THE STABILITY OF SHOCK WAVES IN MAGNETOHYDRODYNAMICS

A. I. AKHIEZER, G. Ia. LIUBARSKII, and R. V. POLOVIN

Physico-Technical Institute, Academy of Sciences, Ukrainian S.S.R.

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The regions of instability of a stationary magnetohydrodynamic shock wave with respect to one-dimensional perturbations have been determined. It is shown that two types of stable shock waves exist. The types of shock waves that can follow each other are determined.

1. The purpose of the present paper is the investigation of the problem of the stability of plane magnetohydrodynamic shock waves with respect to small perturbations which depend only on the distance of the surface of discontinuity and on the time.

We shall show that magnetohydrodynamic shock waves can be unstable and must disintegrate into several shock waves if the total number of magnetohydrodynamic, magnetoacoustic, and entropy waves emanating from the front of the discontinuity is not equal to six.

2. The method that we shall employ consists of the following: we shall write down a set of equations of magnetohydrodynamics describing the perturbations in question in the form

$$\sum_{k=1}^{n}\left\{X_{ik}(u)\frac{\partial u_k}{\partial x}+T_{ik}(u)\frac{\partial u_k}{\partial t}\right\}=0;\quad i=1,2,\ldots,n, \tag{1}$$

Where u_k represents any hydrodynamic quantity (the velocity v, the magnetic field H, the density ρ, the entropy s), $X_{ik}(u)$ and $T_{ik}(u)$ are certain functions of $u_1, u_2, \ldots, u_n$; x is the distance to the surface of discontinuity and t is the time (the direction of the x axis is chosen so that the projection of the velocities on it will be positive). These equations possess discontinuous solutions that satisfy the n given conditions on the surface of discontinuity. We denote the constant values of u_k on the two sides of the discontinuity by u_{1k} $(x<0)$ and U_{2k} $(x>0)$ (index 1 corresponds to the region in front of the shock wave, while index 2 corresponds to the region behind the shock). At the incident of time $t=0$, let the quantities u_k undergo small perturbations $\delta u_k(x)$, which depend only on x; it is required to determine the functions $\delta u_k(x,t)=u_k(x,t)-u_k$ at all subsequent times. If the discontinuous solution is stable, then these quantities will remain small, and we can therefore linearize Eq. (1):

$$\begin{aligned}&\sum_{k=1}^{n}\left\{X_{ik}^{(1)}\frac{\partial\delta u_{1k}(x,t)}{\partial x}+T_{ik}^{(1)}\frac{\partial\delta u_{1k}(x,t)}{\partial t}\right\}=0,\\&\sum_{k=1}^{n}\left\{X_{ik}^{(2)}\frac{\partial\delta u_{2k}(x,t)}{\partial x}+T_{ik}^{(2)}\frac{\partial\delta u_{2k}(x,t)}{\partial t}\right\}=0,\end{aligned} \tag{2}$$

where the indices (1) and (2) indicate the values of the function for $u_k=u_{1k}$ and $u_k=u_{2k}$. The resultant system of linearized equations can, however, fail to have a unique solution in a number of cases. On the other hand, since the Cauchy problem in hydrodynamics always has a unique solution, then the absence of a unique solution for the system (2) indicates the impossibility of replacing the exact equations (1) by the linearized equations (2). In turn, this means that the perturbations $\delta u_k(x,t)$, which are small at $t=0$, are not small at any subsequent instant of time. Such a discontinuous change with the time of the quantities $\delta u_k(x,t)$ can take place only in the breaking up of the original shock wave into several waves, which are enclosed between the surfaces of discontinuity that are once again formed.

Therefore, the necessary condition for the stability of the shock wave relative to decomposition is the existence of a unique solution of the linearized Cauchy problem corresponding to the system (2). We shall make use of this criterion in what follows. The problem of the character of the breaking up of the shock wave is not considered in the present work.

3. Let us go on to the solution of Eq. (2). For this purpose we carry out a Laplace transformation in time:

$$\eta_k(x,z)=\int_0^\infty\delta u_k(x,t)\,e^{-zt}\,dt,\qquad \mathrm{Re}\,z>0,$$

which yields

$$\sum_{k=1}^{n}\left\{X_{ik}\frac{d\eta_k}{dx}+zT_{ik}\eta_k\right\}=\sum_{k=1}^{n}T_{ik}\delta u_k(x). \tag{3}$$

For a solution of these equations, we first consider the homogeneous set

$$\sum_{k=1}^{n}\left\{X_{ik}\frac{d\zeta_k}{dx}+zT_{ik}\zeta_k\right\}=0,$$

whose solution has the form

$$\zeta_k = A_{k\mu}e^{\lambda_\mu x}, \qquad (4)$$

where λ_μ are the roots of the characteristic equation

$$|X_{ik}+T_{ik}z/\lambda_\mu|=0. \qquad (5)$$

On the other hand, the phase velocities V_μ of the plane waves

$$\delta u_k(x,\ t)=C_k e^{i(k_\mu x-\omega t)}, \qquad (6)$$

satisfying Eq. (2) are obviously determined from the equation

$$|X_{ik}-V_\mu T_{ik}|=0.$$

Therefore,

$$\lambda_\mu = -z/V_\mu. \qquad (5')$$

This relation shows that for $z > 0$ each plane wave of (6) moving in the positive direction corresponds to a negative λ_μ, while the wave moving in the negative direction corresponds to positive λ_μ.

We now turn to the solution of the inhomogeneous system (3). As is well known, this can be represented in the form

$$\eta_k=\sum_{\mu=1}^{n}A_{k\mu}e^{\lambda_\mu x}g_\mu(x),$$

where

$$g_\mu(x)=\sum_{\nu\gamma\sigma}(A^{-1})_{\mu\nu}(X^{-1})_{\nu\gamma}T_{\gamma\sigma}\int_0^x e^{-\lambda_\mu\xi}\delta u_\sigma(\xi)\,d\xi+g_\mu(0); \qquad (7)$$

A^{-1}, X^{-1} are matrices inverse to the matrices $A = (A_{k\mu})$ and $X = (X_{ik})$. In order that the quantities $\eta_{2k}(x, z)$ be bounded for $x \to \infty$, the functions $g_{2\mu}(x)$ corresponding to positive $\lambda_{2\mu}$ must vanish for $x \to \infty$;

$$g_{2\mu}(+\infty)=0;\ \lambda_{2\mu}>0.$$

These conditions uniquely determine the constants of integration $g_{2\mu}(0)$ for $\lambda_{2\mu} > 0$; so far as the constants $g_{2\mu}(0)$ that correspond to $\lambda_{2\mu} < 0$ are concerned, the conditions at $+\infty$ do not impose any limitations on them. The number of these conditions is evidently equal to the number of types of plane waves (6) propagating in the positive x direction.

Thus, after consideration of the conditions at $+\infty$ in the solution (7), there remain as many arbitrary constants as there are forms of plane waves propagating to the right from the shock front.

Proceeding in exactly the same manner, it is easy to show that the number of arbitrary constants from among the $g_{1\mu}(0)$ is equal to the number of types of plane waves propagating to the left from the shock front.

Thus the total number m of arbitrary constants is equal to the number of solutions of Eq. (5) with $\lambda_{2\mu} < 0$ and $\lambda_{1\mu} > 0$, i.e., the number of waves propagating from the surface of discontinuity in the two directions.

On the other hand, on the surface of the discontinuity there are satisfied n conditions which relate the quantities u_{1k} and u_{2k}. The velocity of the wave front enters into these conditions. If it is eliminated, we obtain $(n-1)$ conditions connecting u_{1k} and u_{2k} at $x = 0$. We can now verify the fact that if the number of arbitrary constants is not equal to the number of these conditions, then the Cauchy problem either has no solution $(m > n-1)$ or it has an infinite number of solutions $(m < n-1)$. In this and in the other case, we must make some conclusion on the instability of the shock wave which is associated with its disintegration.

It is essential to note that neither boundary conditions nor considerations of thermodynamic stability (Zemplen theorem) exclude these conditions, generally speaking (see Sec. 5).

For stability of the shock wave it is necessary that the number of waves emanating from the discontinuity be equal to the number of boundary conditions on the discontinuity. If this condition is satisfied, it is still necessary to verify whether it is possible to satisfy the boundary conditions with the help of the remaining arbitrary constants.*

4. We now turn to the investigation of the stability of magnetohydrodynamic shock waves. As is well known,[2] there exist seven types of one-dimensional plane waves in magnetohydrodynamics: (1) Magnetohydrodynamic waves whose phase velocities are $v_x - V_x$, $v_x + V_x$) where $V_x = H_x/\sqrt{4\pi\rho}$. (2) Magnetoacoustic waves whose phase velocities are $v_x - u_-$, $v_x + u_-$, $v_x - u_+$, $v_x + u_+$, where

$$u_\pm^2=\frac{1}{2}\left[V^2+c^2\pm\sqrt{(V^2+c^2)^2-4c^2V_x^2}\right],\ \mathbf{V}=\mathbf{H}/\sqrt{4\pi\rho}$$

(c is the velocity of sound). (3) An entropy wave whose phase velocity coincides with the velocity of

*This criterion is well-known for ordinary hydrodynamics (see reference 1, p. 405). However, we note that the considerations introduced in reference 1 cannot be applied in the case $m < n - 1$.

the liquid v_x. We note that

$$u_- \leqslant V_x \leqslant u_+. \tag{8}$$

For stability of the shock wave, it is necessary (as was shown above) that the number of emergent waves should be one less than the number of boundary conditions, that is, it must be equal to six.

Waves having phase velocity $v_{1x} + V_{1x}$, $v_{1x} + u_{1+}$, $v_{1x} + u_{1-}$, and v_{1x}, are convergent, while the waves $v_{2x} + V_{2x}$, $v_{2x} + u_{2+}$, $v_{2x} + u_{2-}$, and v_{2x} are divergent. Therefore, in the case of stability of the remaining six waves with velocities $v_{1x} - V_{1x}$, $v_{1x} - u_{1+}$, $v_{1x} - u_{1-}$, $v_{2x} - V_{2x}$, $v_{2x} - u_{2+}$, and $v_{2x} - u_{2-}$, two must be divergent. Taking the inequalities (8) into account, it follows that the shock wave can be stable only under the following three conditions:*

$$\begin{aligned} &\text{A) } u_{-1} < v_{1x} < V_{1x}, \quad v_{2x} < u_{2-};\\ &\text{B) } V_{1x} < v_{1x} < u_{1+}, \quad u_{2-} < v_{2x} < V_{2x};\\ &\text{C) } u_{1+} < v_{1x}, \qquad V_{2x} < v_{2x} < u_{2+} \end{aligned} \tag{9}$$

(see Fig. 1, in which the possible regions of stability are shaded).

We consider the particular case in which the magnetic field is parallel to the wave front: in this case $u_{1-} = u_{2-} = V_{1x} = V_{2x} = 0$; $u_{1+} = \sqrt{c_1^2 + V_1^2}$; $u_{2+} = \sqrt{c_2^2 + V_2^2}$, and therefore, in accord with Eq. (9), the shock wave can be stable on satisfaction of the inequalities

$$v_{1x} > \sqrt{c_1^2 + V_1^2}; \quad v_{2x} < \sqrt{c_2^2 + V_2^2} \tag{10}$$

(see Fig. 2).

5. If the magnetic field is perpendicular to the shock front, then satisfaction of the conditions (9) is insufficient for stability of the wave with respect to the perturbations under consideration, since in this case the boundary conditions break up into three isolated groups. These conditions have the following form:

$$[\rho\delta v_x + v_x\delta\rho - \rho U] = 0, \qquad [2\rho v_x \delta v_x + v_x^2\delta\rho + \delta p] = 0,$$
$$[v_x\delta v_x + \delta w - v_x U] = 0; \tag{11}$$

$$\left[\rho v_x \delta v_y - \frac{H_x}{4\pi}\delta H_y\right] = 0, \quad [v_x\delta H_y - H_x\delta v_y] = 0; \tag{12}$$

$$\left[\rho v_x \delta v_z - \frac{H_x}{4\pi}\delta H_z\right] = 0, \quad [v_x\delta H_z - H_x\delta v_z] = 0 \tag{13}$$

(The set of coordinates is so chosen that $v_{1y} = v_{2y} = v_{1z} = v_{2z} = 0$, U is the velocity of the shock wave, w is the heat function.)

Of the four magnetoacoustic waves at $H_{1y} = H_{2y} = H_{1z} = H_{2z} = 0$, two are longitudinal and two are transverse. The longitudinal waves are ordinary sound waves whose velocities are equal to $v_x \pm c$. The transverse waves are ordinary magnetohydrodynamic waves considered earlier only in the direction of polarization.

In the consideration of the sound waves, it is necessary to take into account only the boundary conditions (11) since conditions (12) and (13) are satisfied identically for them. In the same fashion, in the consideration of the magnetohydrodynamic waves polarized along the y axis, we must take into account only the boundary conditions (12), while for waves polarized along the z axis we need only conditions (13).

Instead of investigating the stability relative to all three types of waves simultaneously, we must consider the stabilities for each type of wave individually. Let us begin with the magnetohydrodynamic waves. The number of boundary conditions for them is equal to 2 [see (12) or (13)]. Since these conditions do not contain the velocity of the wave front, then the presence of two diverging waves is necessary for stability. The wave with phase velocity $v_{2x} + V_{2x}$ diverges, while the wave

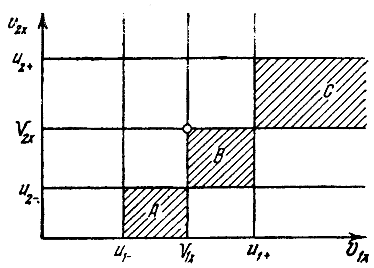

FIG. 1

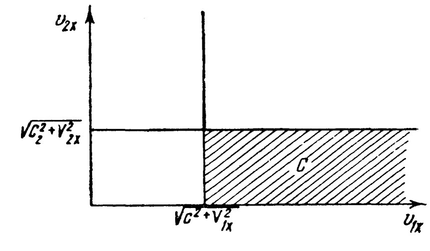

FIG. 2

*Note added in proof (August 6, 1958). Actually it must be considered that (as was pointed out to us by S. I. Syrovatskii) in the first approximation, the aggregate of boundary conditions in the case of shock waves (but not for rotational discontinuities) breaks up into two groups which contain quantities characterizing the magnetohydrodynamic waves (v_z, H_z) and magnetoacoustic waves (ρ, p, v_x, v_y, H_y; H lies in the xy plane). Therefore, computation of the number of diverging waves ought to be carried out not only for the entire aggregate of variables but also for each group separately. From this, we easily conclude that B is not a region of stability and is contracted to a point (small circle in Fig. 1; compare Secs. 5 and 6).

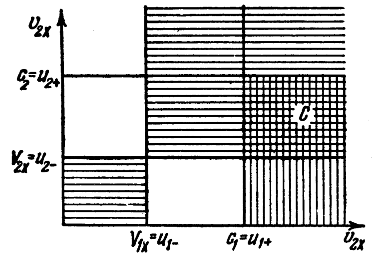

FIG. 3

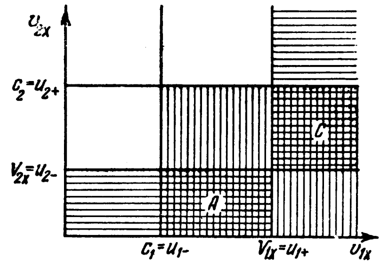

FIG. 4

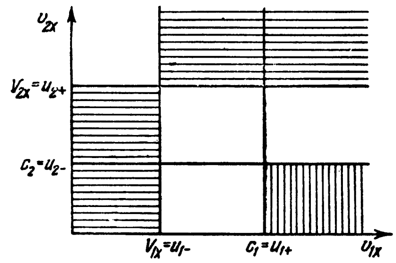

FIG. 5

with phase velocity $v_{1x} + V_{1x}$ converges. Of the remaining waves, with velocity $v_{1x} - V_{1x}$ and $v_{2x} + V_{2x}$, one must be diverging. This is possible in two cases: (1) $v_{1x} > V_{1x}$, $v_{2x} > V_{2x}$; (2) $v_{1x} < V_{1x}$, $v_{2x} < V_{2x}$ (see the horizontally shaded regions in Figs. 3 to 6).

Consideration of the sound waves leads to the following stability conditions:

$$v_{1x} > c_1, \quad v_{2x} < c_2$$

(see the vertically shaded regions in Figs. 3 to 6.)

The region of stability of shock wave relative to arbitrary single one-dimensional perturbations is the intersection of the regions of stability of the two separate waves. It is shown in Figs. 3, 4, and 6 by the doubly-shaded regions. In the case shown in Fig. 5, there is no region of stability. However, this cannot take place if

$$\left(\frac{\partial^2}{\partial p^2}\frac{1}{\rho}\right)_s > 0.$$

We note that under the conditions in Figs. 3 to 6 the region for stability can be formally obtained from (9).

We emphasize that the inclusion of a magnetic field perpendicular to the wave front always leads to a decrease in the region of stability. This is explained by the fact that in such a field there are two independent transverse waves, which produce additional instability.

We note that in the case under consideration

$$V_{1x} > c_1, \; V_{2x} > c_2 \tag{14}$$

(see Fig. 6) the Cauchy problem corresponding to the system (2) has no solutions if

$$v_{1x} > V_{1x}, \; v_{2x} < c_2 \tag{15}$$

(see the region of vertical shading in Fig. 6.)

One might think that the inequalities (14) and (15) are incompatible with Zemplen's theorem, by virtue of which

$$v_{2x} < v_{1x}; \; \rho_2 > \rho_1, \tag{16}$$

and with the boundary conditions. However, it is seen in the example of a monatomic ideal gas that

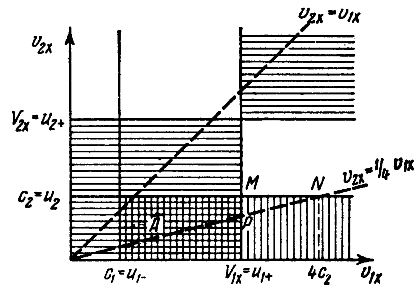

FIG. 6

this is not so. Actually the boundary conditions in this case can be written in the form

$$p_1 = (4v_{2x} - v_{1x})\rho_1 v_{1x}/5, \quad p_2 = (4v_{1x} - v_{2x})\rho_1 v_{1x}/5,$$
$$\rho_2 = \rho_1 v_{1x}/v_{2x},$$

from which it follows that

$$v_{2x} > {}^1\!/_4\, v_{1x}; \; \rho_2 < 4\rho_1. \tag{17}$$

The inequalities (14) to (17) determine the triangle MNP (see Fig. 6). From this it is seen that for their simultaneous fulfilment it is necessary to have $V_{1x} < 4c_2$ and $V_{2x} > c_2$. These inequalities are satisfied if the magnetic field H_x lies within the limits

$$c_2\sqrt{4\pi\rho_2} < H_x < 4c_2\sqrt{4\pi\rho_2}, \tag{18}$$

which is always possible by virtue of (17).

6. Let us examine in particular Alfven rotational shock waves[2] which are the kind of shock waves most frequently considered. In these waves the velocities are characterized by the relations

$$v_{1x} = V_{1x} = v_{2x} = V_{2x}.$$

In Fig. 1, there corresponds to the Alfven wave a point surrounded by a circle. This point lies on the boundary of the region of stability. Therefore the problem of stability of the Alfven discontinuities requires additional consideration.

First we note that in our case two phase velocities vanish, namely, $v_{1x} - V_{1x}$ and $v_{2x} - V_{2x}$. With this, in accord with Eq. (5′), one of the roots λ of the characteristic equation becomes infinite. This means that the determinant $|X_{ik}|$ is equal to zero, i.e., the set (3) reduces to an $(n-1)$-fold differential equation and one algebraic relation among the quantities $u_1, u_2, \ldots, u_n$. Therefore the general solution of (3) contains $(n-1)$ constants of integration. Thus the presence of a "wave" with phase velocity equal to zero leads to the loss of a single arbitrary constant. Consequently, in the calculation of arbitrary constants, one must add such waves to the convergent ones.

Applying this conclusion to Alfven discontinuities, it is easy to see that they are stable relative to the perturbations being considered.

7. The necessary conditions that we have obtained for stability permit us to carry out a classification of shock waves. In the general case, when the magnetic field is inclined to the plane of the discontinuity, only three types of stable shock waves can exist, corresponding to the regions A, C of Fig. 1 and to a rotational discontinuity. If the jumps of all the quantities tend to zero on the surface of discontinuity, then the velocities of the shock waves approach the phase velocities of the linearized waves; in this case, the velocities of waves of type A tend to the phase velocity $v_x \pm u_-$ of the slow magnetoacoustic wave, the velocities of type C tend to the velocities $v_x \pm u_+$ of the fast magnetoacoustic wave. In this connection, it is appropriate to subdivide shock waves into slow and fast magnetoacoustic waves.

If the magnetic field is parallel to the plane of discontinuity, only the fast magnetoacoustic waves can be stable (see Fig. 2).

We note several consequences from the conditions for shock wave stability. If two shock waves of the same type follow one another, then the rear one overtakes the forward one. For example, let us consider two slow magnetoacoustic waves. The velocity of the forward wave, relative to the liquid contained between the waves is equal to v_{2x}, while the velocity of the rear wave is $-v_{1x}$. As follows from Fig. 1, for waves of this type, $v_{1x} > u_{1-}$ and $v_{2x} < u_{2-}$. Since the velocities u_{1-} and u_{2-} relate to the same region of space, then $u_{1-} = u_{2-}$ and, consequently, $v_{1x} > v_{2x}$.

So far as waves of different types are concerned, a rotational discontinuity will overtake a slow magnetoacoustic wave, while a fast magnetoacoustic wave will overtake all types of discontinuities. In the same way, we can show that a shock wave overtakes a weak discontinuity if it belongs to the same type as the shock wave itself, or to a slower type. A weak discontinuity overtakes a shock wave of the same type and shock waves of slower types.

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