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RESONANCE EFFECTS IN THE SCATTERING OF PARTICLES NEAR A REACTION THRESHOLD

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The energy dependence of the cross-section for the scattering X (aa) X near the threshold E_{th} of the reaction X (ab) Y (X, a, b, Y are arbitrary particles) is studied on the assumption that there are long-range attractive forces between particles b and Y. It is shown that if these forces are capable of producing bound states of the particles b + Y there are resonances in the scattering cross-section. These resonances lie below the threshold of the reaction X (ab) Y. A detailed study is made of the case in which attractive Coulomb forces act between b and Y. For this case the number of resonances is infinite, and the density of the resonances approaches infinity as the threshold is approached from below.

INTRODUCTION

In a previous paper[1] the writer has studied the energy dependence of the cross-section for the scattering X (aa) X near the threshold of the reaction X (ab) Y. It was found that if there is no Coulomb interaction between b and Y the scattering cross-section has a characteristic singularity at the threshold point. In this connection it is interesting to examine the case in which long-range (for example, Coulomb) forces exist between b and Y. The present paper is devoted to the study of this problem.

Let us look at the statement of the problem in more detail. We shall study the energy dependence of the cross-section for the scattering X (aa) X near the threshold E_{th} for the reaction X (ab) Y. No restrictions are placed on the nature of the particles X, a, b, Y, except that for simplicity we shall regard them as spinless. Let us introduce one more simplification: we shall assume that there are no other inelastic processes besides the reaction X (ab) Y in the range of energies with which we are concerned. We shall free ourselves from this restriction in Sec. 3.

The reason that some sort of anomalies in the scattering cross-section are to be expected near E_{th} is as follows. Suppose the long-range potential between b and Y is an attractive one and is capable of leading to the formation of a bound state with binding energy Δ. If such a state exists, then it can be formed in the collision of particles X and a (because the reaction X (ab) Y exists), and consequently the scattering cross-section must have a resonance at the energy of formation of the bound state, $E = E_{th} - \Delta$. Breit[2] was the first to call attention in the literature to the possibility of such an effect. It is clear that it is possible only if the size of the long-range potential is much larger than r_0, the radius outside which one can neglect specifically nuclear forces. We shall always assume that this condition is fulfilled.

The general theory of the effect is presented in Sec. 1. The second section is devoted to a detailed study of the case in which an attractive Coulomb potential acts between b and Y. This situation is encountered whenever differently charged particles can arise from the reaction. Section 3 contains a discussion of the effects that appear when one of the particles formed in the reaction has a finite lifetime.

1. GENERAL THEORY

Let us consider the general problem of the behavior of the cross-section for the elastic scattering X (aa) X near the threshold E_{th} for a reaction X (ab) Y. We consider a state with a given total angular momentum l; in the region outside the action of nuclear forces $(r > r_0)$ the wave function has the form

$$\Psi = \Phi(a, X)\,(R_k^{(-)}(r) - S_l R_k^{(+)}(r))\,Y_{l0} - \Phi(b, Y)\,M_l\,\mathcal{R}_{k_1}^{(+)}(r) Y_{l0}, \tag{1.1}$$

where $\Phi(a, X)$ and $\Phi(b, Y)$ are the internal wave functions of particles a and X and of b and Y, respectively; $R_k^{(+)}(r)$ is the radial function for the relative motion of particles a and X with the wave number k (the signs ± refer to diverging and converging waves); $\mathscr{R}_{k_1}^{(+)}(r)$ is the radial function for the diverging particles b and Y with the wave number k_1; Y_{l0} is the angular part of the wave functions, normalized to unity; and S_l and M_l are the elements of the scattering matrix, whose energy dependence we have to determine. The wave numbers k and k_1 are connected by the relation

$$h^2k^2/2\mu = h^2k_1^2/2\mu_1 + E_{th}$$

[μ and μ_1 are the reduced masses of (a+X) and (b+Y)].

The radial functions $R_k^{(+)}$ are determined in the usual way. They are combinations of Bessel functions if a and X can be regarded as free in the region $r > r_0$, and Coulomb functions if the particles a and X are charged. As for $\mathscr{R}^{(+)}$, this is the function that describes the motion of particles b and Y in the field of the long-range potential that acts between them. At infinity it must behave like e^{ik_1r}/r. Hereafter we shall always suppose that $R^{(+)}$ and $\mathscr{R}^{(+)}$ are normalized to unit flux.

The scattering cross-section is given by

$$\sigma(\theta, E) = \frac{1}{4k^2}\left|\sum_l (2l+1)(S_l - 1)P_l(\cos\theta)\right|^2.$$

In order to determine the dependence of S_l on the energy, we use a formula proved in the Appendix, according to which

$$S_l = (a + b\varkappa_b^{(+)})/(a^* + b^*\varkappa_b^{(+)}), \tag{1.2}$$

where the quantities a and b can be supposed independent of the energy near the threshold, and

$$\varkappa_b^{(+)} = \frac{d}{dr}\ln\mathscr{R}_{k_1}^{(+)}\Big|_{r=r_0} \tag{1.3}$$

Thus for our problem it suffices to determine the energy dependence of the function $\kappa_b^{(+)}$. We shall first consider the range of energies $E < E_{th}$.

For $r > r_0$ the function $\mathscr{R}_{k_1}^{(+)} \equiv \chi/r$ satisfies the usual Schrödinger equation

$$d^2\chi/dr^2 + [k_1^2 - l(l+1)/r^2 - (2\mu_1/h^2)V(r)]\chi = 0, \tag{1.4}$$

where V is the long-range potential between particles b and Y. We are interested in the solution of this equation that behaves like e^{ik_1r} at infinity. Since $E < E_{th}$ and k_1 is pure imaginary, this asymptotic form assures the finiteness of χ for $r \to \infty$. As is well known from the general theory of the Schrödinger equation,[3] the function χ for small r can be represented in the form

$$\chi = \xi[(k_1r)^{l+1} + i\varepsilon(k_1r)^{-l}], \tag{1.5}$$

where ξ and ϵ are functions of k_1; the concrete form of these functions is determined by the potential V. The function ϵ is real, since both the equation (1.4) and the boundary condition at infinity, $\chi \sim e^{ik_1r} = e^{-|k_1|r}$, are real. Some general conclusions can be drawn about the energy dependence of the function ϵ. Indeed, let us continue equation (1.4) into the region $r < r_0$. If it admits of the existence of bound states with binding energies Δ_m, then the function ϵ must go to zero at the values $k_1 = (k_1)_m = (\Delta_m/2\mu_1h^2)^{1/2}$, since only under this condition is χ regular at the origin. Thus as a function of the energy E of the particles a + X the quantity ϵ has the form shown schematically in Fig. 1. Sometimes, as for example in the case of the Coulomb field, considered in Sec. 2, ϵ goes to infinity in the intervals between the values Δ_m. This makes no difference, however, in the following arguments.

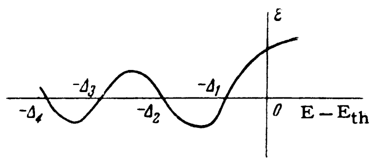

FIG. 1

Let us calculate the logarithmic derivative $\kappa_b^{(+)}$. From Eqs. (1.3) and (1.5) we get

$$\varkappa_b^{(+)} = \frac{1}{r_0}\left[l - \frac{i(2l+1)\varepsilon}{i\varepsilon + (k_1r_0)^{2l+1}}\right]. \tag{1.6}$$

Let us examine the behavior of $\kappa_b^{(+)}$ near one of the bound states. Since we are considering the region near the threshold, where (k_1r) is small, it follows from Eq. (1.6) that the function $\kappa_b^{(+)}$ is practically constant and equal to $-(l+1)/r_0$ everywhere except in narrow intervals of energy around the energies Δ_m of the bound states. In these intervals, defined by the condition $|\epsilon| \lesssim |(k_1r_0)^{2l+1}|$, $\kappa_b^{(+)}$ changes very rapidly, running through the entire range of values from $-\infty$ to $+\infty$. This is shown schematically in Fig. 2.

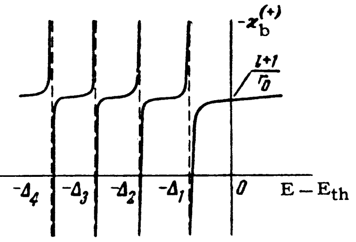

FIG. 2

Substituting Eq. (1.6) in the formula (1.2) for S_l, and noting that $\kappa_b^{(+)}$ is real for $E < E_{th}$, we can write S_l in the form $e^{2i\varphi}$, where φ is the real phase of the scattering S_l.

The energy dependence of φ can be seen from the formula

$$\varphi = \tan^{-1} \frac{\operatorname{Im}(a + b\varkappa_b^{(+)})}{\operatorname{Re}(a + b\varkappa_b^{(+)})}. \tag{1.7}$$

Using the energy dependence of $\kappa_b^{(+)}$ obtained above (see Fig. 2), we easily see that the scattering phase shift φ remains constant everywhere except in narrow regions around the bound-state energies $E = E_{th} - \Delta_m$, where it changes by π. As can be seen from the formula for the scattering cross-section

$$\sigma_l = 4\pi k^{-2}(2l+1)\sin^2\varphi,$$

to such changes of the phase shift there correspond resonance peaks of the scattering cross-section at energies close to $E_m = E_{th} - \Delta_m$, and the scattering cross-section has the shape shown in Fig. 3.

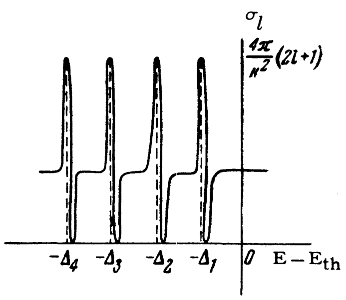

FIG. 3

We can determine the width and shape of the resonance peaks. For this purpose we rewrite S_l in the following way:

$$S_l = \frac{b}{b^*}\frac{c+\varepsilon}{c^*+\varepsilon} = \frac{b}{b^*}\left[1 - \frac{2ic_2}{(c_1+\varepsilon)+ic_2}\right],$$
$$c \equiv c_1 - ic_2, \tag{1.8}$$

where the complex constants c and b are expressed in an obvious way in terms of the constants appearing in Eqs. (1.2) and (1.6). We note further that near the m-th bound state the quantity ϵ can be expanded in a series in powers of the energy difference,

$$\varepsilon = A[E - (E_{th} - \Delta_m)] + \ldots \tag{1.9}$$

Substituting Eq. (1.9) in Eq. (1.8), we get for S_l

$$S_l = \frac{b}{b^*}\left[1 - \frac{2i\Gamma_e}{(E - E_m') + i\Gamma_e}\right],$$
$$\Gamma_e = \frac{c_2}{A}, \quad E_m' = E_{th} - \Delta_m - \frac{c_1}{A}. \tag{1.10}$$

This equation has the form of the usual Breit-Wigner formula (cf. reference 3), with Γ_e playing the role of the elastic width, which in our case is the total width, and the constant c_1/A is the shift of the resonance maximum relative to the energy of the bound state. Thus the general shape of the resonance peaks is described by the Breit-Wigner formula. The parameters of the resonances (Γ_e and c_1/A) cannot be calculated, since through the constants c_1 and c_2 they depend on the form of the wave function for $r < r_0$.

This picture corresponds exactly to the resonance scattering that is well known from the theory of nuclear reactions. The part of the quasistationary states is played by the bound states of particles b + Y in the field of the long-range potential. There is no absorption, since for $E < E_{th}$ there are no inelastic processes, and therefore at its maxima the cross-section reaches the maximum value $4\pi k^{-2}(2l+1)$.

Up to now we have spoken only of the region $E < E_{th}$. What will happen at the threshold point itself and above it? Here we must distinguish two cases: for large r the long-range potential falls off either a) faster than 1/r, or b) as 1/r. In the first case the theory developed in reference 1 applies, and at the threshold point the differential cross-section has a singularity (a peak, an inverted peak, or a finite discontinuity), and above the threshold it is a smooth function of the energy. The width of the singularity at the threshold point is very small: $|k_1| < R^{-1}$, where R is the radius of the long-range potential. Case b) is treated in the next section. Looking ahead, we can state that the threshold is a point of condensation of the resonances, and above the threshold the cross-section is constant.

2. CASE OF ATTRACTIVE COULOMB FORCE BETWEEN b AND Y

The case of a Coulomb attraction can be studied exhaustively, since the analytic properties of the Coulomb functions are known. Using the method of reference 4, we find without difficulty that for $|k_1| \to 0$ the following formulas hold:

$$\mathscr{R}_{k_1}^{(+)} = \frac{1}{r}(G_l + iF_l) = \frac{1}{r}C_0^{-1}[-2A(\ln(-iz)$$
$$+ f(x) - C_0^2/2x) + B], \tag{2.1}$$

where

$$x = ie_1e_2\mu_1/h^2k_1, \quad z = 2ik_1r,$$
$$A = \frac{(-)^l}{2}\sum_{n=0}^{\infty}(-xz)^{n+l+1}/n!\Gamma(2l+n+2),$$
$$B = \sum_{t=0}^{2l}\frac{(xz)^{t-l}}{t!}\Gamma(2l+1-t) + \sum_{n=1}^{\infty}\frac{(-xz)^{n+l+1}}{n!}\frac{(-)^{l+1}}{\Gamma(2l+2+n)}$$
$$\times\left[-2\gamma + \sum_{s=1}^{n}\frac{1}{s} + \sum_{s=1}^{n+2l+1}\frac{1}{s}\right],$$
$$C_0 = [\pi x e^{i\pi x}/\sin\pi x]^{1/2}, \quad 2f(x) = \psi(x) + \psi(-x),$$

e_1 and e_2 are charges of particles b and Y, γ is the Euler constant, and ψ is the logarithmic derivative of the Γ function. As can be seen from the formulas, A and B are real and do not depend on the energy (the product xz does not contain k_1). Let us consider the quantity

$$T \equiv \ln(-iz) + f(x) - C_0^2/2x.$$

From the properties of the ψ function[5]

$$\psi(y) - \psi(-y) \approx -\pi \cot \pi y,$$
$$\psi(y) \approx \ln y \quad \text{for } |y| \to \infty; \ |\arg y| < \pi$$

it follows that

$$T = \ln(xz) + \begin{cases} -i\pi, & E > E_{\text{th}}, \\ -\pi \cot \pi x, & E < E_{\text{th}}. \end{cases} \tag{2.2}$$

Substituting Eq. (2.2) in Eq. (2.1) and calculating the logarithmic derivative, we get

$$\varkappa_b^{(+)} = \begin{cases} (p + is)/r_0(p_1 + is_1), & E > E_{\text{th}}, \\ (p + s\cot\pi x)/r_0(p_1 + s_1 \cot \pi x), & E < E_{\text{th}}, \end{cases} \tag{2.3}$$

where p, p_1, s, and s_1 are constants that can be expressed simply in terms of A, B, and their derivatives with respect to (xz).

Substituting Eq. (2.3) in Eq. (1.2), we get for S_l

$$S_l = \begin{cases} (\alpha + i\beta)/(\alpha^* + i\beta^*), & E > E_{\text{th}}, \\ (\alpha + \beta \cot \pi x)/(\alpha^* + \beta^* \cot \pi x), & E < E_{\text{th}}. \end{cases} \tag{2.4}$$

Here α and β are constants made up from a, b, p, s, p_1, and s_1. We now have assembled all the formulas needed for the study of the behavior of the elastic scattering.

Let us consider the region above the threshold. Here two processes are possible: elastic scattering and the reaction. The cross-section for elastic scattering is given by

$$\sigma_s = \frac{\pi}{k^2}(2l+1)|S_l - 1|^2 = \frac{\pi}{k^2}(2l+1)\left|\frac{\alpha + i\beta}{\alpha^* + i\beta^*} - 1\right|^2, \tag{2.5}$$

and the reaction cross-section by

$$\sigma_r = \frac{\pi}{k^2}(2l+1)[1 - |S_l|^2] = \frac{\pi}{k^2}(2l+1)\left[1 - \left|\frac{\alpha + i\beta}{\alpha^* + i\beta^*}\right|^2\right]. \tag{2.6}$$

The total cross-section is

$$\sigma_t = \sigma_r + \sigma_s = \frac{2\pi}{k^2}(2l+1)\left[1 - \operatorname{Re}\frac{\alpha + i\beta}{\alpha^* + i\beta^*}\right]. \tag{2.7}$$

Since α and β do not depend on the energy, all three cross-sections are constants. For σ_r this has been known for a long time.[6]

Let us now go on to the region below the threshold. From Eq. (2.4) we get

$$\sigma_s = \frac{2\pi}{k^2}(2l+1)\left[1 - \operatorname{Re}\frac{\alpha + \beta\cot\pi x}{\alpha^* + \beta^*\cot\pi x}\right],$$
$$x = \frac{e_1 e_2}{h}\sqrt{\frac{\mu_1}{2(E_{\text{th}} - E)}} \tag{2.8}$$

and the scattering cross-section has the form shown in Fig. 4. When the cotangent is large, the cross-section is constant (the plateaus between the resonances in Fig. 4), and is given by

$$\sigma_{s0} = 4\pi k^{-2}(2l+1)(\sin\beta)^2|\beta|^{-2}.$$

The sharp changes in the cross-section occur only for $\cot \pi x \approx -\operatorname{Re}(\alpha/\beta)$; in such a place the cross-section runs through the whole range of possible values from zero to $4\pi k^{-2}(2l+1)$. Corresponding to the nature of the Coulomb spectrum the cross-section has an infinite number of resonances, which are more and more closely spaced as we approach

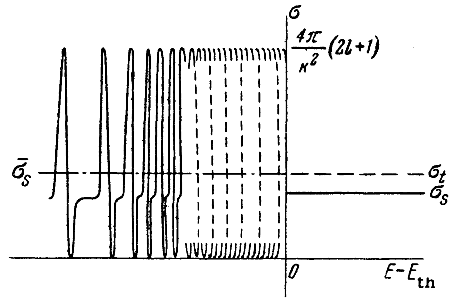

FIG. 4

the threshold point. The width of the resonance region is obviously given by the energy of the first Coulomb level, i.e., the quantity $e_1^2 e_2^2 \mu_1/2h^2$. For heavy particles the width can be large. Thus if b and Y are singly charged and their masses are of the order of the nucleon mass the width is ~50 kev. A quantity of interest is the average value of the cross section:

$$\bar{\sigma}_s = \frac{1}{2\Delta}\int_{E-\Delta}^{E+\Delta}\sigma_s dE = \frac{2\pi}{k^2}(2l+1)\operatorname{Re}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\left[1 - \frac{\alpha + \beta y}{\alpha^* + \beta^* y}\right]\frac{dy}{1+y^2}\right\} = \frac{2\pi}{k^2}(2l+1)\left[1 - \operatorname{Re}\frac{\alpha + i\beta}{\alpha^* + i\beta^*}\right] = \sigma_t. \tag{2.9}$$

In the integration one must use the inequality $i(\alpha\beta^* - \alpha^*\beta) > 0$, which follows from the fact that above the threshold the condition $|S_l| \leq 1$ must hold. Equation (2.9) leads to a remarkable result: the average scattering cross-section below the threshold is equal to the total cross-section above the threshold. Thus the total cross-section (below the threshold it is just the elastic cross-section) is in a certain sense continuous as we go through the threshold point, whereas the average elastic cross-section decreases discontinuously at this point.

3. CASE OF PRODUCTION OF AN UNSTABLE PARTICLE IN THE REACTION

It has been assumed so far that the particles b and Y are stable. We shall now consider the case in which one of them (Y) has a finite lifetime τ. It is not hard to see that the instability of Y can decidedly change the shape of the cross-section for the elastic scattering X(aa)X below the threshold. In fact, let us recall how the scattering process occurs: in the collision of a with X there is formed a certain intermediate state of the particles a, X and b, Y, which then decays to a state a + X (below threshold) or to a + X and b + Y (above threshold). If Y is unstable, it can decay during the time t of existence of the intermediate state. As a rule, however, this can be neglected, since t is usually small (of the order of nuclear times). The only exception is the case of formation of a bound state of the particles b + Y, about which we spoke in Sec. 1. The time of existence of this state can reach large values; if it is comparable with τ, an appreciable part of the Y in the intermediate state has time to decay, and this leads to a corresponding decrease of the scattering cross section. This effect is important only near the resonance maxima of the scattering cross-section and can produce a decided smoothing-out of these maxima. We also note one other conclusion from these considerations: the cross-section of the reaction

$$X(ab)Y^* \tag{3.1}$$

(Y* means the products of the decay of particle Y) will have maxima near the energies for formation of the bound states of the particles b + Y.

It is not hard to obtain formulas for the scattering cross-section and the cross-section for the reaction (1.3). To do so we must substitute in Eq. (1.10) in place of E_{th} the quantity $E_{th} - i\Gamma$ (Γ is the energy width of particle Y). We then get

$$S_l = \frac{b}{b^*}\left[1 - \frac{2i\Gamma_e}{(E - E_m^1) + i(\Gamma_e + \Gamma)}\right]. \tag{3.2}$$

This formula, like Eq. (1.10) has the usual Breit-Wigner form, but now the total width is not just the elastic width. The cross-sections for scattering and for the reaction (3.1) are obtained by substitution of Eq. (3.2) in Eqs. (2.5) and (2.6). They are just the usual Breit-Wigner formulas, and we shall not write them out here. We note only that if $\Gamma > \Gamma_e$, then at the maximum the cross-section for the reaction (3.1) will be larger than the scattering cross-section.

Like Eq. (1.10), Eq. (3.2) is valid only in the immediate neighborhood of the energy of a bound state: $E - (E_{th} - \Delta_m) \ll \delta$ (δ is the distance between resonances). Far from the resonances, however, the instability of Y can be neglected, as we have seen. The cross-section of the reaction (3.1) is equal to zero for such energies, and the scattering cross-section is constant. The approximate character of Eq. (3.2) also shows itself in the fact that it is true only if $\Gamma < \delta$.

It is easy to generalize the theory of Sec. 1 to the case in which for $E < E_{th}$ some other reactions are possible besides elastic scattering. To do this we must take the point of view of the usual resonance theory of nuclear reactions and regard the bound states of the particles b + Y as intermediate states of our system. Then introducing the partial widths for decay of a bound state into various channels, we easily obtain the generalization of the results of Sec. 1: the cross-sections for all the processes will have resonances at the energies $E \approx E_{th} - \Delta_m$; the heights of the resonances will be determined by the partial widths for the respective channels; and the width of each resonance will be the total width.

In conclusion I would like to thank Ya. A. Smorodinskiĭ for his constant interest in this work and L. A. Maksimov for reading the manuscript of the paper and making a number of comments.

APPENDIX

In the region outside the radius of action of nuclear forces ($r > r_0$) the most general wave function with a given angular momentum l has the form

$$\Psi = (R^{(-)} - SR^{(+)})\,\Phi(a, X)\,Y_{l0} + D\,(\mathscr{R}^{(-)} - \Omega\mathscr{R}^{(+)})\,\Phi(b, Y)\,Y_{l0}, \tag{A.1}$$

where S, Ω, and D are certain constants. The coefficient of $R^{(-)}$ is set equal to unity, since Ψ can be multiplied by any number. For $r = r_0$ this function must join on to the internal wave function. Then in general the internal wave function will not satisfy the conditions of regularity. These conditions will be satisfied only if S, Ω, and D take certain definite values. Corresponding to the two possible channels (scattering and the reaction) there must exist two and only two sets of values of S, Ω, and D for which Ψ is regular. Let us denote these values by S_i, Ω_i, D_i ($i = 1, 2$), and the two corresponding regular functions by Ψ_1 and Ψ_2. The quantities S_i, Ω_i, and D_i can be expressed in terms of the values of the wave function and its derivative at the surface of the nucleus. To do this we must consider separately the radial functions for the particles a + X and b + Y in Eq. (A.1), and match their logarithmic derivatives, and also the ratio of these two functions, at $r = r_0$ to the corresponding quantities formed from the internal wave function. We thus get

$$\left.\frac{R^{(-)'} - S_i R^{(+)'}}{R^{(-)} - S_i R^{(+)}}\right|_{r=r_0} = \tau_i, \quad \left.\frac{\mathscr{R}^{(-)'} - \Omega_i \mathscr{R}^{(+)'}}{\mathscr{R}^{(-)} - \Omega_i \mathscr{R}^{(+)}}\right|_{r=r_0} = \rho_i,$$
$$\left.\frac{R^{(-)} - S_i R^{(+)}}{D_i(\mathscr{R}^{(-)} - \Omega_i \mathscr{R}^{(+)})}\right|_{r=r_0} = \sqrt{\frac{\mu}{\mu_1}}\,\sigma_i, \tag{A.2}$$

where the prime indicates differentiation with respect to r; τ_i, ρ_i, and σ_i are determined by the internal function; and μ and μ_1 are the reduced masses for the particles a + X and b + Y. From Eq. (A.2) we easily get

$$S_i = \frac{\varkappa_a^{(-)} - \tau_i}{\varkappa_a^{(+)} - \tau_i}, \quad \Omega_i = \frac{\varkappa_b^{(-)} - \rho_i}{\varkappa_b^{(+)} - \rho_i},$$
$$D_i = \frac{1}{\sigma_i}\sqrt{\frac{\mu}{\mu_1}}\,\frac{\mathscr{R}^{(+)}}{R^{(+)}}\frac{\varkappa_b^{(+)} - \rho_i}{\varkappa_a^{(+)} - \tau_i}, \tag{A.3}$$

where

$$\varkappa_a^{(\pm)} = (R^{(\pm)'}/R^{(\pm)})|_{r=r_0}, \quad \varkappa_b^{(\pm)} = (\mathscr{R}^{(\pm)'}/\mathscr{R}^{(\pm)})|_{r=r_0}.$$

We note that by using the law of conservation of numbers of particles and the reversibility of the time one can show that the quantities τ_i, ρ_i, and σ_i are real, and that furthermore one can choose the functions in such a way that $\tau_i = \rho_i$ and $\sigma_1\sigma_2 = -1$. This must be kept in mind in order to assure ourselves that the scattering matrix obtained below is unitary and symmetric, as a scattering matrix should be.[7]

We can now determine the scattering matrix of our problem. To do this we take the general wave function

$$\begin{aligned} A_1\psi_1 + A_2\psi_2 = {} & \Phi(a, X)[(A_1 + A_2)R^{(-)} \\ & - (A_1S_1 + A_2S_2)R^{(+)}]Y_{l0} \\ & + \Phi(b, Y)[(A_1D_1 + A_2D_2)\mathscr{R}^{(-)} \\ & - (A_1D_1\Omega_1 + A_2D_2\Omega_2)\mathscr{R}^{(+)}]Y_{l0} \end{aligned} \tag{A.4}$$

and require that it describe the processes occurring in the collision of particles a + X. To this statement of the problem there corresponds a function that does not contain converging waves of particles b + Y. In order to get such a function, we impose on the amplitudes A_1 and A_2 the condition

$$A_1D_1 + A_2D_2 = 0. \tag{A.5}$$

Then, setting $A_1 + A_2 = 1$, we can write the function (A.4) in the form

$$[R^{(-)} - SR^{(+)}]\Phi(a, X)Y_{l0} - M\mathscr{R}^{(+)}\Phi(b, Y)Y_{l0},$$

where the quantities S and M that are the amplitudes of the outgoing waves of particles a + X and b + Y are given by:

$$S = \frac{D_1S_2 - D_2S_1}{D_1 - D_2}, \quad M = -\frac{\Omega_1 - \Omega_2}{D_1 - D_2}D_1D_2. \tag{A.6}$$

Substituting Eq. (A.3) in Eq. (A.6), we get for S:

$$S = \left.\frac{(\tau_2 - \varkappa_a^{(-)})(\tau_1 - \varkappa_b^{(+)}) + \sigma_1^2(\tau_2 - \varkappa_b^{(+)})(\tau_1 - \varkappa_a^{(-)})}{(\tau_2 - \varkappa_a^{(+)})(\tau_1 - \varkappa_b^{(+)}) + \sigma_1^2(\tau_2 - \varkappa_b^{(+)})(\tau_1 - \varkappa_a^{(+)})}\frac{R^{(-)}}{R^{(+)}}\right|_{r=r_0}. \tag{A.7}$$

From this formula we can determine the energy dependence of S near the reaction threshold. We begin by noting that near the threshold the entire energy dependence is determined by the function $\kappa_b^{(+)}$, and all the other quantities in Eq. (A.7) can be regarded as constants. Let us prove this. By definition $\kappa_b^{(+)}$ is the logarithmic derivative of the radial function of particles b + Y at $r = r_0$. This function has a singularity at $k_1r_0 \to 0$, and consequently $\kappa_b^{(+)}$ changes rapidly even for small values of k_1r_0. As for the other quantities in S, $\kappa_a^{(\pm)}$ and $R^{(\pm)}$ by definition have no singularity at the threshold and depend only on k, which near the threshold is a quadratic function of k_1:

$$k \approx \sqrt{\frac{2\mu E_{th}}{h^2}}\left(1 + \frac{1}{2}\frac{h^2k_1^2}{2\mu_1 E_{th}}\right).$$

The situation is also similar for the quantities τ_1, τ_2, and σ_1. They are determined by the form of the wave function inside the sphere $r < r_0$, where the specifically nuclear forces are large, and therefore these quantities cannot change much for small changes of the energy of the system near the threshold. This conclusion is invalid only if there exists near the reaction threshold E_{th} some level of the system that owes its occurrence to the nuclear forces.

Changing the designations of the constants, we now get the formula for S_l given in Sec. 1.

This formula for S_l is valid both above and below the threshold. In fact, only two assumptions have been made in its derivation: 1) the quantities τ_i, ρ_i, and σ_i are real and related as stated above, and 2) the complete wave function must not contain $\mathscr{R}^{(-)}$. Strictly speaking the first assumption is valid only above the threshold, but owing to the constancy of the quantities τ_i, ρ_i, and σ_i, which has been proved, it remains true in a small region for $E < E_{th}$. The second condition holds equally well for both regions, above and below the threshold. For $E > E_{th}$ it corresponds to the absence of converging waves of b + Y, and for $E < E_{th}$

(k_1 imaginary) it is required for the finiteness of the wave function at infinity.

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