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THEORY OF RELATIVISTIC MAGNETOHYDRODYNAMIC WAVES

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One-dimensional simple waves in relativistic magnetohydrodynamics and relativistic hydrodynamic discontinuities (contact, tangential, Alfvén, and fast and slow shock waves) are considered. The Zemplen theorem is proved for shock waves of arbitrary intensity.

THE general formulation of the problem of discontinuous solutions in relativistic magnetohydrodynamics has been given by de Hoffman and Teller,[1] who obtained an equation for the nonrelativistic shock adiabat and showed that the shock wave is always plane. These authors proved a theorem which is the inverse of the Zemplen theorem for an ideal gas in the particular case in which a magnetic field is parallel to the surface of the discontinuity. However, neither this work or other work published in this field has, to the best of our knowledge, been concerned with the stability of a relativistic magnetohydrodynamic shock wave. The Zemplen theorem has not been proved and the behavior of the magnetic field in the shock wave has been investigated only in particular cases. Relativistic magnetohydrodynamic discontinuities (contact, tangential, Alfvén, fast and slow shock waves) have not been classified. All of these problems are treated in the present paper.

As in ordinary hydrodynamics, in relativistic magnetohydrodynamics a shock wave results from a simple wave as a consequence of the fact that the points of the liquid which have the highest density are displaced with the highest velocity.

Simple waves are related to the low-amplitude waves which have been investigated in relativistic magnetohydrodynamics by Khalatnikov,[2] Zumino,[3] and Harris.[4] We start with an investigation of simple plane waves in relativistic magnetohydrodynamics in which all quantities are given in the form of functions of one of the quantities which, in turn, is a function of the coordinate x and the time t.

1. SIMPLE WAVES*

The complete system of relativistic magnetohydrodynamic equations for zero viscosity and infinite electrical conductivity is as follows:

$$\partial T_{ik} / \partial x_k = 0, \tag{1.1}$$

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{H} / \partial t, \qquad \operatorname{div} \mathbf{H} = 0, \tag{1.2}$$

$$\mathbf{E} = -[\mathbf{v} \times \mathbf{H}], \tag{1.3}$$

$$\partial (n u_k) / \partial x_k = 0, \tag{1.4}$$

where $T_{ik} = T_{ik}^{h} + T_{ik}^{em}$, and

$$T_{ik}^{h} = n w u_i u_k + p \delta_{ik}, \tag{1.5}$$

$$T_{\alpha\beta}^{em} = \frac{1}{4\pi}\left\{-H_\alpha H_\beta - E_\alpha E_\beta + \frac{1}{2}\delta_{\alpha\beta}(H^2 + E^2)\right\},$$

$$T_{\alpha 4}^{em} = \frac{i}{4\pi}[\mathbf{E}\times\mathbf{H}]_\alpha, \quad T_{44}^{em} = -\frac{1}{8\pi}(E^2 + H^2), \tag{1.6}$$

and the u_i are the components of the four-dimensional velocity, w and n are the internal energy per particle and the density in the reference system that moves with the wave, p is the pressure (the velocity of light is taken as unity).

Since the magnetohydrodynamic equations (1.1) — (1.6) constitute a hyperbolic system of first-order linear and homogeneous partial differential equations, we can apply the results obtained in reference 7; in particular, the differential equations which relate the magnetohydrodynamic variables in a simple wave can be obtained from the relations between the amplitudes of these quantities in waves of infinitesimally small amplitude. These relations are as follows:

Alfvén wave

$$d\mathbf{v}_t = -\varepsilon (V_{Ax} / H_x)\, d\mathbf{H}_{0t}, \qquad H_{0y} dH_{0y} + H_{0z} dH_{0z} = 0,$$
$$dv_x = dW = dp = dH_{0x} = 0; \tag{1.7}$$

magnetoacoustic waves

$$dv_x = \varepsilon \frac{V_\pm dW}{W(1 + c^2)}, \qquad d\mathbf{v}_t = -\varepsilon \frac{V_\pm}{W(1+c^2)} \frac{U_{Ax}\mathbf{U}_{At}}{U_\pm^2 - U_{Ax}^2} dW,$$

$$dH_{0x} = 0, \qquad d\mathbf{H}_{0t} = \frac{4\pi \mathbf{H}_{0t}(V_\pm^2 - c^2)}{(1 + c^2)\mathbf{H}_{0t}^2 (1 - V_\pm^2)} dW, \tag{1.8}$$

where the quantities $\mathbf{V}_A$, $\mathbf{U}_A$, $V_\pm$, and $U_\pm$ are defined by the relations

*Simple waves have been considered by Stanyukovich[5,6] for the particular case in relativistic magnetohydrodynamics in which $H_x = 0$. A number of papers[7-13] have been devoted to studies of simple waves in non-relativistic magnetohydrodynamics.

$$\mathbf{V}_A = \mathbf{U}_A / \sqrt{1 + \mathbf{U}_A^2}, \quad V_\pm = U_\pm / \sqrt{1 + U_\pm^2}, \quad (1.9)$$

$$\mathbf{U}_A = \mathbf{H}_0 / \sqrt{4\pi W}, \quad (1.10)$$

$$V_\pm = \left\{ \frac{c^2(1 + U_{Ax}^2) + \mathbf{U}_A^2 \pm \sqrt{[c^2(1 + U_{Ax}^2) - \mathbf{U}_A^2]^2 + 4c^2\mathbf{U}_{At}^2}}{2(1 + \mathbf{U}_A^2)} \right\}^{1/2}, \quad (1.11)$$

$\mathbf{H}_0$ is the magnetic field in the reference system which moves with the wave; W = nw; c is the velocity of sound; $\mathbf{U}_{At}$ is the component of the vector $\mathbf{U}_A$ perpendicular to the direction of propagation of the wave (i.e., perpendicular to the x axis); the quantity ϵ is (+1) if the wave propagates in the positive x direction and (−1) if the wave propagates in the opposite direction; the plus sign in Eq. (1.11) corresponds to the fast magnetoacoustic wave and the minus sign refers to the slow magnetoacoustic wave.

Equations (1.7) and (1.8) are not to be considered as relations between the amplitudes dW, and dH, ... for constant W, H, ..., but as differential equations from which all the quantities may be expressed as functions of one of these quantities, for example W.

The equations in (1.8) apply only in the moving reference system. Hence they can be used only in the case in which the velocity of the liquid **v** is small compared with the velocity of light.

The quantity W satisfies the equation

$$\partial W / \partial t + V \partial W / \partial x = 0, \quad (1.12)$$

where V is the phase velocity of propagation of a wave of infinitely small amplitude which corresponds to the given simple wave. The phase velocity is a function of the magnetohydrodynamic quantities (W, **H**, ... etc.). For this reason points corresponding to different values of W are displaced with different velocities and in general there is a distortion of the wave shape.

In the simple Alfvén wave the phase velocity remains constant since the quantities v_x, H_x, $\mathbf{H}_t$, and W do not change. Hence the Alfvén wave propagates without distortion.

In order to determine the manner in which the shape of the magnetoacoustic wave changes it is necessary to compute the derivative dV_p/dW, where V_p is the phase velocity, given by

$$V_p = v_x + V_\pm \quad (1.13)$$

(we may recall that the velocity **v** is nonrelativistic).

Differentiating the equation

$$V_\pm^4 (1 + \mathbf{U}_A^2) - V_\pm^2 [\mathbf{U}_A^2 + c^2(1 + U_{Ax}^2)] + U_{Ax}^2 c^2 = 0$$

and making use of Eq. (1.8) we have

$$\frac{dV_p}{dW} = \frac{1}{4V_\pm W(1 + c^2)A}\Big[(V_\pm^2 - c^2)V_\pm^2(1 - c^2) + 2(V_\pm^2 \mathbf{U}_A^2 - c^2 U_{Ax}^2) + 2Bc^4 + BWn^2c^6\left(\frac{\partial^2(w/n)}{\partial p^2}\right)_s\Big], \quad (1.14)$$

where

$$A = V_\pm^2(1 + \mathbf{U}_A^2) - \tfrac{1}{2}[\mathbf{U}_A^2 + c^2(1 + U_{Ax}^2)],$$
$$B = V_\pm^2(1 + U_{Ax}^2) - U_{Ax}^2. \quad (1.15)$$

For fast magnetoacoustic waves the following inequalities hold:

$$V_+ > c, \quad A > 0, \quad B > 0, \quad V_+ > cU_{Ax}/U_A,$$

whereas for slow waves the following hold:

$$V_- < c, \quad A < 0, \quad B < 0, \quad V_- < cU_{Ax}/U_A.$$

It follows from Eq. (1.14) that both for fast and slow waves, when the following condition holds:

$$(\partial^2(w/n)/\partial p^2)_s > 0 \quad (1.16)$$

(s is the entropy per particle), we have

$$dV_p/dW > 0. \quad (1.17)$$

Under these conditions points characterized by higher densities are displaced more rapidly than points of low density.* This means that at compression points the density gradient is increased whereas at rarefaction points the density gradient is reduced.

The further analysis of simple waves is essentially the same as that for nonrelativistic magnetohydrodynamics.[8] At the compression points discontinuities arise; the self similar waves are always rarefaction waves.

It follows from Eq. (1.8) that in a fast self-similar magnetoacoustic simple wave the magnetic field falls off; in a slow wave it increases.

2. DISCONTINUITIES

The following boundary conditions[1] hold at the surface of a discontinuity:

$$\{nwv_x^2/(1 - v^2) + p + (\mathbf{H}_t^2 - E_x^2)/8\pi\} = 0, \quad (2.1)$$

$$\{nwv_x v_y/(1 - v^2) - H_x H_y/4\pi\} = 0, \quad (2.2)$$

$$\{nwv_x v_z/(1 - v^2) - H_x H_z/4\pi\} = E_z\{E_x\}/4\pi, \quad (2.3)$$

$$\{nwv_x/(1 - v^2)\} = \{H_x(\mathbf{v}_t \mathbf{H}_t) - v_x \mathbf{H}_t^2\}/4\pi, \quad (2.4)$$

$$j = n_1 v_{1x}/\sqrt{1 - v_1^2} = n_2 v_{2x}/\sqrt{1 - v_2^2}, \quad (2.5)$$

*Because of the thermodynamic relation $(\partial W/\partial n)_s > 0$, an increase in W corresponds to an increase in density.

$$E_{1y} = E_{2y} = v_{2x}H_{2z} - v_{2z}H_x = 0, \qquad (2.6)$$

$$E_{1z} = E_{2z} = -v_{1x}H_{1y} = -v_{2x}H_{2y} + v_{2y}H_x, \qquad (2.7)$$

$$\{H_x\} = 0. \qquad (2.8)$$

The subscript "1" refers to the region in front of the discontinuity while the subscript "2" refers to the region behind the discontinuity. The x axis is taken normal to the discontinuity and points from region 1 to region 2.

We shall use a reference system which moves with the discontinuity which is such that the following relation holds in region 1:

$$v_{1y} = v_{1z} = H_{1z} = 0. \qquad (2.9)$$

The boundary conditions can be simplified considerably if we go over to a reference system in which the vectors $\mathbf{v}$ and $\mathbf{H}$ at both sides of the discontinuity surface are parallel. In this case the electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ and the right sides of Eqs. (2.3) and (2.4) all vanish.

It should be noted that a reference system in which the velocity of the liquid is parallel to the magnetic field does not necessarily exist for all discontinuities. The quantity v_{1x} is fixed (this is the velocity of the discontinuity with respect to the liquid in region 1). The quantity $\mathbf{v}_{1t}$ is determined from the condition that the vectors $\mathbf{v}_1$ and $\mathbf{H}_1$ be parallel:

$$\mathbf{v}_{1t} = v_{1x}\mathbf{H}_{1t} / H_x.$$

In order that the total velocity $\sqrt{\mathbf{v}_{1x}^2 + \mathbf{v}_{1t}^2}$ be smaller than the velocity of light the following relation must hold

$$v_{1x}^2\mathbf{H}_1^2 / H_x^2 < 1. \qquad (2.10)$$

As we shall see below, in general this relation is not satisfied for fast shock waves* (for $H_x \neq 0$).

The classification of discontinuities in relativistic magnetohydrodynamics is similar to the classification in nonrelativistic magnetohydrodynamics.[14] The following types of discontinuities exist:

1) Discontinuities that are at rest with respect to the liquid, $j = 0$.

a) Contact discontinuities: $H_x \neq 0$.

From the boundary conditions it follows that:

$$\mathbf{v} = 0, \quad \mathbf{E} = 0, \quad \{\mathbf{H}\} = 0, \quad \{p\} = 0.$$

b) Tangential discontinuities: $H_x = 0$.

From the boundary conditions it follows that

$$v_x = 0, \quad \mathbf{E}_t = 0, \quad \{p + (\mathbf{H}_t^2 - E_x^2) / 8\pi\} = 0;$$

*The assertion of de Hoffman and Teller that for any shock wave there is a reference system in which the velocity of the liquid is parallel to the magnetic field is not true in general.

the discontinuities in the remaining thermodynamic quantities are arbitrary.

2) Discontinuities that move with respect to the liquid, $j \neq 0$.

From the boundary conditions (2.1) — (2.9) it follows that:

$$[n_2 w_2 v_{2x}^2 / (1 - v_2^2) - H_x^2 / 4\pi + E_{2z}^2 / 4\pi] H_{2z} = 0. \qquad (2.11)$$

If the first factor vanishes the discontinuity is called an Alfvén discontinuity; if the second factor vanishes the discontinuity is called a shock wave. We consider these in greater detail.

a) Alfvén discontinuities: $H_{2z} \neq 0$.

$$n_2 w_2 v_{2x}^2 / (1 - v_2^2) - H_x^2 / 4\pi + E_{2z}^2 / 4\pi = 0. \qquad (2.12)$$

The following similar relation holds for region 1:

$$\frac{n_1 w_1 v_{1x}^2}{1 - v_1^2} - \frac{H_x^2}{4\pi} + \frac{v_{1x}^2 H_{1y}^2}{4\pi} = 0. \qquad (2.13)$$

Converting to the moving reference system we find that the Alfvén discontinuity is displaced with respect to the liquid in region 1 with a velocity V_{Ax}, which is defined by Eq. (1.9).

For the Alfvén discontinuity the relation in (2.10) is satisfied; hence we can convert to a reference system in which the velocity of the liquid is parallel to the magnetic field. In this reference system, it follows from the boundary conditions that

$$\{w / n\} = 0, \quad \{p + H_x^2 n^2 / 8\pi j^2\} = 0. \qquad (2.14)$$

In view of the continuity of H_x and j at the discontinuity, the independent thermodynamic quantities can be taken as w/n and $p + H_x^2 n^2/8\pi j^2$. It follows from Eq. (2.14) that at the Alfvén discontinuity all the thermodynamic quantities are continuous

$$\{n\} = \{w\} = \{p\} = 0. \qquad (2.15)$$

The vectors associated with the magnetic field and the velocity of the liquid do not change in absolute magnitude but rotate through the same angle about the x axis. We emphasize that this rotation of the vectors $\mathbf{v}$ and $\mathbf{H}$ takes place only in the reference system in which the velocity of the liquid is parallel to the magnetic field.

b) Shock waves: $H_{2z} = 0$.

In this case, in accordance with Eq. (2.6) we find that $v_{2z} = 0$, i.e., the shock wave is a plane discontinuity and the vectors $\mathbf{v}$ and $\mathbf{H}$ lie in the xy plane.

We now consider the stability of shock waves.

To have a stable shock wave it is necessary that the number of waves of infinitely small intensity which diverge from the surface of the discontinuity be equal to the number of independent boundary conditions which relate the magnetohydrodynamic quantities at both sides of the discontinuity surface. On the basis of the considerations given in reference 15 we find that there are two kinds of stable magnetohydrodynamic shock waves: slow shock waves, for which

$$V_{1-} < v_{1x} < V_{1Ax}, \qquad v_{2x} < V_{2-} \tag{2.16}$$

and fast shock waves, for which

$$V_{1+} < v_{1x}, \qquad V_{2Ax} < v_{2x} < V_{2+}. \tag{2.17}$$

In accordance with Eq. (2.16), for slow shock waves there is a reference system in which the velocity of the liquid is parallel to the magnetic field.

In the case of fast shock waves the inequality in (2.10) need not necessarily hold and in this case such a reference system does not exist.

3. ZEMPLEN THEOREM*

We now show that in relativistic magnetohydrodynamics, just as in ordinary relativistic hydrodynamics,[19] there is a Zemplen theorem, according to which the pressure and density in a shock wave increase if the conditions in (1.16) hold† and if

$$(\partial s / \partial p)_{w/n} > 0. \tag{3.1}$$

The proof goes as follows.

In the absence of a magnetic field $H = 0$, along the thermodynamically stable portion of the shock adiabat ($s_2 > s_1$) the following relations hold:[19] $p_2 > p_1$, $w_2/n_2 < w_1/n_1$.

There is no portion of the shock adiabat for which $w_2/n_2 > w_1/n_1$, $p_2 < p_1$ since this corresponds to a shock wave for which the entropy is reduced ($s_2 < s_1$). By virtue of the inequality in (3.1) this section of the shock adiabat lies in the $(w/n, p)$ plane below the Poisson adiabat ($s = \text{const}$). We show that in the presence of a magnetic field the portion of the shock adiabat which corresponds to a rarefaction shock wave $w_2/n_2 > w_1/n_1$ lies still lower. From this it will follow that on this section the entropy is reduced ($s_2 < s_1$), and this is impossible.

*The Zemplen Theorem was proved by Iordanskiĭ[16] and Polovin and Lyubarskiĭ[17,18] in nonrelativistic magnetohydrodynamics for shock waves of arbitrary intensity.

†For a relativistic ideal gas

$$\left(\frac{\partial^2 (w/n)}{\partial p^2}\right)_s = \frac{2(2-\gamma)}{\gamma(\gamma-1)}\frac{1}{pn^2}, \quad \left(\frac{\partial s}{\partial p}\right)_{w/n} = \frac{2-\gamma}{\gamma-1}\frac{1}{2nT}.$$

These expressions are positive since the quantity γ lies in the interval[20] $1 < \gamma \leqslant 5/3$. In addition to the inequalities in (1.16) and (3.1), just as in references 19 and 21 we show that the pressure increases monotonically along a shock adiabat.

For the proof we write the equation of the shock adiabat in the presence of a magnetic field

$$w_2^2 - w_1^2 - (p_2 - p_1)(w_1/n_1 + w_2/n_2) = Q, \tag{3.2}$$

$$Q = \frac{H_{1y}^2 (w_2/n_2)^2 (v_{1x}/v_{2x} - 1)^2 (w_1/n_1 - w_2/n_2)}{8\pi (w_2/n_2 - H_x^2/4\pi j^2)^2}. \tag{3.3}$$

Equations (3.2) and (3.3) are obtained from the boundary conditions (2.1) — (2.9) by eliminating the variables. When $w_1/n_1 < w_2/n_2$ the quantity Q is negative if $H_{1y} \neq 0$, whereas when $H_{1y} = 0$ the quantity Q vanishes.

We shall now investigate the behavior of the curve (3.2), (3.3) (in the w_2/n_2, p_2 plane) if Q, which we take as a parameter, vanishes. Differentiating Eq. (3.3) with respect to p_2 at constant w_2/n_2, we have

$$\left(\frac{\partial Q}{\partial p_2}\right)_{w_2/n_2} = 2w_2\left(\frac{\partial w_2}{\partial p_2}\right)_{w_2/n_2} - \left(\frac{w_1}{n_1} + \frac{w_2}{n_2}\right).$$

On the other hand we have the thermodynamic relation

$$\left(\frac{\partial w}{\partial p}\right)_{w/n} = \frac{1}{n} + T\left(\frac{\partial s}{\partial p}\right)_{w/n},$$

whence

$$\left(\frac{\partial Q}{\partial p_2}\right)_{w_2/n_2} = \left(\frac{w_2}{n_2} - \frac{w_1}{n_1}\right) + 2w_2 T_2 \left(\frac{\partial s_2}{\partial p_2}\right)_{w_2/n_2}$$

By virtue of (3.1), when $w_2/n_2 > w_1/n_1$, this quantity is positive. The fact that the derivative $(\partial Q/\partial p_2)_{w_2/n_2}$ is positive indicates that with fixed w_2/n_2 a reduction of Q means a reduction in p_2. In other words, a curve of the shock adiabat (3.2) and (3.3) for $H_{1y} \neq 0$ actually lies lower than the shock adiabat in the absence of the magnetic field.

Thus, in the shock wave the pressure and the quantity n/w increase

$$p_2 > p_1, \quad w_2/n_2 < w_1/n_1. \tag{3.4}$$

Using the Zemplen theorem we can draw certain conclusions concerning the behavior of the magnetic field in shock waves.* In a slow shock wave it is possible to go over to a reference system in which the velocity of the liquid is parallel to the magnetic field. In this case, from the boundary conditions we have

*Landau and Lifshitz[21] have investigated the change of magnetic field in nonrelativistic magnetohydrodynamic shock waves of low intensity and in nonrelativistic waves of arbitrary intensity for the case in which $H^2 \ll p$. An increase in the magnetic field in fast nonrelativistic magnetohydrodynamic shock waves for $H^2 \ll p$ has been noted by Helfer.[22]

The change in the magnetic field in nonrelativistic magnetohydrodynamic shock waves has been considered by Lyubarskiĭ and Polovin.[18] The relativistic case for which $H_x = 0$ has been treated by Stanyukovich.[23]

$$H_{2y} = H_{1y} \frac{w_1 / n_1 - H_x^2 / 4\pi j^2}{w_2 / n_2 - H_x^2 / 4\pi j^2}. \quad (3.5)$$

From the Zemplen theorem (3.4), the condition of stability for a slow shock wave (2.16), and the relation in (3.5) it follows that in a slow shock wave the tangential magnetic field does not change direction and is reduced. This statement is valid also for the moving reference system since the magnetic field is the same in this system as in the reference system in which the velocity of the liquid is parallel to the magnetic field.

For a fast shock wave, as has already been indicated, there may not be a reference system in which **v** is parallel to **H**. Hence the relation in (3.5) must be replaced by the more complicated relations

$$H_{2y} = H_{1y} \frac{w_2 v_{1x} / n_2 v_{2x} - H_x^2 / 4\pi j^2}{w_2 / n_2 - H_x^2 / 4\pi j^2}, \quad (3.6)$$

$$v_{1x}^2 H_{1y}^2 = \frac{4\pi j^2 (w_1 / n_1 - w_2 v_{1x} / n_2 v_{2x}) (w_2 / n_2 - H_x^2 / 4\pi j^2)}{(w_2 / n_2)(v_{1x} / v_{2x} - 1)}, \quad (3.7)$$

which follow from the boundary conditions in (2.2), (2.4), (2.7), and (2.9). Since the shock wave is a compression wave ($w_2 > w_1$, $n_2 > n_1$) from (2.5), (2.9), and (3.7) it follows that

$$v_1 > v_2, \quad v_{1x} > v_{2x}. \quad (3.8)$$

Equation (3.6) together with the stability conditions (2.17) indicate that in a fast shock wave the tangential magnetic field does not change direction and is increased. The last statement applies in a reference system in which the equality in (2.9) is satisfied. Carrying out a Lorentz transformation to the moving reference system and using (3.8) we find that the magnetic field also increases in this system.

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