STATIONARY RELATIVISTIC MOTIONS OF A GAS IN A CONDUCTING MEDIUM

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Quasi-unidimensional relativistic stationary flow of a gas is considered in a medium with infinite conductivity and with a magnetic field perpendicular to the velocity. Particular attention is devoted to cylindrically symmetric flow. We calculate the total momentum which an expanding gas can acquire per unit time (equal to the "reactive" force), both with a field and without it. This calculation is performed, in particular, for escape of an ultrarelativistic gas.

CONSIDER the quasi-unidimensional stationary flow of a gas in a magnetic field, assuming the conductivity to be infinite.

By quasi-unidimensional flow we mean a flow with a smoothly varying cross section containing given flow lines. In particular, we shall treat the cylindrically symmetric case.

The basic equations, assuming adiabatic flow, are[1]

$$w^* / \theta = w_0^* = \text{const}; \tag{1}$$

$$\Delta sa / \theta V = \Delta \dot{M} = \text{const}, \tag{2}$$

$$\theta = (1 - a^2 / c^2)^{1/2}; \quad w^* = pV + \rho V c^2 + \mu H^2 V / 4\pi,$$

where a is the velocity, w_0^* is the rest heat content, p is the pressure, V is the specific volume, ρ is the density of the medium (including the microscopic energy), H is the magnetic field strength (we consider H to be perpendicular to a) which, for infinite conductivity, is related to the specific volume by the expression

$$HV = b = \text{const}; \tag{3}$$

and $\Delta\dot{M}$ is the amount of mass crossing the area Δs per unit time. (The area Δs is a function of r.)

For cylindrical symmetry, we have

$$s = 2\pi r; \quad \dot{M} = 2\pi r a / \theta V; \quad HV = br. \tag{4}$$

If there is no field, we can assume point symmetry, and then

$$\rho = 4\pi r^2; \quad \dot{M} = 4\pi r^2 a / \theta V.$$

In general,

$$\Delta\dot{M} / \Delta s = f(r). \tag{5}$$

The energy equation for adiabatic flow gives σ = const (where σ is the entropy), or isoentropic flow.

Let us now write Eqs. (1) and (2) in the form

$$(pV + \rho V c^2 + b_0 / V) / \theta = p_0 V_0 + \rho_0 V_0 c^2 + b_0 / V_0 = w_0; \quad \Delta\dot{M} = a \Delta s / \theta V, \tag{6}$$

where $b_0 = \mu b^2 / 4\pi$.

If the equation of state of a gas

$$pV = RT, \tag{7}$$

(where T is the temperature) and the constant-entropy equation

$$pV^k = A = \text{const}, \tag{8}$$

are satisfied, then

$$w = pV + \rho V c^2 = \alpha c^2 + kpV / (k-1), \tag{9}$$

where

$$\alpha = \rho_a V_a - p_a V_a / (k-1) c^2; \quad w^* = w + b_0 / V.$$

For an ordinary gas $\alpha = 1$, while for an ultrarelativistic gas $\alpha = 0$, and

$$pV = RT = (k-1) \rho V c^2. \tag{10}$$

The first equation of (6) then becomes

$$\alpha c^2 + \frac{k}{k-1} A V^{1-k} + \frac{b_0}{V} = w_0 \theta, \tag{11}$$

thus relating V and a. Further, from the second equation of (6) we have

$$(a / V) \Delta s = \theta \Delta\dot{M}. \tag{12}$$

Relations (11) and (12) lead to equations which can be used to find a and V as functions of r. These are

$$\alpha c^2 + \frac{kA}{k-1} \left(\frac{\theta \Delta\dot{M}}{a \Delta s} \right)^{k-1} + b_0 \frac{\theta \Delta\dot{M}}{a \Delta s} = \theta w_0; \tag{13}$$

$$\left(\alpha c^2 + \frac{kA}{k-1} V^{1-k} + \frac{b_0}{V} \right)^2 \left[1 + \frac{V^2 (\Delta\dot{M})^2}{c^2 (\Delta s)^2} \right] = w_0^2. \tag{14}$$

In the cylindrically symmetric case $\Delta\dot{M}/\Delta s = \dot{M}/2\pi r$. Using this, it is a simple matter to find $a(r)$ and $V(r)$ both for an ordinary gas with high energy and for an ultrarelativistic gas. In the classical limit the cylindrically symmetric flow becomes the same as that given by ordinary gas dynamics.

Since $\Delta s a/V\theta = \Delta\dot{M} = \text{const}$, we have

$$-\frac{d\Delta s}{\Delta s} = -\frac{dV}{V} + \frac{da}{a(1-a^2/c^2)}. \qquad (15)$$

Since

$$dw^*/w^* = -(a^2/c^2)\,da/\theta^2 a, \qquad (16)$$

Eq. (15) becomes

$$\frac{d\Delta s}{\Delta s} = -\frac{da}{a\theta^2} + \frac{dV}{V} = \frac{dw^*}{w^*}\frac{c^2}{a^2} + \frac{dV}{V}. \qquad (17)$$

We also have

$$-(c^2/\omega^{*2})\,dw^*/w^* = dV/V, \qquad (18)$$

where

$$\omega^{*2}/c^2 = dp^*/c^2 d\rho^*, \qquad (19)$$

and ω^* is the magnetohydrodynamic velocity of sound. Comparing (16), (17), and (18), we arrive at

$$-\frac{d\Delta s}{\Delta s} = \left(1 - \frac{a^2}{\omega^{*2}}\right)\frac{da}{a(1-a^2/c^2)}. \qquad (20)$$

At the minimum (critical) cross section, when $d\Delta s = 0$, we have $\omega^* = \pm a$, indicating critical flow in which the velocity of the medium is equal to the velocity of sound.

In the more general case, as in classical gas flow, one can have motion with $a \geq \omega^*$ if $d\Delta s \geq 0$, or motion with $a \leq \omega^*$ if $d\Delta s \leq 0$.

Let us now calculate the momentum that the flow can attain in escaping into a rarefied volume (this is the case with $a \geq \omega^*$ and $d\Delta s \geq 0$).

The time rate of change of momentum (or the reactive force acting on an area Δs) is given by

$$\Delta F = \Delta\dot{J} = \Delta s\left[\frac{a^2}{c^2\theta^2}\left(p + \rho c^2 + \frac{\mu H^2}{4\pi}\right) + p + \frac{\mu H^2}{4\pi}\right]. \qquad (21)$$

(In the cylindrical case the total momentum in all directions over a circle vanishes.) Let us write (21) in a different form. Since

$$(p + \rho c^2 + \mu H^2/4\pi)/\theta = w^*/\theta V = \dot{w}_0/V; \quad a\Delta s/\theta V = \Delta\dot{M},$$

we have

$$\Delta s\,(p + \rho c^2 + \mu H^2/4\pi)(a/c^2\theta^2) = \dot{w}_0 a\Delta\dot{M}/c^2;$$

further

$$(p + \mu H^2/8\pi)\,\Delta s = (\Delta\dot{M}\theta/a)(pV + b_0/2V),$$

so that

$$\Delta F = a\Delta\dot{M}\left[\frac{\dot{w}_0}{c^2} + \frac{\theta}{a^2}\left(pV + \frac{b_0}{2V}\right)\right], \qquad (22)$$

or

$$\Delta F = \Delta\dot{M}a\left[\alpha + \frac{kAV_0^{1-k}}{(k-1)c^2} + \frac{b_0}{c^2V_0} + \frac{\theta}{a^2}\left(AV^{1-k} + \frac{b_0}{2V}\right)\right]. \qquad (23)$$

For the ordinary case we have $\alpha = 1$, and since $\rho_0 V_0 = 1$, we may write

$$\Delta F = \Delta\dot{M}a\left[1 + \frac{kAV_0^{1-k}}{(k-1)c^2} + \frac{b_0V_0^{-1}}{c^2} + \frac{\theta}{a^2}\left(AV^{1-k}(a) + \frac{b_0}{2V(a)}\right)\right]; \qquad (24)$$

as $p \to 0$, we have $V \to \infty$, $pV = 0$, and

$$\Delta F_\infty = \Delta\dot{M}a_\infty\,[1 + kAV_0^{1-k}/(k-1)c^2 + b_0V_0^{-1}/c^2], \qquad (25)$$

where

$$a_\infty = c\{1 - [1 + kAV_0^{1-k}/c^2(k-1) + b_0V_0^{-1}/c^2]^{-2}\}^{1/2}; \qquad (26)$$

Therefore

$$\Delta F_\infty = \Delta\dot{M}c\,[(1 + kAV_0^{1-k}/(k-1)c^2 + b_0V_0^{-1}/c^2)^2 - 1]^{1/2}. \qquad (27)$$

In the classical limit we have

$$\Delta F = \Delta\dot{M}a\,[1 + a^{-2}(AV^{1-k} + b_0/2V)]; \qquad (28)$$

$$a = \left[\frac{2kA}{k-1}(\rho_0^{k-1} - \rho^{k-1}) + 2b_0(\rho_0 - \rho)\right]^{1/2}. \qquad (29)$$

As $V \to \infty$,

$$\Delta F_\infty = \Delta\dot{M}a_\infty = \Delta\dot{M}\,[2kp_0/(k-1)\rho_0 + 2b_0\rho_0]^{1/2}. \qquad (30)$$

In the ultrarelativistic case with $\alpha = 0$ we have $p_0 = (k-1)\rho_0 c^2$ and

$$\Delta F = \Delta\dot{M}a\,[k\rho_0V_0 + b_0/c^2V_0 + \theta a^{-2}(AV^{1-k} + b_0/2V)]. \qquad (31)$$

From Eq. (11) we find that for an ultrarelativistic gas V and a are related by

$$\frac{k}{k-1}AV^{1-k} + \frac{b_0}{V} = \theta\dot{w}_0 = \theta\left[kc^2\rho_0V_0 + \frac{b_0}{V_0}\right]. \qquad (32)$$

this gives

$$AV^{1-k} = \frac{k-1}{k}\left[\theta\left(kc^2\rho_0V_0 + \frac{b_0}{V_0}\right) - \frac{b_0}{V}\right],$$

so that

$$\Delta F = \Delta\dot{M}a\left[\left(1 + (k-1)\frac{c^2}{a^2}\right)\left(\rho_0V_0 + \frac{b_0}{kc^2V_0}\right) + \frac{2-k}{2k}\frac{b_0\theta}{a^2V(a)}\right]. \qquad (33)$$

Equation (33) is more convenient to write in the form

$$\Delta F = \Delta\dot{M}c\left[\left(\frac{a}{c} + (k-1)\frac{c}{a}\right)\left(\rho_0V_0 + \frac{b_0}{kc^2V_0}\right) + \frac{2-k}{2k}\frac{b_0\theta}{caV(a)}\right]. \qquad (34)$$

As $V \to \infty$ and $a \to c$,

$$\Delta F_\infty = \Delta \dot{M} c\,(k\rho_0 V_0 + b_0 / c^2 V_0)$$
$$= \Delta \dot{M} c\,(k\rho_0 V_0 + \mu H_0^2 V_0 / 4\pi c^2). \qquad (35)$$

If there is no field $(b_0 = 0)$,

$$\Delta F = \Delta \dot{M} c\,[a / c + (k - 1)\, c / a]\, \rho_0 V_0. \qquad (36)$$

In the limit, $\Delta F_\infty = k\Delta \dot{M} c\rho_0 V_0$. If there is no gas $(A = 0$ and $p = 0)$ but there exists only a field, we write (22) in the form

$$\Delta F = \Delta \dot{M} c \left(\frac{\mu H_0^2}{4\pi\rho_0 c^2} \frac{a}{c} + \frac{\mu b^2 \theta}{8\pi c^2 V} \frac{c}{a} \right). \qquad (37)$$

Since $(1 - a^2/c^2)^{1/2} = w^*/w_0^* = H^2 V / H_0^2 V_0 = H/H_0$, we find that $a/c = (1 - H^2/H_0^2)^{1/2}$, so that (37) becomes

$$\Delta F = \frac{\mu H_0^2 \Delta \dot{M}}{4\pi c^2 \rho_0} \frac{1 - H^2 / 2H_0^2}{(1 - H^2 / H_0^2)^{1/2}}. \qquad (38)$$

As $p \to 0$, however, with $a = c$, these relations have meaning only if $V \to \infty$ and $H = 0$; then $F_\infty = \mu H_0^2 \Delta \dot{M} / 4\pi c_0^2 = \Delta \dot{E}/c$, where $\Delta \dot{E} = (\mu H_0^2 / 4\pi\rho_0)\, \Delta \dot{M}$ is the mean energy flux.

The expressions given in the present paper can be used in studying the interaction of bodies emitting either streams of gas or fields.

[1] K. P. Staniukovich, Dokl. Akad. Nauk SSSR **119**, 251 (1958), Soviet Phys. "Doklady" **3**, 299 (1958).

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