ON A CERTAIN VERSION OF NONLOCAL ELECTROMAGNETIC FIELD THEORY

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Submitted to JETP editor September 10, 1958


A nonlocal theory is considered which corresponds to a modification of the Lienard-Wiechert potential (2). It is shown that if, in this version of the theory, the potential is assumed to satisfy the usual type of equation (3), the consistent relativistic treatment (many-time formalism) leads to self-contradictions already within the framework of the classical theory.

At various times a number of authors\textsuperscript{1-3} investigated one of the versions of nonlocal electromagnetic field theory from different points of view. This version is based on a modification of the form of the Lienard-Wiechert potentials. Instead of the usual way of writing,

\[ A_{\mu} = -\frac{e}{c} \left( u_{\mu} (r') / R, u_{\mu} (r') \right) R_{\mu \nu} = 0 \]

(1)

(where \( u_{\mu} \) is the four-velocity of the charge, \( R_{\mu \nu} = r_{\mu} - r_{\nu} \) is the difference of the coordinates of the point of observation and the charge), one starts from the following expression:

\[ A_{\mu} = -\frac{e}{c} \left( u_{\mu} (r') / R, u_{\mu} (r') \right) R_{\mu \nu} + a R_{\mu \nu} = 0 \]

(2)

Here \( a \) is a new constant with the dimension of a length. It is further assumed that the potential satisfies the usual equation

\[ \nabla A_{\mu} = -\frac{4\pi}{c} j_{\mu} \]

(3)

which essentially defines the current. In the static case \( u_{\mu} = (p, 0, 0, 1) \), and (3) leads to the charge density

\[ \rho (r - r') = \frac{e}{4\pi c} \frac{3a^2}{(R^2 + a^2)^2} \]

(4)

i.e., the point charge is spread out. Analogously, the current is spread out in a more complicated way. In this theory the field at the location of the charge is finite, and so is the self energy.

The aim of the present paper is to show that this approach is unsatisfactory, and that self-contradictions arise from the adoption of the usual type of equation for the potential leading to a spreading out of the charge: the classical system of equations for charged particles is internally inconsistent. For the proof we make use of the many-time formalization of the classical electrodynamics of a system of charged particles.\textsuperscript{4} We assume that each particle, and the field, has its own time, different from that of the others. The interaction for such a system may be written in the form

\[ S = \sum_{n=1}^{N} \left\{ \int_0^\tau \left[ \frac{1}{\beta_n^2} + \nabla A_{\mu n} \right] \frac{e}{c} \left( \nabla A_{\mu n} \right) \right\} d\tau + \int_0^\tau \rho_n (r - r_n) A_{\mu} (r, \tau) \frac{1}{c} \left[ \frac{1}{\beta_n^2} + \nabla A_{\mu n} \right] \frac{e}{c} \left( \nabla A_{\mu n} \right) d\tau \]

\[ + \int_0^\tau \rho_n (r - r_n) A_{\mu} (r, \tau) \frac{1}{c} \left[ \frac{1}{\beta_n^2} + \nabla A_{\mu n} \right] \frac{e}{c} \left( \nabla A_{\mu n} \right) d\tau \]

(5)

The Hamiltonian of the particles is

\[ H_{\text{p}} = \int_0^\tau \rho_n (r - r_n) \frac{1}{c} \left[ \frac{1}{\beta_n^2} + \nabla A_{\mu n} \right] \frac{e}{c} \left( \nabla A_{\mu n} \right) d\tau \]

(6)

the Hamiltonian of the field is

\[ H = \frac{1}{8\pi} \sum_{1}^{N} \left( \frac{\partial A_{\mu n}}{\partial x_i} + (4\pi)^2 \frac{c^2}{\beta_n} \frac{\partial A_{\mu n}}{\partial x^2} \right) d\tau \]

(7)

where

\[ \Pi_{\mu} = \left( \frac{1}{4\pi c^2} \right) \frac{\partial A_{\mu}}{\partial \tau} \]

We note that, in the many-time formalism, the electromagnetic field is described by the free Hamiltonian and the corresponding free field equation \( \nabla A_{\mu n} = 0 \). A current at the right hand side of the field equation appears only if the transition to a single time is made.

For the following it is convenient to go over to different canonical field variables. Let

\[ A_{\mu} (r, \tau) = V^{-\mu}_k \sum_k P_k k \cos (k \cdot r - kct) + C Q_k k \sin (k \cdot r - kct) \]

\[ 4\pi \Pi_{\mu} (r, \tau) = (V^{-\mu}_k \sum_k \frac{1}{\beta_k} P_k k \cos (k \cdot r - kct) \]

\[ - Q_k k \cos (k \cdot r - kct) \]

(8)

\[ P_k^\nu \] and \( Q_k^\nu \) are the new canonical variables.
Expressions (8) explicitly give the time dependence of \( P_k \) and \( Q_k \).

The field Hamiltonian in the new canonical variables has the form

\[
H = \frac{1}{2\pi} \sum_k \left[ (P_k)^2 + c^2 k^2 (Q_k)^2 \right].
\]  

(10)

The terms in the Hamiltonian for the particles which depend on the field are conveniently rewritten in the following way:

\[
\sum_k \left[ \int \rho_n(r - r_{kn}) A_n(r, t_n) \, dr \right.
\]

\[
\left. + cQ_k \sin (k \cdot r_{kn} - kct_n) \right] \right] \, dr
\]  

\[
= \sum_k \left[ \int \rho_n(R) \left[ V^{-1} \sum_k \left( \frac{P_k}{k} \cos (k \cdot R) + kP_k \right) \right. \right.
\]

\[
+ cQ_k \sin (k \cdot r_{kn} - kct_n) \right] \right] \, dR.
\]

Since \( \rho_n(R) \) depends only on \( R^2 \), this expression becomes

\[
V^{-1} \sum_k \left[ \left( \int \rho_n(R) \cos k \cdot R \, dR \right) \frac{P_k}{k} \cos (k \cdot r_{kn} - kct_n) \right.
\]

\[
\left. + cQ_k \sin (k \cdot r_{kn} - kct_n) \right]
\]  

\[= V^{-1} \sum_k f(k) \times \left( \frac{P_k}{k} \cos (k \cdot r_{kn} - kct_n) + cQ_k \sin (k \cdot r_{kn} - kct_n) \right), \]

(11)

where

\[
f(k) = \frac{1}{\pi} \int \rho_n(R) \cos k \cdot R \, dR.
\]

The Hamilton–Jacobi equations for our system of charged particles are written in the following form:

\[
\partial S(t_1 \ldots t_N) / \partial t_n = -H_n(t_n, t_n, \partial S / \partial t_n, Q_k, \partial S / \partial Q_k),
\]

\[ (n = 1 \ldots N). \]

(13)

The condition of consistency for such a system of equations is (see, e.g., reference 5)

\[
(\partial^2 / \partial t_n^2 - \partial^2 / \partial t_n \partial t_n') S(t_1 \ldots t_N) = 0.
\]  

(14)

According to (13) this leads to

\[
\partial H_n \left( t_n, t_n, \partial S / \partial t_n, Q_k, \partial S / \partial Q_k \right) / \partial t_{n'}
\]

\[- \partial H_n' \left( t_{n'}, t_{n'}, \partial S / \partial t_{n'}, Q_k, \partial S / \partial Q_k \right) / \partial t_n = 0
\]

or

\[
\frac{\partial H_n}{\partial (\partial S / \partial t_n)} \frac{\partial S}{\partial t_n} + \frac{\partial H_n'}{\partial (\partial S / \partial Q_k)} \frac{\partial S}{\partial Q_k} = 0.
\]

(10)

We note that \( \partial S / \partial t_n \) and \( \partial S / \partial Q_k \) correspond to the canonical momenta of the particles and the field and that the order of differentiation with respect to the time and to the coordinates (of the particles and the field) can be interchanged if the condition of consistency is satisfied. The last expression can then be rewritten in form

\[
\frac{\partial H_n}{\partial \rho_n} \frac{\partial H_n}{\partial \rho_n'} + \frac{\partial H_n'}{\partial \rho_p} \frac{\partial H_n}{\partial \rho_p'} - \frac{\partial H_n}{\partial \rho_p} \frac{\partial H_n'}{\partial \rho_p'} = 0.
\]

Since always \( \partial H_n / \partial \rho_n' = \partial H_n / \partial \rho_n' \), we finally get instead of (14)

\[
[H_n, H_n'] = 0.
\]

(15)

Here \( [H_n, H_n'] \) is the classical Poisson bracket.

It is clear from the foregoing that in our case the Poisson bracket is calculated from the canonical field variables only.

From (6) and (11) we have

\[
[H_n, H_n'] = \frac{\epsilon_n \epsilon_n' c}{V} \sum_k \left[ \frac{P(k)}{k} \cos (k \cdot r_{kn} - kct_n) \right]
\]

\[
\times \sin (k \cdot r_{kn} - kct_n) - \sin (k \cdot r_{kn} - kct_n) \cos (k \cdot r_{kn} - kct_n) \right]
\]

\[
\times \left[ \frac{\delta_{n1}}{1} - \frac{3}{\pi} \sum_{\gamma=1} \delta_{\gamma n} \frac{\partial^3 S_{\gamma n}}{\partial t_{n}^3} \right] = \frac{\epsilon_n \epsilon_n' c}{1 - \beta_n \beta_{n'}} \left[ 1 - \beta_n \beta_{n'} \right]
\]

\[
\times \int \frac{P(k)}{k} \sin [k \cdot (r_n - r_{n'}) - kct_n] \, dk.
\]

(16)

In the case of a point charge, \( f(k) = 1 \), and the expression

\[
- (2\pi)^{-3} \int \sin [k \cdot (r_n - r_{n'}) - kct_n] \, dk = 0
\]

\[
= (2\pi)^{-3} \int \cos k \cdot (r_n - r_{n'}) \sin kct_n - \frac{\partial k}{kct} = D(x_n - x_{n'})
\]

agrees with the well-known commutator function. For point particles the conditions of consistency are therefore satisfied everywhere outside the light cone and hence, what is especially important, in the space-like region.
In our case \( f(k) \) is equal to unity, and

\[
\int K_1(ka) \frac{dR}{R} = \int K_1(ka) \frac{a^2}{(R^2 + a^2)^{3/2}} dR
\]

Hence \( K_1 \) is the Hankel function with imaginary argument. Since \( f(k) \) depends only on the absolute value of \( k \), the integral expression in (16) becomes

\[
\frac{4\pi}{|r_n - r_n'|} \int P(k) \sin k |r_n - r_n'| \sin kc (t_n - t_n') \, dk
\]

where \( a^2 \) is the absolute value of \( a \).

and change the order of integration in the integral

(18). Using the notation

\[
|r_n - r_n'| + c(t_n - t_n') = \Delta_1, |r_n - r_n'| - c(t_n - t_n') = \Delta_2;
\]

we have

\[
\int k^2 K_0^2(ka) |\cos k\Delta_n - \cos k\Delta_0| \, dk
\]

\[
= \frac{a^2}{4} \int k^2 R \left( \frac{(\Delta_1 + R)^2 + a^2}{a^2} \right)^{3/2} dR
\]

\[
+ \left( \frac{(\Delta_1 - R)^2 + a^2}{a^2} \right)^{3/2} - \left( \frac{(\Delta_2 + R)^2 + a^2}{a^2} \right)^{3/2}
\]

\[
- \left( \frac{(\Delta_2 - R)^2 + a^2}{a^2} \right)^{3/2}
\]

We note that \( \Delta_1/a \) may be quite large for \( \Delta_2/a \ll 1 \), and vice versa, for \( |r_n - r_n'| \) and \( c(t_n - t_n') \) may be taken as large as one pleases. Let us assume that \( \Delta_2/a \ll 1 \), and let us neglect the terms with \( \Delta_1 \) in view of what has just been said. Making a series expansion, we then obtain from (19)

\[
- (3\pi^2/32a^2)(1 - 15\Delta_2^2/32a^2).
\]

With \( \Delta_2/a \ll 1 \), \( \Delta_1/a \gg 1 \) we then have

\[
[H_n, H_{n'}] = \frac{3\varepsilon_n \varepsilon_{n'} c}{125a} |r_n - r_{n'}| (1 - \beta_n \beta_{n'}) \left( 1 - \frac{15\Delta_2^2}{32a^2} \right)
\]

\[
= -0.023 \frac{\varepsilon_n \varepsilon_{n'} c}{a |r_n - r_{n'}|} (1 - \beta_n \beta_{n'}) \left( 1 - \frac{15\Delta_2^2}{32a^2} \right),
\]

i.e., this expression is different from zero on the cone and near the cone in the space-like region.

An analogous calculation can be carried out for the case \( \Delta_1/a \ll 1 \), \( \Delta_2/a \gg 1 \). Furthermore, we calculated the integral numerically for the value \( \Delta_2/a = 10 \), i.e., in a region which is comparatively far from the cone (\( \Delta_1/a \) was assumed to be sufficiently large, so that the terms with \( \Delta_1 \) could be omitted). The result is

\[
[H_n, H_{n'}] = -0.0066 \frac{\varepsilon_n \varepsilon_{n'} c}{a |r_n - r_{n'}|} (1 - \beta_n \beta_{n'}).
\]

Hence \( [H_n, H_{n'}] \) is different from zero on the light cone and everywhere in the space-like region, and, although it decreases fast with the distance from the cone, it nowhere reduces to zero, except at infinity (\( \Delta_1/a = \infty \), \( \Delta_2/a = \infty \)).

In other words, the conditions of consistency of the system (13) are not fulfilled in this version of the theory.

In the process of quantization the classical Poisson bracket goes over into the commutator of the corresponding quantities:

\[
[H_n, H_{n'}]_{cl} = \frac{i}{\hbar} [\hat{H}_n, \hat{H}_{n'}]_{qu} = \frac{i}{\hbar} [\hat{H}_n, \hat{H}_{n'} - \hat{H}_{n'} \hat{H}_n].
\]

In the case of point particles this leads to (with \( \beta_n \to \alpha_n \), \( \beta_{n'} \to \alpha_{n'} \), where \( \alpha_n \), \( \alpha_{n'} \) are the Dirac matrices)

\[
[H_n, H_{n'}]_{cl} = -i \hbar \varepsilon_n \varepsilon_{n'} (\alpha_n \alpha_{n'} - 1) D(x_n - x_{n'}).
\]

the well-known expression for the Bloch condition. The failure of \( [H_n, H_{n'}]_{cl} \) to reduce to zero in the space-like region in this nonlocal version of the theory must lead to similar difficulties in the quantum region.

2. A. Landé, Phys. Rev. 56, 482 (1939) and 76, 1176 (1949).

*This may also be seen qualitatively from expression (18), for as \( \Delta_1/a \) and \( \Delta_2/a \) increase, the cosine begins to oscillate rapidly, thus making the value of the integral smaller.