ON THE THEORY OF SCATTERING VIA QUASI-STATIONARY STATES

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The continuous-spectrum wave function that describes S scattering is expanded in series in terms of a system of wave functions of quasi-stationary states with complex energies. The validity of the expansion is established by a study of the analytic properties of the solution. The resulting dispersion formulas express the energy dependence of the cross section in a more convenient way than that provided by previous theories.

THE theory generally used at present to explain the existence of resonances in the cross sections for scattering and nuclear reactions is that of Wigner and Eisenbud.[1,2] Only a formal significance, however, attaches to the energy levels of the compound system and the widths of the resonance curves as obtained in this theory, since the system of basis functions used for the expansion is chosen arbitrarily, without any correspondence to the physical meaning of the problem. It would be natural to try to expand the solution in terms of the actual wave functions of the quasi-stationary states that satisfy the Schrödinger equation with complex energies.[3] There are difficulties here, however, owing to the fact that the system of such functions (nonorthogonal and linearly dependent) may be incomplete. In the present paper these difficulties are overcome by means of a study of the asymptotic properties of the solution in the complex-energy region.

THE PROPERTIES OF THE ANALYTIC CONTINUATION ψ (x, k)

We write down the Schrödinger equation for our case:

$$(d^2/dx^2+k^2)\psi = V\psi. \qquad (1)$$

Here $\psi(x, k)$ is the solution of the scattering problem for complex k, and V is the interaction operator. Since the nuclear forces fall off sharply at a certain boundary, we assume that V vanishes at the distance x_0; the kernel $V(x, x') = 0$ for x or $x' > x_0$.

We prescribe the boundary conditions in the form

$$\psi(0, k) = 0, \quad \psi(x, k) - \sin kx \sim e^{ikx} \quad \text{for } x > x_0. \qquad (2)$$

Equation (1) and the boundary conditions lead to two symmetry relations:

$$\psi(x, k) = -S(k)\psi(x, -k), \qquad \psi(x, k) = -\psi^*(x, -k^*). \qquad (3)$$

Here $S(k) = e^{2i\delta}$ is the one-dimensional scattering matrix. At a point k_λ of the complex k plane where S(k) has a pole, $\psi(x, k)$ also has a pole, and the function $[\operatorname{Res}\psi(x, k)]_{k_\lambda}$ contains only an outgoing wave and describes the quasi-stationary state λ. For convenience in what follows we choose the wave functions of the quasi-stationary states in the following way:

$$c_\lambda\psi_\lambda(x) = [\operatorname{Res}\psi(x, k)e^{ikx_0}]_{k_\lambda}, \qquad \psi_\lambda(x_0) = 1. \qquad (4)$$

We retain this definition of the function $\psi_\lambda(x)$ and coefficients c_λ also for the nondecaying stationary states, which are characterized by poles in the upper half of the complex k plane. From Eq. (3) it follows that the poles occur in pairs, so that to every pole k_λ there corresponds a pole $-k_\lambda^*$. Corresponding to the function $c_\lambda\psi_\lambda(x)$ is the function $c_\lambda^*\psi_\lambda^*(x)$.

The coefficients c_λ are determined by the relations

$$2k_\lambda\int_0^{x_0}\psi_\lambda^2(x)\,dx + i = -k_\lambda/c_\lambda. \qquad (5)$$

If a pole in the lower half-plane comes close to the real axis, $\operatorname{Im} k_\lambda = 0$, resonances occur in the cross section. From Eq. (1) we can derive the relation for the functions $\psi_\lambda(x)$ and $\psi_\lambda^*(x)$:

$$2\operatorname{Im}k_\lambda\int_0^{x_0}|\psi\ (x)|^2\,dx + 1 = 0. \qquad (6)$$

It can be seen that for $\operatorname{Im}(k_\lambda x_0) \ll 1$ the wave function is relatively large in the inner region $x < x_0$. Let us separate the function $\psi_\lambda(x)$ into its real and imaginary parts:

$$\psi_\lambda(x) = \psi_R(x) + i\psi_I(x).$$

Applying Green's formula, we get

$$\psi_R(x)\frac{d\psi_I(x)}{dx}-\psi_I(x)\frac{d\psi_R(x)}{dx}=2\,\mathrm{Re}\,k_\lambda\,\mathrm{Im}\,k_\lambda\int_0^x|\psi_\lambda(x)|^2\,dx. \quad (7)$$

In the approximation of small width of the resonance level,

$$|\mathrm{Re}(k_\lambda x_0)\,\mathrm{Im}(k_\lambda x_0)|\ll 1 \quad (\Gamma_\lambda\ll\hbar^2/mx_0^2),$$

it follows from Eq. (7) and the stipulation (4) that $\psi_\lambda(x)$ is real. This corresponds physically to the formation of a standing wave inside the nucleus. For the case in which $\psi_\lambda(x)$ is real right up to the very edge, we can find from Eqs. (5) and (6) the approximate value of c_λ:

$$c_\lambda\approx k_\lambda\,\mathrm{Im}\,k_\lambda/\mathrm{Re}\,k_\lambda. \quad (8)$$

For stationary states the c_λ are easily calculated by means of the integral equation of Gell-Mann and Goldberger (cf., e.g., reference 4).

In what follows we shall assume that $\psi(x, k)$ for $|k| < \infty$ has only simple poles. The correctness of this proposition has been proved only for potentials with infinitely small radius of action.[5] The condition for the presence of a pole of higher order is

$$2k_\lambda\int_0^{x_0}\psi_\lambda^2(x)\,dx+i=0.$$

Comparing Eqs. (5) and (6) we can see that poles near the real axis are not of order higher than the first. As for distant poles of nth order, they can be split up into sets of n closely spaced poles of first order by a small change of the potential.

THE ASYMPTOTIC PROPERTIES

The behavior of $\psi(x, k)$ for $k\to\infty$ in the complex plane depends on the differential properties of the kernel $V(x, x')$, or of the function $V(x)$, if $V = V(x)$. The integral equations for the scattering hold in the upper half-plane, and the asymptotic properties of $\psi(x, k)$ for $k\to\infty$ are determined by the Born approximation

$$\psi(x,k)\approx\sin kx+\iint G(x,x')V(x',x'')\sin kx''\,dx'\,dx'', \quad (9)$$

where

$$G(x,x')=\frac{i}{2k}\left[e^{ik(x+x')}-e^{ik|x-x'|}\right].$$

The passage to the lower half of the complex plane is made by means of Eq. (3). By using Eq. (9) we can also find out the asymptotic properties of the S matrix. For $k\to\infty$ along upward rays ($\mathrm{Im}\,k/\mathrm{Re}\,k$ = const, $\mathrm{Im}\,k > 0$) the main term of the function S(k) has the form

$$S(k)\approx-\frac{1}{2}\frac{e^{-2ikx_0}}{(ik)^{m+n+3}}\left[\frac{\partial^{m+n}V(x,x')}{\partial x^m\partial x'^n}\right]_{x=x'=x_0}; \quad (10)$$

if $V = V(x)$, then

$$S(k)\approx\frac{e^{-2ikx_0}}{(2ik)^{n+2}}\left[\frac{\partial^n V(x)}{\partial x^n}\right]_{x_0}. \quad (11)$$

The right members of Eqs. (10) and (11) contain the first nonvanishing derivatives, $m, n \geqslant 0$. For the passage to the lower half-plane we have only to use the formula $S(k)S(-k) = 1$. Equation (9) enables us to find the asymptotic distribution of the zeroes of S(k) in the upper half-plane, and at the same time the poles for $\mathrm{Im}\,k_\lambda < 0$:

$$\mathrm{Re}\,k_\lambda\approx\pm\pi\lambda/x_0,\quad \mathrm{Im}\,k_\lambda\approx-s\ln(\pi|\lambda|)/2x_0,\quad \lambda\to\infty, \quad (12)$$

where λ is an integer, and $s = m + n + 3$ or $n + 2$. Since

$$\psi_\lambda(x)=-2i\exp\{-ik_\lambda x_0\}\,\psi(x,-k_\lambda),$$

we have for $\lambda\to\infty$

$$\psi_\lambda(x)\approx\exp\{ik_\lambda(x-x_0)\}-\exp\{-ik_\lambda(x+x_0)\}. \quad (13)$$

Furthermore, using Eq. (5), we can calculate the index of increase of the coefficients c_λ: $c_\lambda\approx k_\lambda^s$, $\lambda\to\infty$.

In what follows we shall find it advantageous for the coefficients c_λ not to increase too rapidly. On the other hand, physical considerations suggest that the behavior at infinity must not depend in any essential way on the interaction in the region of finite energy. Let us add to the potential-energy operator a small quantity $\delta V = \epsilon\delta(x-x_0)$. The change of the S matrix will then be

$$\delta S=(2\varepsilon/ik)\,[\psi(x_0,k)]^2.$$

The nature of the increase of the coefficients c_λ, however, is decidedly changed. Using Eqs. (5) and (13), we find

$$c_\lambda\approx-ik_\lambda/2x_0\varepsilon. \quad (14)$$

The asymptotic relations that hold for $k_\lambda\to\infty$ along upward rays are:

$$S(k)\,e^{2ikx_0}\sim\mathrm{const}/k;\qquad \psi(x,k)\,e^{ikx_0}\sim\mathrm{const}/k,$$

$$x<x_0;\qquad \psi(x_0,k)\approx-1/2i.$$

Along downward rays:

$$S(k)\,e^{2ikx_0}\sim k;\qquad \psi(x,k)\,e^{ikx_0}\sim\mathrm{const},$$

$$x<x_0;\qquad \psi(x_0,k)\,e^{ikx_0}\sim k.$$

Let us now expand $S(k)\,e^{2ikx_0}$ in a Mittag-Leffler series:

$$S(k)\,e^{2ikx_0}=2i\sum_\lambda kc_\lambda/k_\lambda(k-k_\lambda)+F(k). \quad (15)$$

According to Eqs. (2) and (4) the sum over all λ is the principal part of $S(k)e^{2ikx_0}$ near the poles. Using Eqs. (12) and (14) we can show that this sum increases as $-ik/\epsilon$ for $k\to\infty$ along rays not coinciding with the real axis. F(k) is an integral analytic function of k. For $k \to \infty$ along a ray, F(k) does not increase faster than k, and therefore $F(k) = ik/\epsilon + C$. Setting $k = 0$, we find that $C = 1$. The Mittag-Leffler series for $\psi(x, k)e^{ikx_0}$ has the form

$$\psi(x, k) e^{ikx_0} = \sum_\lambda \frac{k}{k_\lambda} \frac{c_\lambda \psi_\lambda(x)}{k - k_\lambda}. \tag{16}$$

According to Eqs. (12), (13), and (14) the sum on the right is bounded for $x < x_0$ and $k \to \infty$ along a ray; the corresponding integral function goes to zero. Equations (15) and (16) are the desired expansion of $\psi(x, k)$ in terms of the system of wave functions $\psi_\lambda(x)$ for $x \leqslant x_0$, and it follows from Eq. (16) that the system $\psi_\lambda(x)$ is complete. The equation

$$\sum_\lambda c_\lambda \psi_\lambda(x) \psi_\lambda(x') = -\delta(x - x'), \quad x < x_0.$$

can be established by direct verification.

It is interesting to check directly that the wave function (16) that has been constructed satisfies the Schrödinger equation and the boundary conditions. For this purpose we consider the series $\Sigma c_\lambda \psi_\lambda(x)/k_\lambda$. By applying Eqs. (12) — (14) we can show that this series converges for all $x < x_0$, and forms a δ functions for $x = x_0$. The coefficient of $\delta(x - x_0)$ is $- i/\epsilon$. By manipulating the expression

$$\int_0^{x_0} \Big(\sum_\mu c_\mu \psi_\mu(x)/k_\mu + \frac{i}{\varepsilon}\delta(x - x_0)\Big)\psi_\lambda(x)\,dx, \tag{17}$$

we verify that

$$\sum_\mu \frac{c_\mu \psi_\mu(x)}{k_\mu} = -\frac{i}{\varepsilon}\delta(x - x_0). \tag{18}$$

It can be seen from Eqs. (15) and (16) that the wave function that describes scattering at zero energy is given by

$$\psi_0(x) = \lim_{k\to\infty} \frac{\psi(x, k)}{k} = -\sum_\lambda \frac{c_\lambda \psi_\lambda(x)}{k_\lambda^2}, \quad x < x_0, \tag{19}$$

$$\psi_0(x_0) = -\sum_\lambda \frac{c_\lambda}{k_\lambda^2} + \frac{1}{2\varepsilon}. \tag{20}$$

The series (19) converges nonuniformly at $x = x_0$ and may have a discontinuity at this point. Determining the discontinuity from Eq. (18), we find the limit for $x \to x_0 - 0$:

$$\lim \psi_0(x) = -\lim c_\lambda k_\lambda^{-2} \psi_\lambda(x) = \psi_0(x_0).$$

Let us calculate the derivative of Eq. (19) at the point x_0:

$$\lim_{x\to x_0 - 0} \frac{d\psi_0(x)}{dx} = -\lim \sum_\lambda \frac{c_\lambda}{k_\lambda^2}\Big(\frac{d\psi_\lambda(x)}{dx} - ik_\lambda \psi_\lambda(x)\Big) = 1 - \varepsilon\psi_0(x_0).$$

Thus it can be seen that Eq. (19) is continuous at the point x_0 and satisfies the boundary condition (2) to accuracy ϵ.

The function $\psi_0(x)$ must satisfy the Schrödinger equation for zero energy. As the result of formal application of Eq. (1) to Eq. (18) we get

$$\Big(\frac{d^2}{dx^2} - V\Big)\psi_0(x) = \sum_\lambda c_\lambda \psi_\lambda(x).$$

The series on the right diverges; we can, however, introduce the generalized sum (for example, by Abel's method):

$$\sum_\lambda c_\lambda \psi_\lambda(x) = \lim_{\theta\to 1} \sum_\lambda \theta^\lambda c_\lambda \psi_\lambda(x), \quad \theta < 1.$$

Calculation shows that this sum is equal to zero everywhere for $x < x_0$. This completes the proof that $\psi_0(x)$ is the solution of the scattering problem for zero energy. In the case $k \neq 0$ we rewrite Eq. (16) in the form

$$\psi(x, k) = k\psi_0(x) + \sum_\lambda \frac{k^2}{k_\lambda^2} \frac{c_\lambda \psi_\lambda(x)}{k - k_\lambda}. \tag{21}$$

The series converges well enough, and direct verification shows that $\psi(x, k)$ defined by Eq. (20) satisfies the equation (1) and the conditions (2).

THE DISPERSION FORMULAS

According to Eqs. (15) and (20) the scattering amplitude $(2ik)^{-1}$. $(S(k) - 1)$ is given by

$$f(k) = \frac{e^{-2ikx_0} - 1}{2ik} + e^{-2ikx_0}\Big(\psi_0(x_0) + \sum_\lambda \frac{kc_\lambda}{k_\lambda^2 (k - k_\lambda)}\Big). \tag{22}$$

In the case of small widths the coefficients c_λ are determined from Eq. (8). The resulting formula is

$$f(k) = f_0(k) + \frac{e^{-2ikx_0}}{k} \sum_\lambda^{(1/2)} \frac{\Gamma_\lambda/2}{E - |E_\lambda| - i\Gamma_\lambda/2}. \tag{23}$$

Here $E = k^2$, $\Gamma_\lambda = 4k \operatorname{Im} k_\lambda < 0$, and the sum includes the poles located in the half-plane $\operatorname{Re} k_\lambda > 0$; $f_0(k)$ is the remaining slowly varying part of f(k). The cross section for S scattering has the form

$$\sigma(k) = \sigma_0(k) + \frac{4\pi}{E} \sum_\lambda^{(1/2)} \frac{\Gamma_\lambda/2}{(E - |E_\lambda|)^2 + \Gamma_\lambda^2/4} \Big[\frac{\Gamma_\lambda}{2} \cos 2kx_0 - (E - |E_\lambda|)\sin 2kx_0\Big]. \tag{24}$$

The dispersion formula shows that in the case in which the standing wave inside the nucleus occupies the entire region $x < x_0$ the potential part of the scattering corresponds exactly to the scattering by an impentrable sphere of radius x_0. Integrating Eq. (23), we get the dispersion relations

$$\frac{1}{i\pi}\int_{-\infty}^{+\infty} f(k')\,e^{2ik'x_0}\frac{dk'}{k'-k} + f(k)\,e^{2ikx_0} = 2\sum_{\mathrm{Im}\,k_\lambda>0}\frac{c_\lambda}{k_\lambda(k-k_\lambda)},$$

$$f(0) = -\frac{1}{i\pi}\int_{-\infty}^{+\infty} f(k)\,e^{2ikx_0}\frac{dk}{k} - 2\sum_{\mathrm{Im}\,k_\lambda>0}\frac{c_\lambda}{k_\lambda^2}.$$

In these relations the oscillating exponential plays an essential part; this makes them useless in practice for calculations in nuclear physics. In special cases, however, one can write dispersion relations without the exponential factor.[6] From Eqs. (15) and (20) there follows the sum rule

$$k_\lambda\sum_\mu c_\mu / k_\mu^2(k_\lambda + k_\mu) = 1/2ik_\lambda + x_0 + f(0),$$

and for the poles located near the origin

$$f(0) + x_0 \approx -\sum_\mu c_\mu k_\mu^{-2}.$$

The present expansions (22) and (24) differ from the corresponding formulas of Wigner and Eisenbud[1] by their simplicity and intuitive physical meaning. The positions and widths of the resonance levels are directly connected with the energies and lifetimes of the decaying states. The specific difficulties of the expansion in terms of nonorthogonal functions reduce mainly to the study of the asymptotic behavior of the solution for large energies, which is not dealt with at all in references 1 and 2. The development of an analogous formalism for the general case of an arbitrary nuclear reaction involving charged particles and higher spins and angular momenta would evidently make it possible to construct a version of the formal theory of resonance nuclear reactions without using the concept of the R matrix.

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378