ANGULAR DISTRIBUTION AND POLARIZATION OF β PARTICLES IN SECOND FORBIDDEN TRANSITIONS

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Explicit formulas for the polarization and angular distribution of β electrons in second forbidden transitions involving V and A coupling are derived. The angular correlations in unique transitions are examined in the case of arbitrary order of forbiddenness. Unique second forbidden transitions are treated in detail.

1. SECOND FORBIDDEN TRANSITIONS $\Delta j = 2$ (no)

THE results of a great number of recent papers indicate that the time reversal invariant vector and axial vector interactions play the fundamental role in β decay processes. Together with the further refinement of these results it becomes of interest to investigate the forbidden β transitions with the aim of determining the nuclear matrix elements. Many papers have been devoted to the study of forbidden β transitions. However, detailed explicit formulas for the angular distribution and polarization of β particles have been given only for transitions of first order of forbiddenness. In the present paper we consider β processes of second and higher order of forbiddenness. Since the method of calculation has already been described in a series of papers,[1] we give at once the final formulas.

a. Angular Distribution of β Electrons for Second Forbidden β Transitions in Oriented Nuclei

We shall characterize the orientation of the nuclear spins by the quantity

$$\rho_g = \sum_{\mu_0} C^{j_0\mu_0}_{j_0\mu_0 g_0} w(\mu_0), \qquad \sum_{\mu_0} w(\mu_0) = 1, \tag{1}$$

where j_0 is the angular momentum of the initial nucleus, μ_0 is its projection on the axis along which most of the nuclear spins are oriented (chosen as the Z axis), $w(\mu_0)$ is the probability that the nucleus has a spin projection with the value μ_0, and $C^{\cdots}_{\cdots}$ are Clebsch-Gordan coefficients.[2] The number g is even for aligned nuclei, and even or odd for polarized nuclei.

$$\sqrt{2g+1}\,\rho_g = \sqrt{2j_0+1}\, j_0^g f_g,$$

where f_g are quantities which are tabulated in the papers of Cox and Tolhoek.[3]

The angular distribution for second forbidden transitions with a change of the nuclear spin by two units ($|j_0 - j_1| = \Delta j = 2$) and without change of parity is given by the formula

$$w(\mathbf{j}_0\mathbf{p}) = 1 + \frac{p}{E} \sum_{g=1,\ldots,4} r_g(j_0, j_1)\, B_g(E, q, p)\, P_g(\cos\vartheta), \tag{2}$$

where j_1 is the spin of the nucleus after the β decay, $P_g(\cos\vartheta)$ is the Legendre polynomial in the angle of the electron momentum vector $\mathbf{p}(p, \vartheta, \varphi)$, E is the total energy of the electron (including the rest mass), q is the momentum (energy) of the neutrino (we choose units for which $\hbar = m = c = 1$),

$$r_g = U(j_1 2 j_0 g;\ j_0 2)\,\rho_g = \sqrt{5(2j_0+1)}\, W(j_1 2 j_0 g;\ j_0 2)\,\rho_g,$$

where W(abcd; ef) is the Racah function (tables of Racah functions are given in reference 4), and

$$\xi B_g = A_1 + A_2 E + A_3(E+V) + A_4(p^2 + 2EV + V^2)$$

$$+ A_5 E(p^2 + 2p^2 V E^{-1} + V^2) + A_6(p^2 + EV),$$

$$\xi = a_1 + a_2(p^2 + 2p^2 V E^{-1} + V^2) + a_3(p^2 E^{-1} + V). \tag{3}$$

Assuming uniform distribution of the charge over the volume of the nucleus, we have $V = 6\alpha Z/5R$. For the case where the charge is distributed over the nuclear surface, we find $V = \alpha Z/R$ (Z is the nuclear charge, R is the nuclear radius).

The quantities a_i are equal to

$$a_1 = \gamma_1(p^2 + q^2) + (\gamma_2 q^4 + \gamma_3 p^4 + \gamma_4 p^2 q^2) + \gamma_5 q\left(p^2 + \tfrac{3}{5} q^2\right),$$

$$a_2 = \eta\left(p^2 + \tfrac{25}{9} q^2\right),$$

$$a_3 = \zeta_1\left(q^2 + \tfrac{3}{5} p^2\right) + \zeta_2 q(p^2 + q^2). \tag{4}$$

The quantities A_i for different values of g have the following values:

$g = 1$:

$$A_1 = (\varphi_1 q^4 + \varphi_2 p^4 + \varphi_3 p^2 q^2) + \varphi_4 q (p^2 + q^2) + \varphi_5 p^2 + \varphi_6 q,$$
$$A_2 = q(\lambda_1 q^2 + \lambda_2 p^2) + (\lambda_3 q^2 + \lambda_4 p^2),$$
$$A_3 = \varphi_7 q (p^2 + \tfrac{5}{9} q^2) + \varphi_8 (p^2 + \tfrac{25}{9} q^2),$$
$$A_4 = \varphi_9 \left(\frac{125}{\sqrt{6}} q^2 + p^2 \right),$$
$$A_5 = 0, \quad A_6 = \lambda_5 q^2 + \lambda_6 p^2;$$

$g = 2$:

$$A_1 = p [q (\chi_1 q^2 + \chi_2 p^2) + (\chi_3 q^2 + \chi_4 p^2)],$$
$$A_2 = p [(\omega_1 q^2 + \omega_2 p^2) + \omega_3 q + \omega_4],$$
$$A_3 = A_4 = 0, \quad A_5 = \omega_5 p, \quad A_6 = p(\omega_6 q + \omega_7);$$

$g = 3$:

$$A_1 = -2\sqrt{7/3}\, p^2 \left[\left(-\frac{5}{42} \varphi_2 q^2 + \frac{1}{\sqrt{6}} \varphi_3 p^2 \right) + \varphi_4 q + \varphi_5 \right],$$
$$A_2 = -2\sqrt{7/3}\, p^2 (\lambda_2 q + \lambda_4),$$
$$A_3 = -2\sqrt{7/3}\, p^2 (\varphi_7 q + \varphi_8),$$
$$A_4 = -2\sqrt{7/3}\, \varphi_9 p^2, \quad A_5 = 0, \quad A_6 = -2\sqrt{7/3}\, \lambda_6 p^2;$$

$g = 4$:

$$A_1 = -6p^3 (\chi_2 q + \chi_4), \quad A_2 = -\tfrac{3}{4} \omega_2 p^3,$$
$$A_3 = A_4 = A_5 = A_6 = 0. \tag{5}$$

The coefficients γ_i, η, ζ_i, φ_i, λ_i, χ_i, and ω_i, which depend on the coupling constants and on the three nuclear matrix elements, are given in Appendix A.

In expression (2) we have made the approximation $(\alpha Z)^2 \ll 1$, $(\alpha Z/p)^2 \ll 1$. In the limiting case where $Z \gg 2A^{1/3} E$ (A is the mass number of the decaying nucleus), the angular distribution no longer depends on the nuclear matrix elements, as in the case of unique transitions. In this case we have

$$w(j_0 p) = 1 - \frac{p}{E} \sum_{g=1,2,3} r_g B_g P_g (\cos \vartheta), \tag{6}$$

where

$$\xi B_1 = \sqrt{6}\left(\tfrac{5}{3} q^2 + \tfrac{9}{25} p^2\right), \quad \xi B_2 = \tfrac{3}{5}\sqrt{14}\, pE,$$
$$\xi B_3 = -\tfrac{18}{25}\sqrt{14}\, p^2, \quad \xi = \tfrac{10}{3} q^2 + \tfrac{6}{5} p^2.$$

Formula (6) corresponds to the two-component theory of the neutrino.

b. Longitudinal Polarization of the β Particles

For β^- transitions with $\Delta j = 2$ (no), the longitudinal polarization vector for the β^- particles in the rest system of the electron can be written in the form

$$\zeta = -\frac{\mathbf{p}}{E}\left(a + \frac{b}{E} + \frac{\alpha Z}{p} c\right). \tag{7}$$

The quantity $a\xi$ is different from ξ in that the constants β_{ik} (which enter in the coefficients γ_i, η, and ζ_i) are replaced by the constants α_{ik}:

$$\alpha_{ik} = C_i C_k'^* + C_i' C_k^*, \quad \beta_{ik} = C_i C_k^* + C_i' C_k'^*,$$
$$i, k = V, A\ T. \tag{8}$$

C_i and C_i' are the coupling constants of the β interaction corresponding to the terms which, respectively, do or do not conserve parity. $b\xi = Va_2 + a_3$, where we also replace the β_{ik} in the quantities a_2 and a_3 on the right hand side of this equation by the α_{ik}.

$$c\xi = 2\,\mathrm{Im}\left[\varkappa_1 q \left(q^2 + \tfrac{1}{2} p^2\right) + \varkappa_2 \left(q^2 + \tfrac{3}{10} p^2\right)\right]. \tag{7'}$$

The energy dependence of ξ, a_2, and a_3 is given by (3) and (4). The coefficients κ_i which depend on the constants C, C′ and the nuclear matrix elements, are given in Appendix A.

If the strong interactions are invariant under time reversal the combinations of matrix elements entering in κ_i are real. A possible violation of time reversal invariance in β interactions is determined by the third term on the right hand side of (7). In the two-component theory of the neutrino $\alpha_{ik} = \beta_{ik}$ and $a = 1$.

In (7) we made the approximation $(\alpha Z)^2 \ll 1$ and $(\alpha Z/p)^2 \ll 1$. If $Z \gg 2A^{1/3} E$ for a given nucleus, the formulas for the longitudinal polarization no longer depend on the nuclear matrix elements: $\boldsymbol{\zeta} = -\mathbf{p}/E$ (here we use the two-component theory of the neutrino).

We note that the general form of (7) is the same for β transitions of arbitrary order of forbiddenness, where the energy dependence of a, b, and c is determined by the order of forbiddenness.

2. UNIQUE TRANSITIONS $\Delta j = N + 1$

The formulas for the angular correlations in forbidden β transitions of the unique type, in which the change of the nuclear spin exceeds the order of forbiddenness by one (i.e., $\Delta j = |j_0 - j_1| = N+1$), do not depend on the nuclear matrix elements. It is therefore impossible to obtain from a study of the unique transitions, any information about the structure of the nucleus, except information on the angular momenta and the parities of the nuclear levels. The latter, though, is the most definite information on these quantities that we have.

The unique transitions are determined by the Gamow-Teller interaction, where apparently only the axial vector (A) coupling gives a significant contribution.

In this part of the paper we consider the angular distribution and polarization of β particles for unique β transitions of arbitrary (N-th) order of forbiddenness in oriented and non-oriented nuclei. The detailed form of the formulas obtained is given for the case of second forbidden transitions: $\Delta j = 3$ (no).

a. Longitudinal Polarization of β Electrons

We shall characterize the polarization of the electrons by the angle χ between the electron momentum $\mathbf{p}$ and the direction of the spin ζ. The polarization is then given by the formula

$$w(\zeta\mathbf{p}) = 1 + \zeta\cos\chi.$$

In the approximation $(\alpha Z)^2 \ll 1$, $(\alpha Z/p)^2 \ll 1$ (the exact formulas are given in Appendix B) we have

$$\zeta = \frac{p}{E+\Delta}\left(\frac{\alpha_{TT}-\alpha_{AA}}{\beta_{TT}+\beta_{AA}} - \frac{\alpha Z}{p}\frac{2\,\mathrm{Im}\,\alpha_{AT}}{\beta_{TT}+\beta_{AA}}d_N\right),$$
$$\Delta = 2(\mathrm{Re}\,\beta_{TA})/(\beta_{TT}+\beta_{AA}), \tag{9}$$

where α_{ik} and β_{ik} are given by (8). The weakly energy dependent coefficient d_N is different for different orders of forbiddenness of the β transition, N:

$$\sigma d_N = \sum_j \mathcal{D}_j^N (j+\tfrac{1}{2})^{-1}, \qquad \sigma = \sum_j \mathcal{D}_j^N,$$
$$\mathcal{D}_j^N = \frac{4^N(2N+3)(N!)^2}{(2N+1)!}\,\frac{p^{2j-1}q^{2N-2j+1}}{(2j)!\,(2N-2j+2)!}. \tag{10}$$

The summation in (10) goes over all half odd integer values of j with $j < N+1$.

It is seen from (9) that the interference terms (between the interactions T and A) violate the time reversal invariance. However, this interference is apparently absent in the β interaction.

For "pure" T or A coupling we obtain:*

$$\zeta = \pm\frac{p}{E}\,\frac{2\,\mathrm{Re}\,CC'^*}{|C|^2+|C'|^2}. \tag{9'}$$

b. Angular Distribution of the Electrons

If we characterize, as before, the orientation of the nuclear spins by the quantity ρ_g [see formula (1)], the angular distribution of the electrons emitted by oriented nuclei in unique transitions has the form

$$w(\mathbf{j}_0\mathbf{p}) = 1 + \sum_{g=1,\ldots}^{2N+1} r_g(j_0, j_1)\,B_g(E, q, p)\,P_g(\cos\vartheta). \tag{11}$$

Here

$$r_g = U(j_1N+1j_0g;\ j_0N+1)\,\rho_g,$$
$$\sigma B_g = a_g + \frac{p\,\mathrm{Re}\,S}{EQ-X}b_g + \frac{\alpha Z\,\mathrm{Im}\,S}{EQ-X}b'_g, \tag{12}$$
$$a_g = \sum_j \mathcal{D}_j^N \xi_j^g, \qquad b_g = \sum_j \mathcal{D}_j^N \eta_j^g, \qquad b'_g = \sum_j \mathcal{D}_j^N \eta_j^g (j+\tfrac{1}{2})^{-1},$$
$$Q = \beta_{TT}+\beta_{AA}, X = -2\,\mathrm{Re}\,\beta_{TA},$$
$$\mathrm{Re}\,S = \alpha_{TT}-\alpha_{AA},\ \mathrm{Im}\,S = -2\,\mathrm{Im}\,\alpha_{AT}. \tag{13}$$

In the general case of N-th order of forbiddenness the numerical coefficients ξ_j^g and η_j^g are given in terms of the Racah functions and the Clebsch-Gordan coefficients:

$$\xi_j^g = (-)^{j-1/2}\sqrt{(2j+g+1)(2j-g)(2g+1)(2N+3)}$$
$$\times C^{g0}_{j-1/2\,0\,j-1/2\,0}\,W(N-j+1\;N+1\;jg;\ jN+1),$$
$$g\text{ — even;}$$
$$\eta_j^g = (-)^{j-1/2}\sqrt{g(g+1)(2g+1)(2N+3)}\,C^{g0}_{j-1/2\,0\,j+1/2\,0}$$
$$\times W(N-j+1\;\;N+1\;jg;\ jN+1),$$
$$g\text{ — odd.} \tag{14}$$

For "pure" T or A coupling we have:

$$\sigma B_g = a_g \pm \frac{p}{E}\,\frac{2\,\mathrm{Re}\,CC'^*}{|C|^2+|C'|^2}b_g. \tag{12'}$$

The numerical values of the coefficients $R_j^{(2)}$, ξ_j^g, and η_j^g for second forbidden β transitions (N=2) are given in Appendix C.

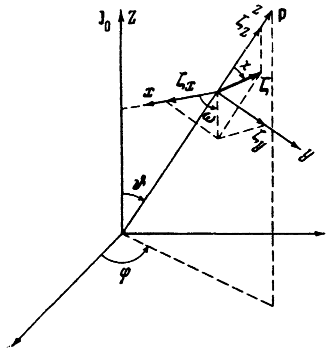
c. Polarization of Electrons Emitted by Oriented Nuclei

The probability for observing an electron with momentum $\mathbf{p}$ and polarization ζ in unique β transitions is

$$w(\mathbf{j}_0, \mathbf{p}, \zeta) = 1 + |\zeta|\cos\chi + \sum_{g=1,\ldots}^{2N+1} r_g B_g P_g(\cos\vartheta)$$
$$+ \sum_{g,s} r_g C_{gs} F_{gs}(\vartheta, \chi, \omega). \tag{15}$$

The directions of the vectors $\mathbf{j}_0$, $\mathbf{p}$, and ζ and the angles between them are given in the figure. The values of $|\zeta|$, r_g, and B_g are determined by the corresponding formulas (9) and (12). The index s in the last term of (15) takes on the values $s = g$, $g \pm 1$, so that the summation over s goes from $s = 0$ to $s = 2N+2$. Furthermore,

*Here, as well as in the analogous formulas below, the upper indices refer to T, and the lower indices to A.



$$\sigma C_{gs} = \left[k^{+}_{gs} + \frac{EX - Q}{EQ - X}\, k^{-}_{gs}\right] + \frac{p}{EQ - X}\,(l_{gs}\,\mathrm{Re}\,S + t_{g}\,\mathrm{Im}\,S) + \frac{\alpha Z}{EQ - X}(-t'_{g}\,\mathrm{Re}\,S + l'_{gs}\,\mathrm{Im}\,S),$$

$$k^{\pm}_{gs} = \sum_{j} \mathcal{D}^{N}_{j} \cdot \tfrac{1}{2}\,(\varepsilon_{j^1/_2} \pm \varepsilon_{j-^1/_2}),$$

$$l_{gs} = \sum_{j} \mathcal{D}^{N}_{j}\zeta_{j}, \qquad l'_{gs} = \sum_{j} \mathcal{D}^{N}_{j}\zeta_{j}\,(j + \tfrac{1}{2})^{-1},$$

$$t_{g} = \sum_{j} \mathcal{D}^{N}_{j}\tau_{j}, \qquad t'_{g} = \sum_{j} \mathcal{D}^{N}_{j}\tau_{j}\,(j + \tfrac{1}{2})^{-1}. \qquad (16)$$

The angular dependence of $F_{gs}(\vartheta, \chi, \omega)$ for different values of s and g is expressed in the following form:

$$F_{gg} = \sqrt{\frac{3(2g+1)}{g(g+1)}}\,\sin\chi\,\sin\omega P^{1}_{g}(\cos\vartheta),$$

$$F_{gg+1} = \sqrt{3(g+1)}\cos\chi P_{g}(\cos\vartheta) + \sqrt{\frac{3}{g+1}}\,\sin\chi\cos\omega P^{1}_{g}(\cos\vartheta),$$

$$F_{gg-1} = -\sqrt{3g}\cos\chi P_{g}(\cos\vartheta) + \sqrt{3/g}\,\sin\chi\cos\omega P^{1}_{g}(\cos\vartheta).$$

$P^{1}_{g}(\cos\vartheta)$ is the first associated Legendre function. The constants Q, X, and S are given by (13).

In the general case of N-th order of forbiddenness the coefficients $\varepsilon_{j\lambda}$, ζ_{j}, and τ_{j} are given in terms of the functions of Racah and Fano[5] and the Clebsch-Gordan coefficients:

$$\varepsilon_{j\lambda} = (-)^{l-1}(2j+1)(2l+1)\sqrt{2(2g+1)(2N+3)}\,C^{s0}_{l0l0} \times W(jN+1\,jg;\ jN+1)X(lls,\ jjg,\ \tfrac{1}{2}\tfrac{1}{2}\,1), \qquad g\text{ — odd},$$

$$\zeta_{j} = (-)^{j+^1/_2}(2j+1)\sqrt{4j(2j+2)(2g+1)(2N+3)} \times C^{s0}_{j-\frac{1}{2}\,0\,j+\frac{1}{2}\,0}\,W(iN+1\,jg;\ jN+1) \times X(j+\tfrac{1}{2}\ j-\tfrac{1}{2}\ s,\ jjg,\ \tfrac{1}{2}\tfrac{1}{2}\,1), \qquad g\text{ — even}$$

$$\tau_{j} = (-)^{j-^1/_2}(2j+1)\sqrt{(2N+3)/3}\,C^{g0}_{j-\frac{1}{2}\,0\,j+\frac{1}{2}\,0} \times W(iN+1\,jg;\ jN+1), \qquad g\text{ — odd}$$

$$l = j + \lambda, \quad \lambda = \pm\tfrac{1}{2}, \quad i = N - j + 1. \qquad (17)$$

A significant polarization of order $\sim p/E$ of the electrons in the direction $[\mathbf{p}\times\mathbf{j}_0]$ can be observed only if time reversal invariance is violated and if there is TA interference. If time reversal invariance holds, or if there is no TA interference, the polarization in this direction is $\sim \alpha Z/E$, i.e., small as compared to unity.

In the case of "pure" T or A coupling the quantity σC_{gs} [formula (16)] is given by

$$\sigma C_{gs} = k^{+}_{gs} - \frac{1}{E}k^{-}_{gs} \pm \frac{p}{E}\,\frac{2\,\mathrm{Re}\,CC'^{*}}{|C|^{2} + |C'|^{2}}\,l_{gs} \mp \frac{\alpha Z}{E}\,\frac{2\,\mathrm{Re}\,CC'^{*}}{|C|^{2} + |C'|^{2}}\,t'_{g}. \qquad (16')$$

The numerical values of the coefficients $\epsilon^{(gs)}_{j\lambda}$, $\zeta^{(gs)}_{j}$, and $\tau^{(g)}_{j}$ for second forbidden β transitions are given in Appendix C.

All formulas quoted above correspond to β^- decay. In the case of positron decay we must make the following substitutions in these formulas:

$$C_{A} \to -C^{*}_{A}, \quad C'_{A} \to C'^{*}_{A}, \quad C_{V,T} \to C^{*}_{V,T}, \quad C'_{V,T} \to -C'^{*}_{V,T}, \quad \alpha Z \to -\alpha Z.$$

APPENDIX A

Coefficients γ_i, η, and ζ_i in the expressions (4):

$$\gamma_1 = |L_V|^2\beta_{VV}, \qquad \gamma_2 = \tfrac{1}{25}(|K_A|^2\beta_{AA} + |K_V|^2\beta_{VV});$$

$$\gamma_3 = \tfrac{1}{125}(2|K_A|^2\beta_{AA} + 3|K_V|^2\beta_{VV} - 2\sqrt{6}\,\mathrm{Re}\,K_AK^{*}_{V}\beta_{AV});$$

$$\gamma_4 = \tfrac{5}{9}\gamma_2 + \tfrac{25}{9}\gamma_3,$$

$$\gamma_5 = \frac{2}{3\sqrt{5}}(-\sqrt{2}\,\beta_{VV}\,\mathrm{Re}\,K_VL^{*}_{V} + \sqrt{3}\,\mathrm{Re}\,K_AL^{*}_{V}\beta_{AV});$$

$$\eta = \tfrac{1}{125}(3|K_A|^2\beta_{AA} + 2|K_V|^2\beta_{VV} + 2\sqrt{6}\,\mathrm{Re}\,K_AK^{*}_{V}\beta_{AV});$$

$$\zeta_1 = -\frac{2}{3\sqrt{5}}(\sqrt{2}\,\beta_{VV}\,\mathrm{Re}\,K_VL^{*}_{V} + \sqrt{3}\,\mathrm{Re}\,K_AL^{*}_{V}\beta_{AV});$$

$$\zeta_2 = \tfrac{2}{75}(-3|K_A|^2\beta_{AA} + 2|K_V|^2\beta_{VV}).$$

Nuclear matrix elements:

$$K_A = [C^{j_0\mu_0}_{2\Lambda j_1\mu_1}]^{-1}\int \psi^*_{j_1\mu_1}\boldsymbol{\sigma}\mathbf{Y}^{0*}_{2\Lambda}\psi_{j_0\mu_0}r^2\,\mathbf{dr};$$

$$K_V = [C^{j_0\mu_0}_{2\Lambda j_1\mu_1}]^{-1}\int \psi^*_{j_1\mu_1}Y^*_{2\Lambda}\psi_{j_0\mu_0}r^2\,\mathbf{dr};$$

$$L_V = [C^{j_0\mu_0}_{2\Lambda j_1\mu_1}]^{-1}\int \psi^*_{j_1\mu_1}i\boldsymbol{\alpha}\mathbf{Y}^{-1*}_{2\Lambda}\psi_{j_0\mu_0}r\,\mathbf{dr}.$$

In these expressions $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ are the four-rowed Dirac matrices, and $\mathbf{Y}^{\tau}_{L\Lambda}$ is the spherical vector function. Its components are expressed in terms of the spherical functions:

$$[\mathbf{Y}^{\tau}_{L\Lambda}]_{\gamma} = (-)^{1-\gamma}C^{L\Lambda}_{L+\tau,\,M1,\,-\gamma}Y_{L+\tau,\,M},$$
$$M = \Lambda + \gamma, \quad \gamma = 0, \pm 1.$$

The scalar product of two vectors $(\mathbf{AB})$ is here conveniently represented in the form

$$(\mathbf{AB}) = \sum_{\gamma}(-)^{\gamma}A_{\gamma}B_{-\gamma},$$
$$A_0 = A_z, \qquad A_{\pm 1} = \pm(A_x \pm iA_y)/\sqrt{2}.$$

Coefficients φ_i, λ_i, χ_i, and ω_i in the expressions (5):

$$\varphi_1 = -\tfrac{1}{25}\left(\sqrt{\tfrac{1}{6}}|K_A|^2\alpha_{AA} + \tfrac{2}{5}\operatorname{Re}K_AK_V^*\alpha_{AV}\right);$$

$$\varphi_2 = -(2\sqrt{6}/625)\left(|K_A|^2\alpha_{AA} + \tfrac{3}{2}|K_V|^2\alpha_{VV} - \sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV}\right);$$

$$\varphi_3 = -\tfrac{1}{225}\sqrt{\tfrac{2}{3}}\,(7|K_A|^2\alpha_{AA} + |K_V|^2\alpha_{VV} - \sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV});$$

$$\varphi_4 = \tfrac{1}{5}\sqrt{\tfrac{6}{5}}\,(\sqrt{3}\operatorname{Re}K_AL_V^*\alpha_{AV} - \sqrt{2}\,\alpha_{VV}\operatorname{Re}K_VL_V^*)\ ;$$

$$\varphi_5 = -\tfrac{1}{5}\sqrt{\tfrac{2}{3}}\left(\tfrac{1}{2}|K_A|^2\alpha_{AA} + \tfrac{1}{3}|K_V|^2\alpha_{VV} + \sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\alpha_{AV}\right)\ ;$$

$$\varphi_6 = -\sqrt{\tfrac{3}{2}}\,|L_V|^2\alpha_{VV};$$

$$\varphi_7 = \tfrac{18}{125}\sqrt{6}\left(\tfrac{1}{2}|K_A|^2\alpha_{AA} - \tfrac{1}{3}|K_V|^2\alpha_{VV} - \sqrt{\tfrac{2}{3}}(\alpha Z/p)\operatorname{Im}K_AK_V^*\alpha_{AV}\right)\ ;$$

$$\varphi_8 = -\tfrac{3}{25}\sqrt{\tfrac{6}{5}}\,[\operatorname{Re}(\sqrt{3}K_AL_V^*\alpha_{AV} - \sqrt{2}K_VL_V^*\alpha_{VV}) + (\alpha Z/p)\operatorname{Im}(\sqrt{3}K_AL_V^*\alpha_{AV} + \sqrt{2}K_VL_V^*\alpha_{VV})];$$

$$\varphi_9 = -\tfrac{2}{625}\left(\tfrac{1}{2}|K_A|^2\alpha_{AA} + \tfrac{1}{3}|K_V|^2\alpha_{VV} + \sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\alpha_{AV}\right);$$

$$\lambda_1 = \tfrac{1}{75}\sqrt{\tfrac{2}{3}}\,[|K_V|^2\alpha_{VV} - \sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV} - (\alpha Z/4p)\sqrt{6}(5E + E^{-1})\operatorname{Im}K_AK_V^*\alpha_{AV}]\ ;$$

$$\lambda_2 = (\sqrt{6}/125)\Big[|K_V|^2\alpha_{VV} + 2|K_A|^2\alpha_{AA} - 3\sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV} + (\alpha Z/2p)\sqrt{\tfrac{1}{6}}(7E + E^{-1})\operatorname{Im}K_AK_V^*\alpha_{AV}\Big]\ ;$$

$$\lambda_3 = -(2/\sqrt{15})\,[\alpha_{VV}\operatorname{Re}K_VL_V^* - (\alpha Z/4p)(5E - E^{-1})\alpha_{VV}\operatorname{Im}K_VL_V^*];$$

$$\lambda_4 = (6/25\sqrt{5})[\sqrt{2}\operatorname{Re}K_AL_V^*\alpha_{AV} - \sqrt{3}\,\alpha_{VV}\operatorname{Re}K_VL_V^* + (\alpha Z/12p)(7E - E^{-1})\operatorname{Im}(-\sqrt{2}K_AL_V^*\alpha_{AV} + \sqrt{3}K_VL_V^*\alpha_{VV})];$$

$$\lambda_5 = \tfrac{1}{15}\sqrt{\tfrac{2}{3}}\,[|K_V|^2\alpha_{VV} + \sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV} - (\alpha Z/4p)\sqrt{6}(5E - E_V^{-1})\operatorname{Im}K_AK_V^*\alpha_{AV}];$$

$$\lambda_6 = (3\sqrt{6}/625)\,[-|K_A|^2\alpha_{AA} + 2|K_V|^2\alpha_{VV} - \sqrt{6}\operatorname{Re}K_AK_V^*\alpha_{AV} + (\alpha Z/4p)\sqrt{6}(7E - E_V^{-1})\operatorname{Im}K_AK_V^*\alpha_{AV}];$$

$$E_V \equiv E + p^2V^{-1};$$

$$\chi_1 = (\sqrt{14}/100)\,(\alpha Z/p)\,(|K_V|^2\beta_{VV} - \sqrt{6}\operatorname{Re}K_AK_V^*\beta_{AV});$$

$$\chi_2 = (1/30\sqrt{14})\,(\alpha Z/p)\,(|K_A|^2\beta_{AA} + |K_V|^2\beta_{VV} - 5\sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\beta_{AV});$$

$$\chi_3 = (\sqrt{14}/180)\,(\alpha Z/p)\,[5E|K_V|^2\beta_{VV} + 3(E + V)|K_V|^2\beta_{VV} + 9\sqrt{10}\,\beta_{VV}\operatorname{Re}K_VL_V^*];$$

$$\chi_4 = -(6/125\sqrt{14})\,(\alpha Z/p)\Big[\tfrac{7}{24}E\,(|K_A|^2\beta_{AA} - |K_V|^2\beta_{VV} - 7\sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\beta_{AV}) + \tfrac{5}{12}(E + V)\,(|K_A|^2\beta_{AA} - |K_V|^2\beta_{VV} - 5\sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\beta_{AV}) + \tfrac{5}{6}\sqrt{\tfrac{5}{6}}\,(\sqrt{2}\operatorname{Re}K_AL_V^*\beta_{AV} - \sqrt{3}\operatorname{Re}K_VL_V^*\beta_{VV}) + (\alpha Z/p)^{-1}7/\sqrt{6}\operatorname{Im}K_AK_V^*\beta_{AV}\Big]\ ;$$

$$\omega_1 = -(\sqrt{14}/15)\left(\tfrac{1}{2}|K_A|^2\beta_{AA} + \tfrac{1}{3}|K_V|^2\beta_{VV} - \sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\beta_{AV}\right)\ ;$$

$$\omega_2 = -\tfrac{24}{125}\sqrt{\tfrac{2}{7}}\left(\tfrac{1}{3}|K_A|^2\beta_{AA} + \tfrac{1}{2}|K_V|^2\beta_{VV} - \sqrt{\tfrac{2}{3}}\operatorname{Re}K_AK_V^*\beta_{AV}\right)\ ;$$

$$\omega_3 = \tfrac{1}{3}\sqrt{\tfrac{14}{5}}\,(\sqrt{3}\,\mathrm{Re}\,K_A L_V^* \beta_{AV} - \sqrt{2}\,\beta_{VV}\,\mathrm{Re}\,K_V L_V^*);$$

$$\omega_4 = -\tfrac{2}{9}\sqrt{7}\,|L_V|^2 \beta_{VV};$$

$$\omega_5 = -(3\sqrt{14}/125)\left(\tfrac{1}{2}|K_A|^2\beta_{AA} + \tfrac{1}{3}|K_V|^2\beta_{VV} + \sqrt{\tfrac{2}{3}}\,\mathrm{Re}\,K_A K_V^* \beta_{AV}\right);$$

$$\omega_6 = -(\sqrt{14}/25)\left(-|K_A|^2\beta_{AA} + \tfrac{1}{3}|K_V|^2\beta_{VV}\right);$$

$$\omega_7 = -\tfrac{1}{5}\sqrt{\tfrac{14}{5}}\,(\sqrt{3}\,\mathrm{Re}\,K_A L_V^* \beta_{AV} + \sqrt{2}\,\beta_{VV}\,\mathrm{Re}\,K_V L_V^*).$$

Coefficients κ_i in the expression (7′):

$$\varkappa_1 = \tfrac{2}{25}\sqrt{\tfrac{2}{3}}\,K_A K_V^* \alpha_{AV};$$

$$\varkappa_2 = -(1/3\sqrt{5})\,(\sqrt{2}\,K_V L_V^* \alpha_{VV} + \sqrt{3}\,K_A L_V^* \alpha_{AV}).$$

APPENDIX B

Expressions for the angular correlations in unique transitions have been given in the approximation $(\alpha Z)^2 \ll 1$ and $(\alpha Z/p)^2 \ll 1$. To obtain the exact formulas for arbitrary Z, we must make the following changes.

1. In the expressions for σ in (10) and a_g in (12) we must replace $\mathcal{D}_j^N$ by D_j^N, where

$$D_j^N = [\varkappa(EQ - X)]^{-1}\,\mathcal{D}_j^N\,p^{1-2j}\,[(2j)!!]^2 \times \left\{a^2_{j_{1/2}}(Q + X) + a^2_{j-1/2}(Q - X)\right\}\omega_j,$$

$$\omega_j = 1 - 3(\varkappa R)^2/10\,(j+1). \qquad \text{(B.1)}$$

Here $\kappa^2 + 1 = (E + \alpha Z/R)^2$ in the case of a uniform surface distribution of the nuclear charge and $= (E + 6\alpha Z/5R)^2$ in the case of a uniform volume distribution. The $a_{j\lambda}$ are coefficients determined by the condition of smooth joining of the radial parts of the electron wave function inside and outside the nucleus. The values of $a_{j\lambda}$ are given in the tables of Sliv and Volchek.[6] The expression

$$[k_{gs}^+ + (EX - Q)(EQ - X)^{-1} k_{gs}^-]$$

in formula (16) must be replaced by $\sum_j D_j^N$, but in the curly brackets in the expression for D_j^N [formula (B.1)] we must have

$$\left\{a^2_{j\frac{1}{2}}\varepsilon_{j\frac{1}{2}}(Q + X) + a^2_{j-\frac{1}{2}}\varepsilon_{j-\frac{1}{2}}(Q - X)\right\}.$$

2. The expression

$$(EQ - X)^{-1}\sum_j \mathcal{D}_j^N \eta_j^g\left[p\,\mathrm{Re}\,S + \alpha Z\left(j + \tfrac{1}{2}\right)^{-1}\mathrm{Im}\,S\right]$$

in the formulas (12) must be replaced by $\sum_j \Gamma_j^N \eta_j^g$, where

$$\Gamma_j^N = 2\,[\varkappa(EQ - X)]^{-1}\,\mathcal{D}_j^N\,p^{1-2j}\,[(2j)!!]^2\,a_{j\frac{1}{2}}\,a_{j-\frac{1}{2}} \times \{\cos\delta_j\,\mathrm{Re}\,S - \sin\delta_j\,\mathrm{Im}\,S\}\,\omega_j,$$

$$\sin\delta_j = -\frac{\alpha Z}{p\,(j + 1/2)}\cos\delta_j = -\frac{\alpha Z}{p\,(j + 1/2)}\left[1 + \left(\frac{\alpha Z}{p\,(j + 1/2)}\right)^2\right]^{-1/2}. \qquad \text{(B.2)}$$

Exactly the same changes have to be made in the terms of (16) which contain l_{gs} and l'_{gs} (with the coefficient ζ_j).

In the quantities of (16) containing t_g and t'_g we must make the substitution (B.2) (with the coefficient τ_j), but in the curly brackets in (B.2) we must have

$$\{\cos\delta_j\,\mathrm{Im}\,S + \sin\delta_j\,\mathrm{Re}\,S\}.$$

For example, the exact expression for the longitudinal polarization of the electrons in unique transitions is then written in the form

$$\zeta = \sum_j \Gamma_j^N \Big/ \sum_j D_j^N.$$

APPENDIX C

Numerical values of the coefficients in the expressions (10) to (17) for second forbidden transitions (N = 2):

$$\mathcal{D}_j^{(2)} = R_j^{(2)}\,p^{2j-1}\,q^{5-2j}$$

$R_j^{(2)}$

j	
$\frac{1}{2}$	1
$\frac{3}{2}$	$\frac{10}{3}$
$\frac{5}{2}$	1

η_j^g

j	$g=1$	$g=3$	$g=5$
$\frac{1}{2}$	$\frac{2}{3}\sqrt{3}$	0	0
$\frac{3}{2}$	$\frac{1}{10}\sqrt{3}$	$-\frac{3}{5}\sqrt{6}$	0
$\frac{5}{2}$	$\frac{2}{7}\sqrt{3}$	$-\frac{2}{3}\sqrt{6}$	$\frac{10}{21}\sqrt{33}$

ξ_j^g

j	$g=2$	$g=4$
$\frac{1}{2}$	0	0
$\frac{3}{2}$	$-\frac{2}{5}\sqrt{15}$	0
$\frac{5}{2}$	$-\frac{4}{7}\sqrt{15}$	$\frac{3}{7}\sqrt{22}$

$\zeta_j^{(gs)}$

j	$g=2$		$g=4$	
	$s=1$	$s=3$	$s=3$	$s=5$
$\frac{1}{2}$	0	0	0	0
$\frac{3}{2}$	$\frac{2}{5}\sqrt{\frac{2}{5}}$	$-\frac{2}{5}\sqrt{\frac{3}{5}}$	0	0
$\frac{5}{2}$	$\frac{4}{7}\sqrt{\frac{2}{5}}$	$-\frac{4}{7}\sqrt{\frac{3}{5}}$	$-\frac{2}{21}\sqrt{\frac{22}{3}}$	$\frac{1}{21}\sqrt{\frac{110}{3}}$

$\tau_j^{(g)}$

j	$g=1$	$g=3$	$g=5$
$\frac{1}{2}$	$\frac{2}{3}\sqrt{\frac{2}{3}}$	0	0
$\frac{3}{2}$	$\frac{4}{5}\sqrt{\frac{2}{3}}$	$-\frac{2}{5}\sqrt{\frac{6}{7}}$	0
$\frac{5}{2}$	$\frac{6}{7}\sqrt{\frac{2}{3}}$	$-\frac{2}{3}\sqrt{\frac{6}{7}}$	$\frac{2}{7}\sqrt{\frac{10}{3}}$

$\varepsilon_{j\lambda}^{(gs)}$

j	λ	$g=1$		$g=3$		$g=5$	
		$s=0$	$s=2$	$s=2$	$s=4$	$s=4$	$s=6$
$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	0	0	0	0	0
	$\frac{1}{2}$	$\frac{2}{9}$	$\frac{4}{9}\sqrt{2}$	0	0	0	0
$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{2}{15}\sqrt{2}$	$\frac{1}{5}\sqrt{6}$	0	0	0
	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{2}{5}\sqrt{2}$	$-\frac{1}{35}\sqrt{6}$	$-\frac{12}{35}\sqrt{2}$	0	0
$\frac{5}{2}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{4}{21}\sqrt{2}$	$\frac{2}{7}\sqrt{6}$	$\frac{2}{21}\sqrt{2}$	$-\frac{2}{21}\sqrt{55}$	0
	$\frac{1}{2}$	$\frac{10}{21}$	$\frac{8}{21}\sqrt{2}$	$-\frac{2}{21}\sqrt{6}$	$-\frac{10}{21}\sqrt{2}$	$\frac{2}{21}\sqrt{\frac{5}{11}}$	$\frac{20}{21}\sqrt{\frac{6}{11}}$

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