MOTION OF A CHARGED PARTICLE IN AN ANISOTROPIC MEDIUM

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Expressions are derived for the electromagnetic field components and the total energy losses are determined for a charged particle moving in an anisotropic gyroelectric and gyromagnetic medium.

1. The energy losses of a charged particle moving in an anisotropic dielectric medium have been considered in a number of papers.[1-3] Sitenko and Kolomenskii[4,5] generalized this work for the case where the medium has optical activity besides the anisotropy (gyroelectric, anisotropic medium). Later Pafomov[6] discussed the Cerenkov radiation in an anisotropic ferrite, using a method which was first applied to the problem of Cerenkov radiation in an anisotropic dielectric by Ginzburg.[1] That paper also contains a discussion of the simplest case of twofold anisotropy (anisotropic ϵ and μ). In the present paper we determine the components of the electromagnetic field and the energy losses of a charged particle moving in a medium with twofold anisotropy (anisotropic ϵ and μ), using the Fourier method;[5] moreover, the medium is assumed to be gyrotropic with respect to its electric and magnetic[7] properties.

This problem may be of interest for the use of anisotropic ferrodielectrics to generate microradiowaves.

2. The electromagnetic field arising in the medium during the motion of the point charge q with velocity **v** is determined by the Maxwell equations:

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} q\mathbf{v}\,\delta(\mathbf{r}-\mathbf{v}t);$$
$$\operatorname{div}\mathbf{B} = 0, \ \operatorname{div}\mathbf{D} = 4\pi q\delta(\mathbf{r}-\mathbf{v}t). \qquad (1)$$

We shall find a solution to this system of equations by the Fourier method in writing

$$\mathbf{E}(\mathbf{r},t) = \iint \mathbf{E}(\mathbf{k},\omega)\, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} d\mathbf{k}d\omega, \qquad (2)$$

etc. Using the relation between the Fourier components

$$D_i(\mathbf{k},\omega) = \varepsilon_{ik}(\omega) E_k(\mathbf{k},\omega), \quad \varepsilon_{ik} = \varepsilon^*_{ki},$$
$$B_i(\mathbf{k},\omega) = \mu_{ik}(\omega) H_k(\mathbf{k},\omega), \quad \mu_{ik} = \mu^*_{ki}, \qquad (3)$$

we obtain the following equation for the Fourier components of the electric field intensity:

$$T_{ik}E_k = -i\frac{q}{2\pi^2}\frac{v_i}{\omega^2}\delta\left(\frac{n}{c}\varkappa_j v_j - 1\right), \qquad (4)$$

where

$$T_{ik} = n^2 \varepsilon_{iab}\varepsilon_{klm}\varkappa_a\varkappa_m\mu^{-1}_{bl} + \varepsilon_{ik},$$
$$n^2 = k^2c^2/\omega^2, \quad \varkappa_i = k_i/k,$$

and ϵ_{ikl} is the completely antisymmetric unit tensor of third rank.

The solution of Eq. (4) can be written in the form

$$E_i = -i\frac{q}{2\pi^2\omega^2} T^{-1}_{ik} v_k\,\delta\left(\frac{n}{c}v_j\varkappa_j - 1\right), \qquad (5)$$

We obtain for the total energy loss per unit length due to the remote collisions the expression

$$-\frac{d\mathcal{E}}{dl} = i\frac{q^2}{2\pi^2 v}\int_{-\infty}^{\infty}\int_0^{k_m}\int_{4\pi} T^{-1}_{ik} v_i v_k \delta\left(\frac{n}{c}v_j\varkappa_j - 1\right)\frac{d\omega}{\omega^2}k^2 dk do, \qquad (6)$$

where k_m is the maximal value of k, which is of order 1/b, where b is the minimum parameter of remote collisions.

3. We apply (6) to the motion of a point charge in an optically-active uniaxial crystal, for which the tensors for the dielectric constant ϵ_{ik} and the magnetic permeability μ_{ik} have the form

$$\varepsilon_{ik} = \begin{pmatrix} \varepsilon_1, & -i\varepsilon_2, & 0 \\ i\varepsilon_2, & \varepsilon_1, & 0 \\ 0, & 0, & \varepsilon_3 \end{pmatrix}; \quad \mu_{ik} = \begin{pmatrix} \mu_1, & -i\mu_2, & 0 \\ i\mu_2, & \mu_1, & 0 \\ 0, & 0, & \mu_3 \end{pmatrix} \qquad (7)$$

For the tensor μ^{-1}_{ik} (the reciprocal of the magnetic permeability tensor μ_{ik}) we obtain*

$$\mu^{-1}_{ik} = \begin{pmatrix} a, & -ib, & 0 \\ ib, & a, & 0 \\ 0, & 0, & g \end{pmatrix};$$
$$a = \mu_1/(\mu_1^2 - \mu_2^2), \quad b = \mu_2/(\mu_2^2 - \mu_1^2), \quad g = 1/\mu_3. \qquad (8)$$

*We assume here that (a) we can choose a coordinate system in which ϵ_{ik} and μ_{ik} have the form (7), which is, of course, the case for the simplest media,[6] and (b) the reciprocal tensor μ^{-1}_{ik} exists for all frequencies.

In this case the tensor T_{ik} has the form

$$T_{ik}=\begin{pmatrix} -n^2(\varkappa_3^2 a+\varkappa_2^2 g)+\varepsilon_1, & n^2(\varkappa_1\varkappa_2 g+ib\varkappa_3^2)-i\varepsilon_2, & n^2(\varkappa_1\varkappa_3 a-ib\varkappa_2\varkappa_3)\\ n^2(\varkappa_1\varkappa_2 g-ib\varkappa_3^2)+i\varepsilon_2, & -n^2(\varkappa_3^2 a+g\varkappa_1^2)+\varepsilon_1, & n^2(\varkappa_2\varkappa_3 a+ib\varkappa_1\varkappa_3)\\ n^2(\varkappa_1\varkappa_3 a+ib\varkappa_2\varkappa_3), & n^2(\varkappa_2\varkappa_3 a-ib\varkappa_1\varkappa_3), & -n^2 a(1-\varkappa_3^2)+\varepsilon_3\end{pmatrix}. \tag{9}$$

To find the reciprocal tensor we have to divide the minors corresponding to the elements T_{ik} by the determinant of this tensor:

$$T=\Phi(\vartheta)[n^2-n_1^2][n^2-n_2^2], \tag{10}$$

$$\Phi(\vartheta)=\varepsilon_1 ag\sin^4\vartheta+\varepsilon_1 a^2\cos^2\vartheta\sin^2\vartheta+\varepsilon_3 a^2\cos^4\vartheta \tag{11}$$
$$+\varepsilon_3 ag\cos^2\vartheta\sin^2\vartheta-\varepsilon_3 b^2\cos^4\vartheta-\varepsilon_1 b^2\cos^2\vartheta\sin^2\vartheta,$$

$$n_{1,2}^2=\{(a\varepsilon_1^2+g\varepsilon_1\varepsilon_3-a\varepsilon_2^2)\sin^2\vartheta+2(a\varepsilon_1-b\varepsilon_2)\varepsilon_3\cos^2\vartheta \tag{12}$$
$$\pm[(\varepsilon_1^2 a-\varepsilon_2^2 a-\varepsilon_1\varepsilon_3 g)^2\sin^4\vartheta+4a(a-g)\varepsilon_2^2\varepsilon_3^2\cos^4\vartheta$$
$$+4b^2\varepsilon_1\varepsilon_3(\varepsilon_2^2-\varepsilon_1^2+\varepsilon_1\varepsilon_3)\cos^4\vartheta+4g\varepsilon_1\varepsilon_3(a\varepsilon_2\varepsilon_3-b\varepsilon_1\varepsilon_3)\cos^2\vartheta$$
$$+4ab\varepsilon_2\varepsilon_3(\varepsilon_2^2-\varepsilon_1^2)]^{1/2}\}/2\Phi(\vartheta).$$

$n_{1,2}$ are the refraction indices for the ordinary and extraordinary waves, and ϑ is the angle between the optical axis of the crystal and the direction of propagation of the waves **k**.

4. We apply formula (6) to the first, simplest case: a charge moving along the optical axis. In this case we have

$$-\frac{d\mathcal{E}}{dz}=i\frac{q^2 v}{\pi c^3}\int_{-\infty}^{\infty}\int_0^{n_m}\int_0^{\pi}\{n^4[(a^2-ag-b^2)\cos^2\vartheta$$
$$+ag]\cos^2\vartheta+n^2[(\varepsilon_1 g-2a\varepsilon_1+2\varepsilon_2 b)\cos^2\vartheta-\varepsilon_1 g]+\varepsilon_1^2-\varepsilon_2^2\}$$
$$\times\frac{\delta(n\beta\cos\vartheta-1)\sin\vartheta d\vartheta n^2 dn\omega d\omega}{\Phi(\vartheta)[n^2-n_1^2(\vartheta)][n^2-n_2^2(\vartheta)]}, \tag{13}$$

where we have introduced the variable $n = kc/\omega$ instead of k.

As in reference 5, we choose a coordinate system in which the z axis is directed along the optical axis of the crystal. The particle moves along the z axis. In integrating over the angles we take account of the δ function, where the integration over n is, of course, restricted to the region from $1/\beta$ to $n_m = k_m c/\omega$. The integration yields

$$-\frac{d\mathcal{E}}{dz}=-\frac{q^2}{c^2\pi}\operatorname{Re}i\int_0^{\infty}\{(ag-\varepsilon_1\beta^2 g)n_1^2+(a^2-ag-b^2)/\beta^2$$
$$+(\varepsilon_1 g-2a\varepsilon_1+2b\varepsilon_2)+(\varepsilon_1^2-\varepsilon_2^2)\beta^2\}\{\varepsilon_1 ag\beta^2(n_1^2-n_2^2)\}^{-1}$$
$$\times\ln\{(n_m^2\beta^2-n_1^2\beta^2)/(1-n_1^2\beta^2)\}\omega d\omega \tag{14}$$
$$-\frac{q^2}{c^2\pi}\operatorname{Re}i\int_0^{\infty}\{(ag-\varepsilon_1\beta^2 g)n_2^2+(a^2-ag-b^2)/\beta^2$$
$$+(\varepsilon_1 g-2a\varepsilon_1+2b\varepsilon_2)+(\varepsilon_1^2-\varepsilon_2^2)\beta^2\}\{\varepsilon_1 ag\beta^2(n_2^2-n_1^2)\}^{-1}$$
$$\times\ln\{(n_m^2\beta^2-n_2^2\beta^2)/(1-n_2^2\beta^2)\}\omega d\omega,$$

where

$$n_{1,2}^2=\{(ag-a^2)\varepsilon_1$$
$$+ag(\varepsilon_1-\varepsilon_3)+b^2\varepsilon_1+(\varepsilon_1^2 a-\varepsilon_2^2 a+\varepsilon_1\varepsilon_3 g)\beta^2$$
$$\pm[(\varepsilon_1^2 a-\varepsilon_2^2 a-\varepsilon_1\varepsilon_3 g)^2\beta^4 \tag{15}$$
$$-2a\varepsilon_1(\varepsilon_3 g-\varepsilon_1 a)^2\beta^2+2a^2\varepsilon_2^2(\varepsilon_1 a+\varepsilon_3 g)\beta^2$$
$$+2b^2\varepsilon_1(a\varepsilon_1^2-a\varepsilon_2^2+g\varepsilon_1\varepsilon_3)\beta^2-8abg\varepsilon_1\varepsilon_2\varepsilon_3\beta^2+(g\varepsilon_3-\varepsilon_1)^2 a^2$$
$$+b^2\varepsilon_1(b^2\varepsilon_1-2a^2\varepsilon_1+2ag\varepsilon_3)]^{1/2}\}/2\varepsilon_1 ag\beta^2$$

are the values of the refraction indices in the directions of maximal radiation determined from the equations

$$\cos\vartheta_{1,2}=1/\beta^2 n_{1,2}^2(\vartheta_{1,2}).$$

The right hand side of (14) is different from zero only in two cases: (a) when the argument of the logarithm is negative, and (b) if the expression under the integral sign has poles, i.e., for frequencies such that ϵ_1, ϵ_2, and ϵ_3 are simultaneously zero. Thus we obtain

$$-\frac{d\mathcal{E}}{dz}=-\frac{q^2}{c^2}\int\{(ag-\varepsilon_1\beta^2 g)n_1^2+(a^2-ag-b^2)\beta^{-2}$$
$$+\varepsilon_1 g-2a\varepsilon_1+2b\varepsilon_2$$
$$+(\varepsilon_1^2-\varepsilon_2^2)\beta^2\}\{\varepsilon_1 ag(n_1^2-n_2^2)\beta^2\}^{-1}\omega d\omega-\frac{q^2}{c^2}\int\{(ag-\varepsilon_1\beta^2 g)n_2^2$$
$$+(a^2-ag-b^2)\beta^{-2}+\varepsilon_1 g-2a\varepsilon_1+2b\varepsilon_2 \tag{16}$$
$$+(\varepsilon_1^2-\varepsilon_2^2)\beta^2\}\{\varepsilon_1 ag(n_2^2-n_1^2)\beta^2\}^{-1}\omega d\omega$$
$$+\frac{q^2}{v^2}\sum_i\frac{\omega_i}{|d\varepsilon_1/d\omega|_i}\left(\frac{ag\nu_1^2+\beta^{-2}(a^2-ag-b^2)}{ag(\nu_1^2-\nu_2^2)}\right)_i\ln\frac{n_m^2\beta^2-\nu_1^2\beta^2}{1-\nu_1^2\beta^2}$$
$$+\frac{q^2}{v^2}\sum_i\frac{\omega_i}{|d\varepsilon_1/d\omega|_i}\left(\frac{ag\nu_2^2+\beta^{-2}(a^2-ag-b^2)}{ag(\nu_2^2-\nu_1^2)}\right)_i\ln\frac{n_m^2\beta^2-\nu_2^2\beta^2}{1-\nu_2^2\beta^2},$$

where

$$\nu_{1,2}^2=[\{(ag-a^2)+ag(1-\varepsilon_3/\varepsilon_1)+b^2$$
$$\pm[a^2(g\varepsilon_3/\varepsilon_1-1)^2+b^2(b^2-2a^2-2ag\varepsilon_3/\varepsilon_1)]^{1/2}\}/2ag\beta^2]_{\omega=\omega_i}.$$

The integration in the first two terms of (16) goes over the frequency regions determined by the inequalities

$$n_m^2\beta^2 > n_1^2\beta^2 > 1\,, \quad n_m^2\beta^2 > n_2^2\beta^2 > 1 \qquad (17)$$

respectively. We note that (16) goes over into the corresponding expression of the paper of Sitenko and Kolomenskii[5] for $a = g = 1$ and $b = 0$. In this case the third term, which comes from the magnetic anisotropy, vanishes.

In the case of an anisotropic magneto-active ferrite we obtain ($\epsilon_2 = 0$, $\epsilon_1 = \epsilon_3 = 1$):

$$d\mathscr{E}/dz = (q/c)^2 \int (1/\beta^2 - 1/a)\,\omega d\omega\,. \qquad (18)$$

If $\mu_2 = 0$ (anisotropic ferrite), formula (18) goes over into the corresponding formula of the paper of Pafomov.[6]

5. In order to determine the character of the losses (16) we have to compute the energy flux through the cylindrical surface surrounding the trajectory of the charge. For this purpose we must determine the fields **E** and **H** arising in the medium during the motion of the charge, after which we make use of the Poynting theorem in the usual manner.

We omit these rather cumbersome calculations and remark only that, under the assumption that ϵ_1, ϵ_2, and ϵ_3 do not have any common roots, the losses on account of the Cerenkov radiation are represented by the first two terms in formula (16).

6. We now consider the second case: a charge moving perpendicular to the optical axis of the crystal. We choose the directions of the axes as in reference 5; the z axis is directed along the optical axis of the crystal. The particle moves along the y axis. We then obtain

$$-\frac{d\mathscr{E}}{dy} = i\frac{q^2}{2\pi^2c^2}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_{1/\beta}^{n_m}\frac{En^2 - F}{An^4+Bn^2+C}\,ndnd\varphi\omega d\omega\,, \qquad (19)$$

where

$$A = \beta^2\{(ag - a^2 + b^2)(\varepsilon_1 - \varepsilon_3)\sin^4\varphi$$
$$+ [(a^2 - ag - b^2)\varepsilon_1 + ag(\varepsilon_3 - \varepsilon_1)]\sin^2\varphi + ag\varepsilon_1\};$$

$$B = [(a\varepsilon_1^2 + g\varepsilon_1\varepsilon_3 - a\varepsilon_2^2) + 2b\varepsilon_2\varepsilon_3 - 2a\varepsilon_1\varepsilon_3]\beta^2\sin^2\varphi$$
$$- [(a^2 - ag + b^2)\varepsilon_1$$
$$+ ag(\varepsilon_3 - \varepsilon_1)]\sin^2\varphi - 2(ag - a^2 + b^2)(\varepsilon_1 - \varepsilon_3)\sin^4\varphi;$$

$$C = \beta^2(\varepsilon_1^2\varepsilon_3 - \varepsilon_2^2\varepsilon_3)$$
$$- [(a\varepsilon_1^2 + g\varepsilon_1\varepsilon_3 - a\varepsilon_2^2) + 2b\varepsilon_2\varepsilon_3 - 2a\varepsilon_1\varepsilon_3]\sin^2\varphi$$
$$+ \beta^{-2}(ag - a^2 + b^2)(\varepsilon_1 - \varepsilon_3)\sin^4\varphi;$$

$$E = (ag - a\varepsilon_1\beta^2)\cos^2\varphi + (a^2 - b^2)\sin^2\varphi - \beta^2\varepsilon_3\sin^2\varphi;$$

$$F = (ag - a^2 + b^2)\sin^2\varphi$$
$$- \beta^2(\varepsilon_1 + \varepsilon_3)a\sin^2\varphi + \beta^2\varepsilon_1\varepsilon_3 - \beta^2\varepsilon_3 g;$$

φ is the angle between the x axis and the projection of **k** on the zx plane.

From (19) we obtain for the energy losses the expression

$$-\frac{d\mathscr{E}}{dy} = \frac{q^2}{2\pi^2c^2}\,\mathrm{Re}\,i\int_0^{2\pi}\int_0^{\infty}\frac{En_1^2 - F}{n_1^2 - n_2^2}\ln\frac{n_m^2\beta^2 - n_1^2\beta^2}{1 - n_1^2\beta^2}\,d\varphi\omega d\omega$$
$$+ \frac{q^2}{2\pi^2c^2}\,\mathrm{Re}\,i\int_0^{2\pi}\int_0^{\infty}\frac{En_2^2 - F}{n_2^2 - n_1^2}\ln\frac{n_m^2\beta^2 - n_2^2\beta^2}{1 - n_2^2\beta^2}\,d\varphi\omega d\omega\,, \qquad (20)$$

where

$$n_{1,2}^2 = (-B \pm \sqrt{B^2 - AC})/2A.$$

The conic surfaces corresponding to the ordinary and extraordinary waves are complicated (dependence on φ). The field intensities on the different generatrices of these surfaces are not the same. The integration in (20) can in principle be carried out to the very end if ϵ_{ik} and ϵ_{ik} are given as functions of the frequency.

In the special case $a = g = 1$, $b = 0$ formula (20) goes over into formula (26) of reference 5.

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