AN INVESTIGATION OF THE SUPERFLUID STATE OF AN ATOMIC NUCLEUS

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The variational principle proposed by N. N. Bogolyubov and the physical ideas and mathematical methods developed in the theory of superconductivity are applied to a study of the properties of a heavy nucleus. Using the nuclear shell model we consider the residual interactions of nucleons in an outer shell; this leads to the appearance of a superfluid state of the nucleus. We have evaluated the energy of the superfluid ground state and those of a number of excited states both in the case of an even and in the case of an odd number of nucleons in the shell. We found some regularities in the level spectrum of even-even nuclei and of odd A nuclei.

Changes in the nuclear ground state energy were evaluated for the case where the number of outer shell nucleons was changed by one; this made it possible to conclude that even-even nuclei are more stable than odd-odd nuclei as regards β decay, which agrees numerically with von Weizsäcker's semi-empirical formula. The results obtained depend little on the nuclear model chosen, and they are also valid for highly deformed nuclei.

A certain similarity between the properties of the Fermi-systems in a nucleus and in a metal makes it possible to apply the physical ideas and mathematical methods developed in the theory of superconductivity[1] to a study of nuclear matter and of a finite nucleus.[2-7] Belyaev[6] and the author[5,7] have used the nuclear shell model to show that the interactions between the outer shell nucleons lead to a superfluid nuclear state (a state energetically lower than the one with a completely degenerate Fermi-gas).

In the present paper we continue the study of the superfluid nuclear state and, in particular, the energies of the excited states of shells both with an even and with an odd number of nucleons will be evaluated and the problem of the stability of nuclear isobars as regards β decay will be considered.

1. THE SUPERFLUID GROUND STATE

We shall investigate the superfluid state of an atomic nucleus using a new variational principle proposed by Bogolyubov[8] which is a generalization of Fock's well-known method.[9]

We take the following model of a heavy nucleus: the nucleons which form the inner shells produce a central-symmetrical field which is somewhat distorted by the outer shell nucleons. Let us consider the residual interactions between the nucleons (protons or neutrons) which are in the outer nuclear shells. We shall characterize the nucleon state by a set of quantum numbers s and by the quantum number m of the total angular momentum component along the symmetry axis of the nucleus.

Let us consider the residual interactions of nucleons which are in the immediate vicinity of the energy of the Fermi surface with a Hamiltonian

$$H = \sum \{E(s, m) - \lambda\} a_{m\rho}(s)^+ a_{m\rho}(s)$$
$$+ \frac{1}{4} \sum J(s_1, s_2 s_1', s_2' | \rho_1 m_1, \rho_2 m_2; \rho_1', m_1', \rho_2' m_2') a_{m_1\rho_1}(s_1)^+$$
$$\times a_{m_2\rho_2}(s_2)^+ a_{m_2'\rho_2'}(s_2') a_{m_1',\rho_1'}(s_1'). \tag{1}$$

The summation is here carried out over all values of all numbers $s, s_1, s_2, \ldots$, over positive values of the numbers $m, m_1, m_2, \ldots$ and over the values ± 1 of the numbers ρ which characterize the sign of m, where

$$\rho_1 m_1 + \rho_2 m_2 = \rho_1' m_1' + \rho_2' m_2', \qquad \rho_1 m_1 \neq \rho_1' m_1';$$

$a_{m\rho}(s)^+$, $a_{m\rho}(s)$ are the creation and annihilation operators for the nucleons; $E(s, m)$ is the energy of a nucleon in the outer shell in an s, m state; λ is a parameter which plays the role of the chemical potential and which is determined from the condition

$$n = \sum_{s,m,\rho} \overline{a_{m\rho}(s)^+ a_{m\rho}(s)}. \tag{2}$$

We shall, morevoer, perform a linear canonical transformation of the Fermi amplitudes, as in references 5 and 7

$$a_{m\rho}(s) = u_m(s)\alpha_{m,-\rho}(s) + \rho v_m(s)\alpha_{m\rho}(s)^+, \tag{3}$$

we shall define a new vacuum state $\alpha_{m\rho}(s)\Psi = 0$, and find the average value of the energy operator $\bar{H}$ over this state. We shall determine $u_m(s)$, $v_m(s)$ from requiring $\bar{H}$ to be a minimum, and as a result we get the following equation

$$C_m(s) = -\frac{1}{2}\sum_{s',m'} J(s, s' \mid m, -m; m', -m') \times C_{m'}(s')/\sqrt{C_{m'}(s')^2 + \xi_{m'}(s')^2} \quad (4)$$

where

$$C_m(s) = \sum_{s'm'} J(s, s' \mid m, -m; m', -m')\, u_{m'}(s') v_{m'}(s'), \quad (5)$$

$$\xi_m(s) = E(s, m) - \lambda - {}^1/_2 \sum_{s'm'} \{J(s, s'; s', s \mid m, m'; m', m) + J(s, s'; s', s \mid m, -m', -m', m)\}\, v_{m'}(s')^2, \quad (6)$$

$$\varepsilon(s, m) = \sqrt{C_m(s)^2 + \xi_m(s)^2},$$

$$J(s, s; s', s' \mid \ldots) \equiv J(s, s' \mid \ldots).$$

We note that if the total angular momentum j is a good quantum number we only need sum over m' in (4). The trivial solution of (4) $C_m(s) = 0$ corresponds to the state of the completely degenerate Fermi gas. It was shown in references 4 and 7 that it is necessary that $J < 0$ for energies near the Fermi surface in order that there be a non-trivial solution of (4).

In reference 5 we obtained the asymptotic solution of (4) (as $J \to 0$) and showed that the interactions of nucleons with equal and opposite angular momentum components along the symmetry axis of the nucleus play the leading part. In references 6 and 7 solutions of (4) were found assuming that $J(s, s' \mid m, -m; m', -m')$ did not depend on s, s', m; m', and it was shown that in this crude approximation the main properties of the asymptotic solution were retained.

Let us consider the approximation

$$J = \text{const}, \quad \rho = \text{const}, \quad (7)$$

i.e., J and the level density $\rho(E)$ do not depend on E.

We can neglect the interactions of nucleons with different m, s since it was shown in reference 5 that they will give a small correction which can be evaluated by perturbation theory. We have then

$$\xi_m(s) = E(s, m) - \lambda.$$

The energy of nucleons in the outer shell is in the neighborhood of the Fermi surface, i.e.,

$$E_F - \delta \leqslant E(s, m) \leqslant E_F + \Delta.$$

We go over in (4) from a sum to an integral and get

$$C_m(s) = -\frac{1}{2}\int_{E_F-\delta}^{E_F+\Delta} dE' \rho(E') J(s s' \mid m, -m; m', -m') \times C_{m'}(s')/\sqrt{C_{m'}(s')^2 + (E-\lambda)^2}, \quad (4')$$

where λ is determined from the equation

$$n = \int_{E_F-\delta}^{E_F+\Delta} dE\, \rho(E) \{1 - (E-\lambda)/\sqrt{C_m(s)^2 + (E-\lambda)^2}\}. \quad (8)$$

In the approximation (7) the solutions of (4′) and (8) will be found in the following form

$$C = \frac{\sqrt{(2\Omega - n) n}}{\rho (e^{2/G} - 1)} e^{1/G}, \quad (9)$$

$$E_F - \lambda = \frac{\Omega - n}{\rho (e^{2/G} - 1)} + \delta - \frac{n}{2\rho}, \quad (10)$$

where $G = -\bar{\rho}\bar{J}$, n is the number of nucleons, and Ω the number of levels in the outer nuclear shell. In the approximation

$$G \gg 1, \quad |E_{max} - E_{min}| \ll \Omega G/2\rho \quad (11)$$

we have

$$\varepsilon \approx \Omega G/2\rho. \quad (12)$$

We note that if j is a good quantum number the form of Eqs. (9)—(12) remains the same, but ρ, n, and Ω refer to levels with a given value of j.

2. EXCITED STATES

We shall calculate the energy of the excited nuclear states, taking into account the interaction of the nucleons in the outer shell. It is well known that the wave function Ψ of the new vacuum state is a superposition of states with an even number of nucleons; in this connection the wave function of the ground state of a system with an odd number of particles is $\alpha_{m\rho}(s)^+\Psi$.

We shall evaluate the energy of the excited states of a shell with an even number of nucleons in it. The difference ΔE^I between the energy of the first excited state $\alpha_{m+}(s)^+ \alpha_{m-}(s)^+\Psi$ and that of the superfluid ground state Ψ is of the form

$$\Delta E^I = 2\varepsilon(s, m). \quad (13)$$

The addition to (13) is small in the case of different values of s and m. The first excited state is thus in shells with an even number of nucleons separated by a gap from the ground state, and then there follows a number of levels with successively increasing spin values which are lying close to one another. We shall investigate the problem of the further sequence of excited states of the form*

$$\alpha_{m_1\rho_1}(s_1)^+\alpha_{m_2\rho_2}(s_2)^+\alpha_{m_3\rho_3}(s_3)^+\alpha_{m_4\rho_4}(s_4)^+\Psi.$$

We shall therefore evaluate the difference ΔE^{II} between excited states of such a kind and excited states $\alpha_{m-}(s)^+\alpha_{m+}(s)^+\Psi$, namely

$$\Delta E^{II}=\varepsilon(s,m)+\varepsilon(s',m') +2\frac{C_{m'}(s')\,C_m(s)}{\varepsilon(s',m')\,\varepsilon(s,m)}J(s,s'\,|\,m,-m;m',-m'). \qquad (14)$$

In the approximations of (7) and (11), Eq. (14) is appreciably simplified

$$\Delta E^{II}\approx\frac{G}{\rho}\left(\Omega-2\frac{(2\Omega-n)\,n}{\Omega^2}\right). \qquad (14')$$

Terms which lead to a decrease of ΔE^{II} in the approximation (7), (11) are relatively small, the greatest decrease of ΔE^{II} arising when the levels of the shell are half filled.

We have thus the following picture of the excited states of even-even nuclei: the first excited state is separated from the ground state by a distinctly expressed gap, there follow then a number of excited states with successively increasing spin values, after which there occur levels with small spin values corresponding to the break-up of a second pair, and the lowest of these is separated by ΔE^{II} from the first excited state. Levels with high spin values may not show up experimentally and it may thus look as if there is a second gap. For energies larger than the ground state energy by $\Delta E^{I}+\Delta E^{II}$ the level density must at least increase appreciably.

We shall now calculate the energies of the excited states of a shell with an odd number of nucleons. The ground state wave function is $\alpha_{m_0\rho_0}(s)^+\Psi$ if $E(s_0,m_0)<E(s,m)$. When the system is excited it must go over from the state $\alpha_{m_0\rho_0}(s_0)^+\Psi$ to a state $\alpha_{m_1\rho_1}\Psi$ where $E(s_1,m_1)>E(s_0,m_0)$, and so on, while the spins of the states will also increase. The more strongly excited states will be states of the form

$$\alpha_{m_3\rho_3}(s_3)^+\alpha_{m_2\rho_2}(s_2)^+\alpha_{m_1\rho_1}(s_1)^+\Psi.$$

The difference ΔE between the energy of an excited state of this kind and the ground state energy will be obtained in the following form

$$\Delta E=\varepsilon(s,m)+\varepsilon(s',m') +\frac{C_{m'}(s')\,C_m(s)}{\varepsilon(s',m')\,\varepsilon(s,m)}J(s,s'\,|\,m,-m;m',-m'), \qquad (15)$$

and in the approximations (7) and (11)

$$\Delta E\approx\frac{G}{\rho}\left\{\Omega-\frac{(2\Omega-n)\,n}{\Omega^2}\right\}. \qquad (15')$$

We are thus led to the following picture in the case of odd nuclei: in the immediate neighborhood of the ground state there is a number of excited states with successively increasing spin values. If states with high spin values do not appear experimentally there will not be levels somewhat below the energy of the excited states of an even shell. Moreover, the level density must increase strongly.

It should be noted that in Bogolyubov's new variational principle[8] the form of the canonical transformation picks out the important part of the interaction in the Hamiltonian. In the case considered, the canonical transformation picks out the interactions of nucleons with equal and opposite values of the angular momentum component along the symmetry axis of the nucleus while the other quantum numbers are equal, if j is a good quantum number, and with arbitrary values of the other quantum numbers in the opposite case while the results depend very little on the actual values of the other quantum numbers and thus also on the details of the nuclear model. It is therefore essential both for the results obtained earlier and for the results of the next section to distinguish the quantum number m from all other quantum numbers of the nucleons near the energy of the Fermi surface. The details of the nuclear model are thus inessential and the results obtained can therefore also be applied to the case of highly deformed nuclei.

3. THE STABILITY OF NUCLEAR ISOBARS AS REGARDS BETA DECAY

The fact that even-even nuclei are more stable against β decay than odd-odd nuclei is reflected in von Weizsäcker's semi-empirical formula for nuclear masses by the introduction of a term $\pm\delta/2A$, where the positive sign refers to the odd-odd isobar, and the negative one to the even-even one. We shall investigate the problem of how the stability of isobars, with regard to β decay, is influenced by taking the interaction of the nucleons in the outer nuclear shell into account. Let us calculate for this purpose the change in the ground state energy if the number of nucleons in the outer shell is changed by one. The difference in the energies of shells containing $2n_0\pm 1$ and $2n_0$ nucleons is expressed as follows

$$\mathscr{E}(2n_0\pm 1)-\mathscr{E}(2n_0)=\bar{H}(2n_0\pm 1)-\bar{H}(2n_0)+\varepsilon(s,m). \qquad (16)$$

In the approximations (7) and (11) this expression takes on a very simple form, namely,

*I express my sincere gratitude to Prof. B. S. Dzhelepov who drew my attention to the possibility of the appearance of a "second gap."

$$\mathscr{E}(2n_0+1)-\mathscr{E}(2n_0)=(\Omega-2n_0-{}^1/_2)/\rho\,(e^{2/G}-1), \quad (17)$$
$$\mathscr{E}(2n_0-1)-\mathscr{E}(2n_0)=(\Omega+2n_0-{}^1/_2)/\rho\,(e^{2/G}-1). \quad (17')$$

From (17) and (17′) it is clear that the addition of one nucleon to an even shell does not change the energy of the shell appreciably, while, on the other hand, one needs to spend a considerable energy to take one nucleon away from an even shell. Moreover, the addition of one nucleon to an odd shell causes a considerable release of energy, while one needs spend only a small amount of energy to remove one nucleon from an odd shell.

If we determine G roughly from the magnitude of the gap in even-even nuclei and evaluate by using (17) and (17′) the difference in mass of the even-even and odd-odd isobars, we obtain a rough numerical agreement with the experimental data as reflected in Weizsäcker's formula.

One can easily show that one can not reach any clear-cut conclusions from (17) and (17′) about the relative stability of odd nuclei.

We have considered a rather idealized nuclear model and obtained a number of regularities which follow automatically from the properties of nuclear superfluidity. These regularities should be made more precise and concrete by using a more complicated nuclear model, by improving the approximation, and by comparison with the experimental data.

The investigations carried out in this paper and in references 3—7 confirm the fruitfulness of applying the physical ideas and mathematical methods developed in the theory of superconductivity to a study of the properties of the atomic nucleus.

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