\rho^2/2} \quad (\text{for } \rho \leqslant R);$$

$$K^2(\rho) = \alpha^2(E - ZB + Z^*/\rho)^2 + 2(E - ZB + Z^*/\rho) - \frac{\varkappa(\varkappa+1)}{\rho^2} + \frac{Z^*\varkappa}{\rho^2[Z^* + (E - ZB + 2/\alpha^2)/\rho]} \quad (\text{for } \rho \geqslant R), \tag{6}$$

where $Z^* = Z(1+BR)$ and α is the fine-structure constant;

$$L^2(\sigma) = \alpha^2\left[\varepsilon + \frac{Z}{2R^3}(3R^2 - \sigma^2)\right]^2 + 2\left[\varepsilon + \frac{Z}{2R^3}(3R^2 - \sigma^2)\right] - \frac{\varkappa(\varkappa+1)}{\sigma^2} + \frac{Z\varkappa}{(\varepsilon + 2/\alpha^2 + 3Z/2R)R^3 - Z\sigma^2/2} \quad (\text{for } \sigma \leqslant R);$$

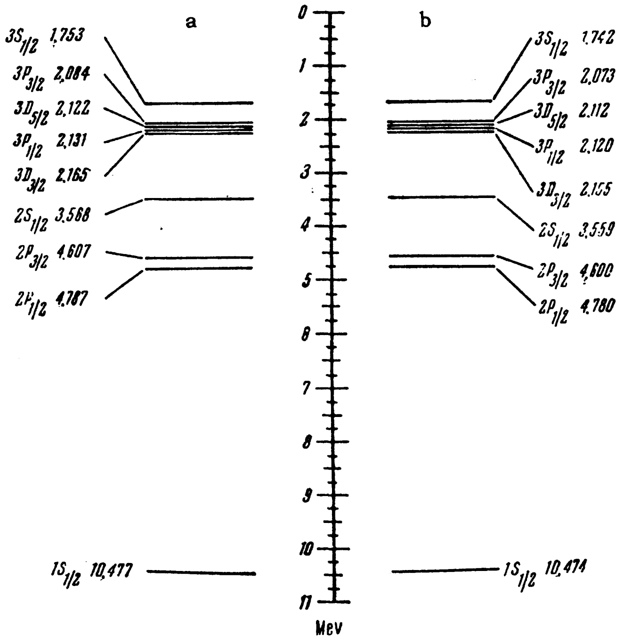
$$L^2(\sigma) = \alpha^2(\varepsilon + Z/\sigma)^2 + 2(\varepsilon + Z/\sigma) - \frac{\varkappa(\varkappa+1)}{\sigma^2} + \frac{Z\varkappa}{\sigma^2[Z + (\varepsilon + 2/\alpha^2)\sigma]} \quad (\text{for } \sigma \geqslant R). \tag{7}$$

In formulas (6) and (7) ϵ is the level energy calculated from the Dirac equations with account of the nuclear volume effect, but neglecting the screening effect, by the method of joining the wave functions at the nuclear boundary;[3] E is the required level energy which includes both the nuclear volume and screening effects. The energies ϵ and E are measured in units of $e^2/r_0 = 5.6366$ kev, where

$$r_0 = \hbar^2/m_\mu e^2 = 2.555 \cdot 10^{-11} \text{ cm};$$

ρ, σ, and R are measured in units of r_0 and the constant B in units of r_0^{-1}. In the functions $K(\rho)$ and $L(\sigma)$ in formulas (6) and (7) we did not include the terms corresponding to the third line of the potential (2), since we do not have to know these terms in the solution of (4) and (5) (the zeros of these functions which are farthest away from the center of the nucleus, r_2 and s_2, lie considerably closer to the nucleus than the point $r = a_0/Z$).

3. The method outlined above was used for the calculation of a number of energy levels of μ mesic lead. The results of the computation are shown in the figure. The comparison of these results with the values of the level energies of μ mesic lead calculated earlier by Pustovalov[3] without account of the screening effect shows that the screening of the nuclear field by the electrons moving at the periphery of the mesic atom leads to an additional level shift which amounts to about 0.03% of the 1S level energy, to 0.15 to 0.25% of the 2P and 2S level energies, and to 0.46 to 0.63% of the energies of the levels 3D, 3P, and 3S. However, this should not lead one to think that these results are unimportant.



Energy levels of μ mesic lead, calculated with account of the nuclear volume effect (for constant density distribution of the protons): a – not including the screening effect, solving the Dirac equations by the method of joining the wave functions at the nuclear boundary;[3] b – including the screening effect, solving the Dirac equations by the quasi-classical method. The numbers near the levels give the values of their energies in Mev. The mass of the meson is 207 m_e, the radius of the nucleus is $1.2 \times 10^{-13} A^{1/3}$ cm, where A = 208.

The results obtained become of definite interest if we compare them with the Lamb shift of the levels in μ mesic lead which is mainly due to the polarization of the electron-positron vacuum. Pustovalov showed[3] that in μ mesic lead the polarization shift of the 1S level is 53 kev, that of the 2P levels is 28 kev, and that of the 2S level 17 kev, if nuclear volume effects are included. The additional level shift due to the screening effect is therefore about 5.7% of the Lamb shift of the 1S level, 25% of the Lamb shift of the 2P levels, and about 53% of the Lamb shift for the 2S level.

A serious shortcoming of the quasi-classical method used in the calculations is the absence of a rigorous criterion for its applicability, as was already pointed out in reference 2. To estimate the accuracy of our calculations we therefore resort to the criterion for the applicability of the usual WKB method in the case of the Coulomb field: the extent of the region which is important in our calculations is mainly given by the potential $Ze[1 - B(r - R)]/r$, which is essentially a potential of the Coulomb type. The above-mentioned criterion (the absolute value of the energy of the μ meson must be small compared with its energy in the first Bohr orbit[4]) is satisfied (poorly at first, but then better and better), starting with the 2P states. The additional shift of 3 kev of the 1S level can be crudely estimated as the energy of the electrostatic interaction (repulsion) between the μ meson inside the nucleus and the $(Z-1)$ electrons which are separated from it by a mean distance of the Bohr radius. This crude estimate gives about 2.2 kev.

In conclusion we note that our method of including simultaneously the nuclear volume and screening effects in the calculation of the energy levels of μ mesic atoms can also be applied in the case of an arbitrary distribution of positive charge over the volume of the nucleus.

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