

THE GREEN'S FUNCTIONS METHOD IN QUANTUM STATISTICS

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A Green's function method in quantum statistics is developed. It is shown that the various methods of statistical physics and the many-body theory, and also their generalization to cases of non-zero temperature (e.g., the methods of Debye-Hückel, Hartree-Fock, Thomas-Fermi, Gell-Mann and Brueckner), are contained in simple form in the resultant equations. A transition to time-dependent Green's functions is considered, and a method is given for determination of the energy spectrum.

INTRODUCTION

RECENTLY, great attention has been paid to the development of methods that differ from the ordinary thermodynamic perturbation theory in statistics, and the usual perturbation theory of the many-body problem. We shall not now review all the researches in this direction,* but shall only remark that the existing methods are manifold and require considerable calculations for their establishment.

It is natural to attempt to find a more general approach to the solution of the problems of quantum statistics, not connected with perturbation theory which, in particular, would give in simple graphic approximation the results of the corresponding methods. Such an approach is given by the Green's function method which had already been applied successfully to the case $T = 0$.² Matzubara³ was the first to formulate the method of Feynman and introduce the Green's function for the case $T \neq 0$. However, the problem of the derivation of closed equations for these Green's functions in x -space has not been brought to solution, and the transformation to p -space, which would facilitate the possibility of a practical solution of the equations, has not been considered.

On the other hand, the development of high-energy physics requires the development of the formalism of quantum statistics, generally with account both of relativistic corrections and the second quantization of the Hamiltonian of the system. Finally, for the investigation of the energy spectrum of the system, it is necessary to know the time dependence of the Green's function as well.

The present article* is devoted to the consideration of these problems (see also reference 5).

1. THE EQUATIONS FOR THE GREEN'S FUNCTION IN QUANTUM STATISTICS

The density matrix of the canonical ensemble has the form

$$\rho = \exp \{-\beta(\hat{H} - \mu\hat{N})\}, \quad (1.1)$$

where $\beta = 1/kT$, \hat{H} is the total Hamiltonian of the system, consisting of the Hamiltonian of the free field (H_0) and the interaction Hamiltonian H_1 ; μ is the chemical potential, and \hat{N} is the operator of an integral number of surviving particles (for example, the difference in the number of electrons and positrons).

Without limiting the generality, we shall carry out our subsequent analysis, for the case of the interaction of a single Fermi field ψ (mass m) with a Bose field (mass κ).

The Hamiltonian of the interaction in this case is equal to

$$\hat{H}_1 = - \int j(x) \varphi(x) d^3x, \quad (1.2)$$

$$j(x) = \frac{1}{2} ig \text{Sp} \gamma [\bar{\psi}(x) \psi(x) - \psi(x) \bar{\psi}(x)],$$

$$\hat{N} = \frac{1}{2} \int \text{Sp} \gamma_4 [\bar{\psi}(x) \psi(x) - \psi(x) \bar{\psi}(x)] d^3x. \quad (1.3)$$

We shall not take a specific form for the interaction γ and the variance of the field φ (for example, for electrodynamics, $\gamma = \gamma_\mu$, $\varphi = A_\mu$). By the usual methods of field theory³ we get

*For the corresponding literature references, see reference 1.

*The authors have been informed that similar questions are considered by Abrikosov, Gorkov and Dzyaloshinskiĭ.¹²

$$\rho(\beta) = \rho_0(\beta) S(\beta), \quad \rho_0(\beta) = \exp\{-\beta(\hat{H}_0 - \mu\hat{N})\},$$

$$S(\beta) = T \exp\left\{-\int_0^\beta dx_4 \int H_1(x_4, \mathbf{x}) d^3x\right\},$$

$$H_1(\mathbf{x}, x_4) = -j(\mathbf{x}, x_4) \varphi(\mathbf{x}, x_4), \quad (1.4)$$

where T denotes the regular position of the operators from right to left in order of increase in time, while the operator $f(\mathbf{x}, x_4)$ is determined from the relation

$$f(\mathbf{x}, x_4) = \rho_0(-x_4) f(\mathbf{x}) \rho_0(x_4). \quad (1.5)$$

By analogy with quantum field theory,⁴ we generalize the interaction Hamiltonian, including the additional interaction with external sources of Bose (J) and Fermi (η) fields, namely,

$$\begin{aligned} \hat{H}_1(\mathbf{x}, x_4) = & -\{[j(\mathbf{x}, x_4) + J(\mathbf{x}, x_4)] \varphi(\mathbf{x}, x_4) \\ & + \bar{\eta}(\mathbf{x}, x_4) \psi(\mathbf{x}, x_4) + \bar{\psi}(\mathbf{x}, x_4) \eta(\mathbf{x}, x_4)\}. \end{aligned} \quad (1.6)$$

From (1.4) and (1.6), we get:

a) for x_4 in the interval from 0 to β :

$$\begin{aligned} \delta\rho(\beta)/\delta J(\mathbf{x}, x_4) &= \rho_0(\beta) S(\beta) S(-x_4) \varphi(\mathbf{x}, x_4) S(x_4) \\ &= \rho(\beta) \hat{\varphi}(\mathbf{x}, x_4), \\ \delta\rho/\delta\bar{\eta}(\mathbf{x}, x_4) &= \rho\hat{\psi}(\mathbf{x}, x_4), \quad \delta\rho/\delta\eta(\mathbf{x}, x_4) = \rho\hat{\bar{\psi}}(\mathbf{x}, x_4), \\ \delta^2\rho/\delta\bar{\eta}(\mathbf{x}, x_4) \delta\eta(\mathbf{x}', x'_4) \\ &= \rho T(\hat{\psi}(\mathbf{x}, x_4) \hat{\bar{\psi}}(\mathbf{x}', x'_4) \text{ etc.}; \end{aligned} \quad (1.7)$$

b) for x_4 outside the interval from 0 to β , all functional derivatives are equal to zero.

The arbitrary operator $f(\mathbf{x}, x_4)$ is connected with the operator $f(\mathbf{x})$ by the relation

$$\hat{f}(\mathbf{x}, x_4) = \rho^{-1}(x_4) f(\mathbf{x}) \rho(x_4). \quad (1.8)$$

Variation according to η and $\bar{\eta}$ is to be understood in the sense of variation to the right and left, respectively, i.e.,

$$\delta\rho = \int d^4x \left(\delta\bar{\eta} \frac{\delta\rho}{\delta\bar{\eta}} + \frac{\delta\rho}{\delta\eta} \delta\eta \right). \quad (1.9)$$

Using the commutation rule for ψ and φ , we can obtain⁵ the following functional equations for the integral of the state:

$$\begin{aligned} (i\hat{p} + m - ig\gamma\delta/\delta J) \delta z / \delta\bar{\eta}(x) &= \eta(x) z, \\ (-\partial^2/\partial x_\nu + x^2) \delta z / \delta J(x) \\ &= J(x) z - ig \text{Sp} \{ \gamma \delta^2 z / \delta\bar{\eta}(x) \delta\eta(x) \}, \end{aligned} \quad (1.10)$$

where $z = \sum_{\mathbf{n}} \Phi_{\mathbf{n}}^* \rho \Phi_{\mathbf{n}} = \text{Tr}(\rho(\beta))$ ($\Phi_{\mathbf{n}}$ is the wave function of the total Hamiltonian); $i\hat{p} = \gamma_\nu \partial_\nu$; $\partial_4 = (\partial/\partial x_4) - \mu$; $\partial_k = \partial/\partial x_k$; $\hbar = c = 1$; $\nu = 1, 2, 3, 4$; $k = 1, 2, 3$. Here the functional derivatives differ from zero only for x_4 in the interval from 0 to β .

From (1.10), by the method of reference 6, we find the following operator solution for the state integral:*

$$\begin{aligned} z &= z_0 \exp\left\{ ig \text{Sp} \gamma \int d^4x \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\bar{\eta}(x)} \frac{\delta}{\delta J(x)} \right\} \\ &\times \exp\left\{ J(x) D_0(x-y) J(y) \right. \\ &\left. + \bar{\eta}(x) G_0(x-y) \eta(y) \right\} d^4x d^4y, \end{aligned} \quad (1.11)$$

where

$$\begin{aligned} z_0 &= \prod_{rk} (1 + \exp\beta(\mu - \varepsilon_r))(1 + \exp[-\beta(\mu + \varepsilon_r)]) \\ &\times (1 - \exp[-\beta\omega_k])^{-1}, \end{aligned} \quad (1.12)$$

$$\begin{aligned} D_0(x-x') &= (2\pi)^{-3} \int \frac{d^3k}{2\omega_k} \{ (f_k + 1) \exp[ik(x-x') \\ &- \omega_k|x_4 - x'_4|] + f_k \exp[-ik(x-x') + \omega_k|x_4 - x'_4|] \}; \\ G_0(x-x') &= (i\hat{p} - m) (2\pi)^{-3} \\ &\times \int \frac{d^3k}{2\varepsilon_k} \left\{ (N_k^+ - 1) \exp[ik(x-x') - (\varepsilon_k - \mu)(x_4 - x'_4)] \right. \\ &\left. + N_k^- \exp[-ik(x-x') + (\varepsilon_k + \mu)(x_4 - x'_4)] \right\}, \quad x_4 > x'_4, \\ &+ (N_k^- - 1) \exp[ik(x-x') + (\varepsilon_k + \mu)(x_4 - x'_4)] \}, \quad x_4 < x'_4; \\ N_k^\pm &= [1 + \exp(\varepsilon_k \pm \mu)\beta]^{-1}; \quad f = [\exp(\omega_k\beta) - 1]^{-1}; \\ \varepsilon_k &= \sqrt{k^2 + m^2}; \quad \omega_k = \sqrt{k^2 + x^2}; \end{aligned} \quad (1.13)$$

N_k^+ (N_k^-) is the mean number of electrons (positrons) in the state with energy ε_k . Eliminating the meson derivatives, we get from (1.11)

$$\begin{aligned} z &= z_0 \exp\left\{ \left[(J(x) + ig\gamma \frac{\delta}{\delta\eta(x)} \frac{\delta}{\delta\bar{\eta}(x)}) \frac{D_0(x-y)}{2} (J(y) \right. \right. \\ &\left. \left. + ig\gamma \frac{\delta}{\delta\eta(y)} \frac{\delta}{\delta\bar{\eta}(y)}) \right] d^4x d^4y \right\} \\ &\times \exp\left\{ \bar{\eta}(x) G_0(x-y) \eta(y) d^4x d^4y \right\}. \end{aligned} \quad (1.14)$$

Differentiating $\ln z$ with respect to the charge, we get the following operator solution for the thermodynamic potential:

$$\Omega = \Omega|_{g=0} - \int_0^g \frac{idg'}{g'\beta z} \text{Sp} \gamma \int d^4x \frac{\delta}{\delta\eta} \frac{\delta}{\delta\bar{\eta}} \frac{\delta}{\delta J} z, \quad (1.15)$$

Equations (1.11) and (1.15) can be expanded in power series in g , in which each term coincides with the corresponding approximation computed by the Feynman method.³

Our method is more suitable, especially for the higher approximations, since it is not necessary here to examine all possible Feynman diagrams; the entire problem reduces to the successive application of functional differentiation in the re-

*Here and below the integral over the fourth component extends from 0 to β .

quired order with respect to the charge, which is simple to perform.

The mean value of any T-regular function of the field operators $f(\psi\bar{\psi}\varphi)$ is calculated by the rule

$$\langle f \rangle = \frac{\text{Tr}(\rho f)}{z} = \frac{1}{z} f \left(\frac{\delta}{\delta \bar{\eta}} \frac{\delta}{\delta \eta} \frac{\delta}{\delta J} \right) z \Big|_{J=\eta=0}, \quad (1.16)$$

For example,

$$\begin{aligned} & \langle T(\psi(x)\psi(x_1)\bar{\psi}(x')\bar{\psi}(x'_1)) \rangle \\ &= \frac{1}{z} \frac{\delta^4 z}{\delta \bar{\eta}(x)\delta \bar{\eta}(x_1)\delta \eta(x')\delta \eta(x'_1)} \Big|_{J=\eta=0}. \end{aligned}$$

We introduce the "one-particle" Green's function of fermions G and bosons D which, in accord with (1.16) can be written in the form*

$$\begin{aligned} G(x, y) &= \frac{1}{z} \frac{\delta^2 z}{\delta \bar{\eta}(x)\delta \bar{\eta}(y)} \Big|_{J=\eta=0}, \\ D(x, y) &= \frac{\delta^2 \ln z}{\delta J(x)\delta J(y)} \Big|_{J=\eta=0}. \end{aligned} \quad (1.17)$$

with the aid of (1.10), we obtain the following set of equations for G and D:

$$\begin{aligned} & (i\hat{p} + m - ig\gamma \langle \varphi(x) \rangle) G(x, y) \\ & + \int \Sigma^*(x, z) G(z, y) d^4z = \delta(x - y), \end{aligned} \quad (1.18)$$

$$\begin{aligned} & (-\partial^2 / \partial x_\nu^2 + \kappa^2) D(x, y) \\ & - \int \Pi(x, z) D(z, y) d^4z = \delta(x - y), \end{aligned} \quad (1.19)$$

$$\begin{aligned} (-\partial^2 / \partial x_\nu^2 + \kappa^2) \langle \varphi(x) \rangle &= -\frac{ig}{2} \text{Sp} \gamma [G(x, x_4; x, x_4 - \varepsilon) \\ & + G(x, x_4; x, x_4 + \varepsilon)]_{\varepsilon \rightarrow 0}, \end{aligned} \quad (1.20)$$

$$\Sigma^*(x, y) = g^2 \int \gamma G(x, z) \Gamma(z, y, y') D(y', x) d^4z d^4y',$$

$$\Pi(x, y) = g^2 \text{Sp} \int \gamma G(x, z) \Gamma(z, y', y) G(y', x) d^4z d^4y',$$

$$\begin{aligned} \Gamma(x, y, y') &= \gamma \delta(z - y') \delta(y - y') \\ & - \delta \Sigma^*(x, y) / \delta (ig \langle \varphi(y') \rangle). \end{aligned} \quad (1.21)$$

"One-particle" Green's functions are defined by the relations (1.19) - (1.21) for x_4 only in the interval from β to $-\beta$. Outside this interval, these functions can be defined in arbitrary fashion.

We apply the condition of periodicity (with period 2β) to the functions and expand them in Fourier series:

*The first Green's functions for $T \neq 0$ were introduced by Matzubara.³ Our Green's functions differ from his in that a complete symmetry is established between \hat{H} and \hat{N} i.e., all the quantities depend on $\hat{H} - \mu \hat{N}$ (while in Matzubara's case, x_4 enters only with the factor \hat{H}). As will be seen from what follows, this permits us to make the transformation to the p-representation in the equations for the Green's functions.

$$G(x) = \frac{1}{\beta (2\pi)^3} \sum_{p_4} \int d^3p G(\mathbf{p}, p_4) \exp(-ip_\nu x_\nu),$$

$$D(x) = \frac{1}{\beta (2\pi)^3} \sum_{p_4} \int d^3p D(\mathbf{p}, p_4) \exp(-ip_\nu x_\nu),$$

$$\begin{aligned} \Gamma(x, y, z) &= \Gamma(x - z, y - z) = \beta^{-2} (2\pi)^{-6} \sum_{p_4, p'_4} \int \Gamma(p, p') \\ & \times \exp\{-ip_\nu(x - z)_\nu + ip'_\nu(y - z)_\nu\} d^3p d^3p'. \end{aligned} \quad (1.22)$$

It can be shown that: 1) the frequencies p_4 for the Fermi particles take the form $(2n + 1)\pi/\beta$ while the frequencies p_4 for the Bose particles take* only the values $2\pi n/\beta$, where n runs through all integral values from $-\infty$ to $+\infty$; 2) for G and D, the following relations hold:

$$\begin{aligned} G(x_4 = \mp 0) &= -G(x_4 = \pm \beta), \quad \dot{D}(x_4 = \mp 0) = \dot{D}(x_4 = \pm \beta), \\ & (\dot{D} = \frac{\partial}{\partial x_4} D). \end{aligned}$$

This follows from the spectral representations for G and D, which have the form

$$\begin{aligned} G(x, x') &= \frac{1}{z} \sum_{m, n} \phi_{m, n}(0) \bar{\psi}_{n, m}(0) \exp\{(\mu N_m - E_m)\beta \\ & + (E_m - E_n + \mu N_n - \mu N_m)(x_4 - x'_4) \\ & + i(\mathbf{p}_m - \mathbf{p}_n)(\mathbf{x} - \mathbf{x}')\}, \quad x_4 > x'_4, \\ G(x, x') &= -\frac{1}{z} \sum_{m, n} \bar{\psi}_{n, m}(0) \phi_{m, n}(0) \exp\{(\mu N_n - E_n)\beta \\ & + (E_m - E_n + \mu N_n - \mu N_m)(x_4 - x'_4) \\ & + i(\mathbf{p}_m - \mathbf{p}_n)(\mathbf{x} - \mathbf{x}')\}, \quad x_4 < x'_4; \end{aligned} \quad (1.23)$$

$$\begin{aligned} D(x, x') &= \frac{1}{z} \sum_{m, n} |\varphi_{n, m}(0)|^2 \exp[-\beta E_n \\ & + (E_n - E_m)|x_4 - x'_4| + i(\mathbf{p}_n - \mathbf{p}_m)(\mathbf{x} - \mathbf{x}')]. \end{aligned} \quad (1.24)$$

The fact that we have only odd frequencies (in units of π/β) for the Green's functions G, and only even ones for the functions D, leads to the appearance of the Kronecker δ in the integration over x_4 .† (The law of conservation of the "quasi-

*The circumstance that, in the case $T \neq 0$, the thermodynamic quantities are expressed in the form of a series in $\varepsilon_n = \varepsilon(2n\pi/\beta)$ was shown by E. M. Lifshitz⁷ in another connection.

†The author thanks I. M. Khalatnikov, who recently called our attention to the work of Ezawa, Tomozawa and Umezawa, where a Fourier expansion has been carried out for the case $\mu = 0$ in the matrix diagram technique. However, the Green's functions of Matzubara for $\mu \neq 0$ would have both even and odd frequencies, and therefore the integration over x_4 does not always lead to the Kronecker δ , and this transformation to the p-representation would not have been effective.

energy" at each angle of the Feynman diagrams.) In the p -representation, the set of equations for the Green's functions is appreciably simplified and takes the following form:⁵

$$\begin{aligned} [-ip_v\gamma_v - \gamma_4\mu + m + \Sigma^*(p)] G(p) &= \sum_n \delta(p_4 - (2n+1)\pi/\beta), \\ [k_v^2 + \kappa^2 - \Pi(k)] D(k) &= \sum_n \delta(k_1 - 2n\pi/\beta), \\ \Sigma^*(p) &= \frac{g^2}{(2\pi)^3\beta} \int_{k_4} \sum_k \gamma G(p+k) \Gamma(p+k, p) D(k) d^3k, \\ \Pi(k) &= \frac{g^2}{(2\pi)^3\beta} \text{Sp} \int_{p_4} \sum_k \gamma G(p+k) \Gamma(p+k, p) G(p) d^3p, \\ \Gamma(p, p') &= \sum_{n, m} \gamma \delta(p_4 - (2n+1)\pi/\beta) \delta(p_4 - (2m+1)\pi/\beta) \\ &\quad + \Lambda(p, p'), \end{aligned} \quad (1.25)$$

where δ is the Kronecker symbol and Λ is the set of all diagrams of the vertex part, except the simple vertex.

As would be expected, it follows from (1.25) that the arguments p_4 and p'_4 in $\Gamma(p, p', p-p')$ (which correspond to the "quasi-energies" of the Fermi particles) take on only the odd values $(2n+1)\pi/\beta$, and their difference (which corresponds to the "quasi-energy" of Bose particles) has only the even values $2n\pi/\beta$.

By means of (1.15), the thermodynamic potential (Gibbs function) is also expressed in terms of the "one-particle" functions

$$\Omega = \Omega|_{g=0} - \frac{V}{(2\pi)^3\beta} \sum_{k_4} \int_0^{\beta} \frac{\Pi(k) dg'}{g'(k^2 - \Pi(k))} d^3k. \quad (1.26)$$

In concluding this section, we note that the transition to the non-relativistic approximation is contained in the following way: we must substitute $-i\gamma_\nu p_\nu - \gamma_\mu\mu + m$ for $-ip_4 - \mu_1 + p^2/2m$ and $G_{\alpha\beta}$ for $G\delta_{\alpha\beta}$ everywhere. In fact, it is easy to see from Eq. (1.25) that it suffices to consider the transition to the non-relativistic limit only for the Green's functions without the interaction:

$$\begin{aligned} G_0 &= \frac{\Sigma\delta(p_4 - (2n+1)\pi/\beta)}{-i\gamma_\nu p_\nu - \gamma_4\mu + m} \\ &= \frac{(i\gamma_\nu p_\nu + \gamma_4\mu + m) \Sigma\delta(p_4 - (2n+1)\pi/\beta)}{(p_4 - i\mu)^2 + p^2 + m^2}. \end{aligned} \quad (1.27)$$

In the non-relativistic approximation, we have $\mu = m + \mu_1$; $\gamma = 0$; $\gamma_4 = \delta_{\alpha\beta}$; and all quantities are small relative to m . Leaving terms of first order of smallness, we obtain the following rule for transition to the non-relativistic case:

$$\begin{aligned} G_0 &= \frac{2m\delta_{\alpha\beta} \Sigma\delta(p_4 - (2n+1)\pi/\beta)}{-2ip_4 - m^2 - 2\mu_1 m + m^2 + p^2} \\ &= \frac{\delta_{\alpha\beta} \Sigma\delta(p_4 - (2n+1)\pi/\beta)}{-ip_4 - \mu_1 + p^2/2m}. \end{aligned} \quad (1.28)$$

2. METHOD OF OBTAINING THE TIME-DEPENDENT GREEN'S FUNCTION

Up to now we have been concerned with temperature dependent Green's functions which make it possible to compute all the thermodynamic characteristics of interest to us. However, for problems on the determination of the spectrum, etc., it is very important to know the time-dependent Green's function. For simplicity we shall carry out the following analysis for a nonrelativistic Hamiltonian.

Thus, our problem is to find the time dependent Green's function in quantum statistics; it is convenient to define the latter differently than was done by Landau,⁸ namely

$$\bar{G}(x, t; x', t') = i \langle T(\psi(x, t)\psi^\dagger(x', t')) \rangle, \quad (2.1)$$

where the dependence on time of any Heisenberg operator $f(\mathbf{x}t)$ is generalized in the following way:

$$\bar{f}(\mathbf{x}, t) = e^{i(\hat{H}-u\hat{N})t} f(\mathbf{x}) e^{-i(\hat{H}-u\hat{N})t}, \quad (2.2)$$

$\langle \rangle$ denotes averaging over the canonical ensemble. It is easy to write down the spectral representation for the time dependent function:⁸

$$\begin{aligned} \bar{G}(x, t; x', t') &= \frac{i}{z} \\ &\times \begin{cases} \sum_{n, m} |\phi_{n, m}(0)|^2 \exp[(\mu N_n - E_n)\beta + i(E_n - E_m) \\ - \mu N_m + \mu N_n)(t - t') - i(\mathbf{p}_n - \mathbf{p}_m)(\mathbf{x} - \mathbf{x}')], & \text{for } t > t' \\ - \sum_{n, m} |\phi_{n, m}(0)|^2 \exp[(\mu N_m - E_n)\beta + i(E_n - E_m \\ - \mu N_n + \mu N_m)(t - t') - i(\mathbf{p}_n - \mathbf{p}_m)(\mathbf{x} - \mathbf{x}')], & \text{for } t < t'. \end{cases} \end{aligned} \quad (2.3)$$

Comparing (1.23) and (2.3), we see that the transition from G to \bar{G} is connected with the analytic continuation of x_4 , that is, $\bar{G}(\mathbf{x} - \mathbf{x}'; t - t')$ in the interval $t - t' > 0$ is the analytic continuation in x_4 ($x_4 \rightarrow it$) of the function $G(\mathbf{x} - \mathbf{x}', x_4 - x'_4)$ in the interval $\beta > x_4 - x'_4 > 0$ and the function $\bar{G}(\mathbf{x} - \mathbf{x}', t - t')$ in the interval $t - t' < 0$ is the analytic continuation of $G(\mathbf{x} - \mathbf{x}'; x_4 - x'_4)$ determined in the interval $0 > x_4 - x'_4 > \beta$.

Thus the problem reduces to the establishment of the form of the Green's function in the intervals (0β) and $(-\beta 0)$. For this purpose, we make use of the spectral representation of the Green's function $G(\mathbf{x}, x_4)$. According to (1.23), we have

$$G(\mathbf{p}, p_4) = \frac{1}{z}$$

$$\times \sum_{n, m} |\phi_{n, m}^{(0)}|^2 \frac{\exp(\mu N_n - E_n)\beta + \exp(\mu N_m - E_m)\beta}{-ip_4 + (E_m - E_n) - \mu}, \quad (2.4)$$

where $p_4 = (2n+1)\pi/\beta$, n is an arbitrary posi-

tive or negative number. It follows from (2.4) that it is always possible to represent G in the form

$$G(\mathbf{p}, p_4) = \int_{-\infty}^{+\infty} \frac{f(w, \mathbf{p}) dw}{-ip_4 - \mu + w}. \quad (2.5)$$

By means of (2.5) and (1.22), we can obtain $G(\hat{\mathbf{p}}, x_4)$ for the values of x_4 in the interval from $-\beta$ to β , that is,

$$G(\mathbf{p}, x_4) = \begin{cases} \int_{-\infty}^{+\infty} dw f(w, \mathbf{p}) \frac{\exp[-(w-\mu)x_4]}{1 + \exp[-(w-\mu)\beta]}, & \text{when } \beta > x_4 > 0, \\ - \int_{-\infty}^{+\infty} dw f(w, \mathbf{p}) \frac{\exp[-(w-\mu)x_4]}{1 + \exp(w-\mu)\beta}, & \text{when } 0 > x_4 > -\beta. \end{cases} \quad (2.6)$$

We then get for the time-dependent Green's function

$$\bar{G}(\mathbf{p}, t) = i \begin{cases} \int_{-\infty}^{+\infty} dw f(w, \mathbf{p}) \frac{\exp[-i(w-\mu)t]}{1 + \exp[-(w-\mu)\beta]} & \text{for } t > 0, \\ \int_{-\infty}^{+\infty} dw f(w, \mathbf{p}) \frac{\exp[-i(w-\mu)t]}{1 + \exp(w-\mu)\beta} & \text{for } t < 0. \end{cases} \quad (2.7)$$

Making a Fourier transformation in t , we get the time-dependent Green's function in the p -representation:*

$$\bar{G}(\mathbf{p}, p_4) = i \int_{-\infty}^{+\infty} dw f(w, \mathbf{p}) \left[\frac{\delta + (p_4 + \mu - w)}{1 + \exp[-(w-\mu)\beta]} - \frac{\delta - (p_4 + \mu - w)}{1 + \exp(w-\mu)\beta} \right], \quad (2.8)$$

where

$$\delta_{\pm}(\alpha) = \int_0^{\infty} \exp(\pm i\alpha t) dt = \pi\delta(\alpha) \pm i/\alpha.$$

Here p_4 runs continuously through a series of values in the interval $-\infty$ to $+\infty$. We shall show that the energy spectrum can be found from the temperature dependent Green's function. In fact, the spectral function $f(w, \mathbf{p})$ is represented in the general case in the form†

$$f(w, \mathbf{p}) = \sum A_n(\mathbf{p}) \delta(w - E_n(\mathbf{p})) + \rho(w, \mathbf{p}). \quad (2.9)$$

Substituting (2.9) in (2.6), we get

$$G(\mathbf{p}, p_4) = \sum \frac{A_n(\mathbf{p})}{-ip_4 + E_n(\mathbf{p}) - \mu} + \int_{-\infty}^{+\infty} \frac{dw \rho(w)}{-ip_4 - \mu + w}, \quad (2.10)$$

*It follows from (2.8) that the dispersion formula for \bar{G} , found by Landau,⁸ is satisfied.

†The δ function can be spread out somewhat, because of the interaction, and the singularity takes the form of a pole for complex values of E_n .

where $p_4 = (2m + 1)\pi/\beta$.

On the other hand, it is evident that the poles of the time dependent Green's function are located at the points $p_4 = E_n - \mu$ where E_n gives the energy spectrum of the system. Hence we obtain the following rule: to find the poles of the time-dependent Green's function (and, consequently, the energy spectrum also) it suffices to find the zeros of the analytic continuation of $G^{-1}(\mathbf{p}, p_4)$ in the complex plane of p_4 .

Consequently, E_n is determined from the following equation:

$$-E_n + \frac{p_4^2}{2m} + \Sigma^*(\mathbf{p}, p_4 = i(\mu - E_n)) = 0. \quad (2.11)$$

In this case, the real part of E_n corresponds to the energy, and the imaginary part to the damping of the n -th excitation. As $\beta \rightarrow 0$, the levels with $E_n < \mu$ correspond to holes in the Fermi distribution, while the levels with $E_n > \mu$ correspond to the production of particles beyond the Fermi surface. It is also easy to determine the quantity

$$A_n(\mathbf{p}) = [1 - \partial \Sigma^* / \partial ip_4]^{-1} |_{ip_4 = E_n - \mu}. \quad (2.12)$$

In the approximation, where the damping is small, i.e., $\Gamma_n \ll E_n^{(0)}$ ($E_n = E_n^{(0)} - i\Gamma_n$), we get from (2.11)

$$-E_n^{(0)} + \frac{p_4^2}{2m} + \Sigma_0^*(\mathbf{p}, ip_4 = E_n^{(0)} - \mu) = 0; \quad \Sigma^* = \Sigma_0^* - i\Sigma_1^*,$$

$$\Gamma_n = \frac{\Sigma_1^*(\mathbf{p}, p_4)}{1 - \partial \Sigma_0^*(\mathbf{p}, p_4) / \partial ip_4} \Big|_{ip_4 = (E_n^{(0)} - \mu)} \quad (2.13)$$

All these results are easily generalized to the case of Bose statistics:

$$\bar{D}(\mathbf{p}, p_4) = i \int_{-\infty}^{+\infty} dw f_1(w, \mathbf{p}) \times \left[\frac{\delta + (p_4 + \bar{\mu} - w)}{1 - \exp[-(w-\mu)\beta]} + \frac{\delta - (p_4 + \bar{\mu} - w)}{\exp[(w-\bar{\mu})\beta] - 1} \right], \quad (2.14)$$

where $\bar{\mu}$ is the chemical potential of Bose particles. The poles of the time-dependent Green's function correspond to the poles of the analytic continuation (in p_4) of the temperature-dependent Green's function. Thus, for example, in the case $\bar{\mu} = 0$, we have the following equation for the spectrum of the energy E_n :

$$-E_n^2 + p^2 + \kappa^2 - \Pi(p^2; -E_n^2) = 0. \quad (2.15)$$

In the case of small damping, we have

$$\Gamma_n = \frac{\Pi_1(p^2; p_4^2)}{2E_n(1 - \partial \Pi^0 / \partial p_4^2)} \Big|_{p_4^2 = -E_n^{(0)2}},$$

$$-E_n^{(0)2} + p^2 + \kappa^2 - \Pi_0(p^2; p_4^2 = -E_n^{(0)2}) = 0, \quad (2.16)$$

where $\Pi = \Pi_0 + i\Pi_1$, $E_n = E_n^{(0)} - i\Gamma_n$.

3. A SYSTEM OF PARTICLES INTERACTING ACCORDING TO COULOMB'S LAW

Let us consider a system of electrons and ions, interacting according to Coulomb's law. In this case, the Hamiltonian of the interaction, with accuracy up to unimportant terms of the eigenenergy, can be written in the form

$$H_1 = -\frac{e^2}{2} \sum_{\lambda_1 \lambda_2} z_{\lambda_1} z_{\lambda_2} \int j_4^{(\lambda_1)}(x) V(x-x') j_4^{(\lambda_2)}(x') d^3x d^3x', \quad (3.1)$$

where z_λ is the charge of particles of type λ : $e^2/4\pi c = 1/137$; $\hbar = 1$. It is not difficult to see, according to Eq. (1.14), that the system under consideration is equivalent to a system of charged particles interacting with the fourth component of the vector field φ_μ , wherein

$$D^{(0)}(x-x') = V(x-x') \delta(x_4-x'_4). \quad (3.2)$$

Therefore we can at once set down the equations for the Green's function for a system of particles interacting according to Coulomb's law.

Let us consider two limiting cases:

1) When the ions are uniformly distributed (a translationally invariant system). In this case, the set of equations for the Green's function has the form

$$[i\hat{p} - \gamma_4 \mu_\lambda + m_\lambda - \Sigma_\lambda^*(p)] G_\lambda(p) = \sum_n \delta(p_4 - (2n+1)\pi/\beta), \quad (3.3)$$

$$D(k) = D_0(k) + D_0(k) \Pi(k) D(k), \quad (3.4)$$

$$\Sigma_\lambda^*(p) = \frac{e^2 z_\lambda^2}{(2\pi)^3 \beta} \int \sum_{k_4} \gamma_4 G_\lambda(p+k) \Gamma_4^{(\lambda)}(p+k, p) D(k) d^3k, \quad (3.5)$$

$$\Pi(k) = \sum_\lambda \frac{e^2 z_\lambda^2}{(2\pi)^3 \beta} \text{Sp} \int \sum_{p_4} \gamma_4 G_\lambda(p+k) \Gamma_4^{(\lambda)}(p+k, p) G_\lambda(p) d^3p, \quad (3.6)$$

$$\Gamma_4^{(\lambda)}(p, p') = \gamma_4 + \Lambda^{(\lambda)}(p, p'), \quad (3.7)$$

When G_n is the "one-particle" Green's function of particles with charge z (for electrons, $z_\lambda = -1$); $D(k)$ is the distribution function of a longitudinal photon with the presence of plasma taken into account.

a) The self-consistent field approximation (Hartree-Fock) in terms of the equations just set forth corresponds to the following approximation: the mass operators are determined by Eq. (3.5) with neglect of Γ and Λ (i.e., it is assumed that $\Pi = \Lambda = 0$).

b) Because of the singularity $D_0(k) = 1/k^2$ for small momenta, there are essential and higher approximations (the so-called "correlation" contribution) and these eliminate the singularity in the $D(k)$ function at low momenta (k).

In a practical case, we consider the principal part of the correlation energy by computing Π in first approximation, i.e.,

$$\Pi^{(1)}(k) = \sum_\lambda \frac{e^2 z_\lambda^2}{(2\pi)^3 \beta} \text{Sp} \int \sum_{p_4} \gamma_4 G_\lambda^{(0)}(p+k) \gamma_4 G_\lambda^{(0)}(p) d^3p. \quad (3.8)$$

With the help of $\Pi^{(1)}$ we can, by means of (1.26), obtain the following expression for the thermodynamic potential in this approximation:

$$\Omega = \Omega|_{e=0} + \frac{V}{2\beta} \frac{1}{(2\pi)^3} \sum_{k_4} \int \ln \left(1 - \frac{1}{k^2} \Pi^{(1)}(k) \right) d^3k. \quad (3.9)$$

Separating the self-consistent part, i.e., $\Pi^{(1)}(k)/k^2$, we obtain the following expression for the "correlation" part of the thermodynamic potential:

$$\Omega_{\text{cor}} = \frac{V}{2(2\pi)^3 \beta} \times \int \sum_{k_4} \left[\ln \left(1 - \frac{1}{k^2} \Pi^{(1)}(k) \right) + \frac{1}{k^2} \Pi^{(1)}(k) \right] d^3k. \quad (3.10)$$

In the classical limit $\hbar \rightarrow 0$ and $e^2 \beta n^{1/3} \ll 1$, we obtain the Debye-Hückel theory. This corresponds to the approximation in which $\Pi_{(k)}^{(1)} = \Pi^{(1)}(0)$; in this case, we get the well-known expression for the Debye radius λ_D ; it is determined by the expression

$$\lambda_D^{-2} = -\Pi^{(1)}(0) = -\frac{2e^2}{(2\pi)^3} \sum_\lambda \int \frac{\partial f^{(\lambda)}(k)}{\partial \epsilon_k^{(0)}} d^3k = e^2 \sum_\lambda \frac{\partial n_\lambda}{\partial \mu_\lambda} z_\lambda^2, \quad (3.11)$$

where n_λ is the mean density of particles of type λ . Finding $\Pi^{(1)}$ we can, by means of (3.10), obtain not only the results of the Debye-Hückel theory, but also all subsequent corrections (in terms of the Debye-Hückel smallness parameter, i.e., $e^2 \beta n^{1/3}$, to the thermodynamic potential.

c) It is important the other limiting case, where the Gell-Mann-Brueckner approximation is valid (the parameter of smallness is $me^2/\hbar^2 n^{1/3}$) also is contained in the given approximation of $\Pi^{(1)}$

In fact, carrying out the necessary calculations,* we get the following results from (3.10) in the non-relativistic approximation:

*We note that the poles of the function $\cot(\alpha\beta/2)[\tan(\alpha\beta/2)]$ are located at $2n\pi/\beta[(2n+1)\pi/\beta]$; therefore, summation over the frequencies of the Bose (Fermi) fields can be changed to a calculation of the contour integral.

$$\Omega_{e, \text{cor}} = V \int \frac{d^3q}{2(2\pi)^3 \beta} \sum_{n=-\infty}^{\infty} \left[\ln \left(1 + \frac{2e^2}{q^2 (2\pi)^3} \int \frac{w_p (f_p - f_{p+q}) d^3p}{w_p^2 + (2n\pi/\beta)^2} \right) - \frac{2e^2}{q^2 (2\pi)^3} \sum \int \frac{w_p (f_p - f_{p+q})}{w_p^2 + (2n\pi/\beta)^2} \right], \quad (3.12)$$

where $w_p = \epsilon_{p+q}^{(0)} - \epsilon_p^{(0)}$; $\epsilon_p^{(0)} = p^2/2m$; $f_p = [1 + \exp(\epsilon_p^{(0)} - \mu)\beta]^{-1}$. By the usual methods, we find from (3.13) that for $\beta \rightarrow \infty$

$$E_{e, \text{cor}} = V \int \frac{d^4q}{2(2\pi)^4} \left\{ \ln \left(1 + \frac{2e^2}{(2\pi)^3} \int \frac{w_p (f_p^{(0)} - f_{p+q}^{(0)}) d^3p}{w_p^2 + q_4^2} \right) - \frac{2e^2}{q^2 (2\pi)^3} \int \frac{w_p (f_p^{(0)} - f_{p+q}^{(0)}) d^3p}{w_p^2 + q_4^2} \right\}, \quad (3.13)$$

where

$$f_p^{(0)} = \begin{cases} 0, & \text{when } \epsilon_p^{(0)} > \mu, \\ 1, & \text{when } \epsilon_p^{(0)} < \mu. \end{cases} \quad (3.14)$$

Expanding (3.14) in the smallness parameter set forth above, we obtain the results of Gell-Mann and Bruekner.

By means of the method applied in Sec. 2, it is not difficult to establish the time dependence of the \bar{D} -function.

d) It is easy to verify that \bar{D} has discrete poles, corresponding to the plasma. In this case, the equation for the determination of the poles α_n of the Green's function \bar{D} , according to (2.15), has the form

$$1 + \frac{2e^2}{q^2 (2\pi)^3} \int \frac{w_p (f_p - f_{p+q}) d^3p}{w_p^2 - \omega_n^2 - i\delta} = 0. \quad (3.15)$$

The continuous spectrum corresponds to pair production, and the discrete spectrum gives the energy of the plasma. Going on to the case $\beta \rightarrow 0$ we obtain the results we had previously.

In conclusion, we note that the best results for the Green's function in the case under consideration are obtained if we solve the set of equations (3.3) - (3.6), setting $\Gamma = \gamma$.

2) The other limiting case of interest to us is the one in which the ions are so correlated with their own electrons that an appreciable violation of the translational invariance takes place, which is the case for example, in the statistical theory of the atom. In this case, we can assume the ions are fixed in their lattice sites (atoms) in first approximation, and the problem can be reduced to the interaction of electrons with the scalar components of the electromagnetic field. [D_0 is determined as before by the equation $\nabla^2 D_0(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$]. Inasmuch as the distribution of the ions is essentially non-uniform, the "one-particle" Green's functions depend not only on the difference of the spatial coordinates (the dependence on the fourth components is differential, as before). The problem consists in obtaining a system of equations

for the Green's function in the presence of an external field, which violates the translational invariance of the system. For simplicity, we consider an approximation in which $\Gamma = \gamma$ (in practice, this is the best approximation for atomic systems). It can be shown that in this case, the equations for the Green's function have the following form:

$$\begin{aligned} & \{-i(p_4 - i\mu + ie\langle\varphi(\mathbf{x})\rangle)\gamma_4 + \gamma\nabla_x\} G(p_4, \mathbf{x}, \mathbf{x}') \\ & + \int \Sigma^*(p_4, \mathbf{x}, \mathbf{y}) G(p_4, \mathbf{y}, \mathbf{x}) d^3y \\ & = \delta(\mathbf{x} - \mathbf{x}') \delta(p_4 - (2n+1)\pi/\beta); \end{aligned} \quad (3.16)$$

$$\begin{aligned} \Delta\langle\varphi(\mathbf{x})\rangle & = e\rho_i + \frac{e}{\beta} \text{Sp} \gamma_4 \sum_{p_4} G(p_4, \mathbf{x}, \mathbf{x}); \\ & - \Delta D(k_4, \mathbf{x}, \mathbf{x}') \end{aligned} \quad (3.17)$$

$$- \int \Pi(k_4, \mathbf{x}, \mathbf{y}) D(k_4, \mathbf{y}, \mathbf{x}') d^3y = \delta(\mathbf{x} - \mathbf{x}'); \quad (3.18)$$

$$\Sigma^*(p_4, \mathbf{x}, \mathbf{y}) = \frac{e^2}{\beta} \sum_{k_4} \gamma_4 G(p_4 + k_4, \mathbf{x}, \mathbf{y}) \gamma_4 D(k_4, \mathbf{y}, \mathbf{x}); \quad (3.19)$$

$$\begin{aligned} & \Pi(k_4, \mathbf{x}, \mathbf{y}) \\ & = \frac{e^2}{\beta} \text{Sp} \sum_{p_4} \gamma_4 G(p_4 + k_4, \mathbf{x}, \mathbf{y}) \gamma_4 G(p_4, \mathbf{y}, \mathbf{x}). \end{aligned} \quad (3.20)$$

We transform to the p -representation relative to the difference in coordinates, i.e.,

$$G(p_4, \mathbf{x}, \mathbf{y}) = \frac{1}{(2\pi)^3} \int G(p_4, \mathbf{x}, p) e^{ip(\mathbf{x}-\mathbf{y})} d^3p. \quad (3.21)$$

Omitting the necessary calculations, we get the final form for the Green's function in the case being considered:

$$\begin{aligned} & (-i\gamma_\nu p_\nu - \gamma_4 \mu + m + e\gamma_4 \langle\varphi(\mathbf{x})\rangle + \Sigma^*(\mathbf{x}, \tilde{p})) G(\mathbf{x}, p) \\ & = \sum_n \delta(p_4 - (2n+1)\pi/\beta); \end{aligned} \quad (3.22)$$

$$\Delta\langle\varphi(\mathbf{x})\rangle = e\rho_i + \frac{e}{(2\pi)^3 \beta} \text{Sp} \int \gamma_4 \sum_{p_4} G(\mathbf{x}, p) d^3p; \quad (3.23)$$

$$[(p - i\nabla)^2 - \Pi(\mathbf{x}, \tilde{p})] D(\mathbf{x}, p) = 1, \quad (3.24)$$

$$\Sigma^*(\mathbf{x}, p) = \frac{e^2}{(2\pi)^3 \beta} \int \sum_{k_4} \gamma_4 G(\mathbf{x}, \tilde{p} + k) \gamma_4 D(\mathbf{x}, k) d^3k; \quad (3.25)$$

$$\Pi(\mathbf{x}, k) = \frac{e^2}{(2\pi)^2 \beta} \sum_{p_4} \text{Sp} \int \gamma_4 G(\mathbf{x}, p + \tilde{k}) \gamma_4 G(\mathbf{x}, p) d^3p, \quad (3.26)$$

where

$$\tilde{p}_k = p_k - i\nabla_k \text{ for } k = 1, 2, 3; \tilde{p}_4 = p_4,$$

Here the operator ∇ acts on the functions to its right.

a) The Hartree approximation in terms of these equations corresponds to $\Pi = \Sigma^* = 0$. In this case, we can write the following symbolic solution for G :

$$G(\mathbf{x}, p) = \frac{\sum_n \delta(p_4 - (2n + 1)\pi/\beta)}{-i\gamma_4 \tilde{p}_4 - \gamma_4 \mu + m + e\gamma_4 \langle \varphi(\mathbf{x}) \rangle}. \quad (3.27)$$

It is not difficult to show that

$$-\frac{1}{\beta} \text{Sp} \gamma_4 \sum_{p_4} G(\mathbf{x}, p) = 2(N^+ - N^-), \quad (3.28)$$

where

$$N^\pm = 1/(1 + \exp(\beta \tilde{\varepsilon}_p^\pm)),$$

$$\tilde{\varepsilon}_p^\pm = \sqrt{(\mathbf{p} - i\nabla)^2 + m^2} \mp (\mu - e \langle \varphi(\mathbf{x}) \rangle). \quad (3.29)$$

(In the classical approximation, $\nabla = 0$).

Thus, in the Hartree approximation, we obtain the following symbolic equation for $\langle \varphi \rangle$:

$$-\Delta \langle \varphi(\mathbf{x}) \rangle = -e\rho_i + \frac{2e}{(2\pi)^3} \int \hat{\rho}(\varepsilon_p) d^3p, \quad (3.30)$$

where $\hat{\rho} = \hat{N}^+ - N^-$, N^\pm and $\tilde{\varepsilon}$ are determined by (3.29). We obtained the generalized Thomas-Fermi model with all the quantum and relativistic corrections.

In the nonrelativistic approximation, Eqs. (3.30) coincide in form with the operator equations obtained by Kirzhnits.⁹

b) Calculation of the exchange energy corresponds in our case to the calculation of the mass operator in first approximation (setting $D = D_0$). In this case we obtain the generalized Thomas-Fermi-Dirac model in operator form with all quantum and relativistic corrections (or the method of Hartree-Fock). The correlation corrections in our case, when the homogeneity of the spatial distribution of the charge is disrupted, have a still greater significance.¹⁰ In order to find them, it suffices to compute the polarization operator in first approximation:

$$\Pi^{(1)}(\mathbf{x}, p) = \frac{e^2}{(2\pi)^3 \beta} \text{Sp} \sum_{k_4} \int \gamma_4 G(\mathbf{x}, \tilde{p} + k) \gamma_4 G(\mathbf{x}, k) d^3k. \quad (3.31)$$

In this case the correlation part of the thermodynamic potential is obtained with the help of (1.26).

In conclusion, we note that in the case in which the potential $V(\mathbf{x} - \mathbf{x}')$ is short range, there are significant higher corrections to the function, and it is more appropriate to express the mass operator in terms of the two-particle Green's function G_{12} :

$$\int \Sigma^*(x, y) G(y, x') d^4y$$

$$= g^2 \int V(\mathbf{x} - \mathbf{y}) \delta(x_4 - y_4) G_{12}(x, y, y, x') d^4y,$$

where $G_{12}(\mathbf{x}, y, \mathbf{x}', y') = \langle T(\psi(\mathbf{x})\psi(y)\bar{\psi}(\mathbf{x}')\bar{\psi}(y')) \rangle$. It is not difficult to obtain the equation for G_{12} (in first approximation, this equation is of the Bethe-Salpeter type). Solving simultaneously for G and G_{12} , we obtain results corresponding to the Bruekner approximation (we obtain the "gaseous" approximation if we solve the equation for G_{12} approximately, setting $G = G_0$ in it). In this approximation superconductivity is also explained.

This question will be discussed by us in more detail in another paper.

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