AN INVESTIGATION OF PARTICLE-LIKE SOLUTIONS OF A NONLINEAR SCALAR FIELD EQUATION

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We investigate how the field energy of the particle-like solutions of a nonlinear complex scalar field equation is related to the parameters of the equation and the frequency of oscillations. For the case of the simplest cubic nonlinear term, the first three particle-like solutions are derived and the relevant mass spectrum is analyzed by numerical integration. Under some natural physical assumptions, the spectrum is found to be bounded.

1. It is of interest, in order to clarify the potentialities of a nonlinear field theory of elementary particles,[1-9] to investigate the particle-like solutions* of the simplest nonlinear complex scalar field equation. This equation can be used as an example to investigate qualitative questions such as the existence and nature of a mass spectrum. It is of course clear that an attempt to construct a quantitative theory giving the relation between mass, charge, spin, coupling constants, radii, and other structural elements of particles can be undertaken only after all possible fields and interactions have been investigated.

2. Consider a complex scalar field whose Lagrangian is

$$L = -\nabla\psi^*\nabla\psi + \frac{\partial\psi^*}{\partial x_0}\frac{\partial\psi}{\partial x_0} - m^2[\psi^*\psi + F(\psi^*\psi)], \tag{1}$$

where $x_0 = ct$, the function $F(v)$ is nonlinear, and m is a parameter whose units are the reciprocal of length (we are using units in which $\hbar = c = 1$, so that on going to ordinary units m must be replaced by $mc/\hbar$).

According to (1), the field equations are

$$\begin{gathered}\nabla^2\psi - \partial^2\psi/\partial x_0^2 - m^2[1 + F'(\psi^*\psi)]\psi = 0,\\ \nabla^2\psi^* - \partial^2\psi^*/\partial x_0^2 - m^2[1 + F'(\psi^*\psi)]\psi^* = 0,\end{gathered} \tag{2}$$

where $F'(v) = dF(v)/dv$.

If T_{ik} is the energy-momentum tensor and J_k is a vector, which we shall provisionally call the current vector, then according to (1) the field energy $E = \int_V T_{00}\,dV$ and the quantity $Q = \int_V J_0\,dV$, which we shall also provisionally call the charge, are given by

$$E = \frac{1}{2}\int_V\left\{\frac{d\psi^*}{dx_0}\frac{d\psi}{dx_0} + \nabla\psi^*\nabla\psi + m^2[\psi^*\psi + F(\psi^*\psi)]\right\}dV, \tag{3}$$

$$Q = \frac{i}{2}\int_V\left(\psi^*\frac{d\psi}{dx_0} - \psi\frac{d\psi^*}{dx_0}\right)dV. \tag{4}$$

For the case of spherical symmetry, the particle-like solution ψ decays exponentially as $r \to \infty$.[6,10] Using this fact and Eq. (2) in the integration indicated in (3), we obtain

$$E = \frac{1}{2}\int_V\left\{\frac{\partial\psi^*}{\partial x_0}\frac{\partial\psi}{\partial x_0} - \psi^*\frac{\partial^2\psi}{\partial x_0^2} + m^2[F(\psi^*\psi) - \psi^*\psi F'(\psi^*\psi)]\right\}dV. \tag{5}$$

3. As did Rosen and Rosenstock,[6] let us restruct our considerations to the simplest form of nonlinear function, namely $F(v) = -\lambda v^2/2$, and let us attempt to find a spherically symmetric particle-like solution of Eq. (2) in the form

$$\psi = u(r)e^{-i\varepsilon x_0}, \quad \psi^* = u(r)e^{i\varepsilon x_0}, \tag{6}$$

where ϵ is a parameter proportional to the frequency. Then the equation for $u(r)$ and the expressions for E and Q become

$$\frac{1}{r}\frac{d^2}{dr^2}(ru) + [\varepsilon^2 - m^2 + \lambda u^2]u = 0, \tag{7}$$

$$E = \varepsilon^2\int_0^\infty u^2 4\pi r^2 dr + \frac{\lambda m^2}{4}\int_0^\infty u^4 4\pi r^2 dr, \tag{8}$$

$$Q = \varepsilon\int_0^\infty u^2 4\pi r^2 dr. \tag{9}$$

If we assume that $\epsilon < m$, we can introduce the dimensionless quantity $\rho = r\sqrt{m^2 - \epsilon^2}$ and the dimensionless function $\eta = \sqrt{\lambda}\,mru$ to transform

*We call a solution particle-like if it decreases monotonically as $r \to \infty$ and has no singularity at the origin.

(7), (8), and (9) to the form*

$$d^2\eta / d\rho^2 = [1 - \eta^2 / \rho^2]\, \eta, \tag{10}$$

$$E = \frac{4\pi}{\lambda m^2}\left[\frac{\varepsilon^2}{\sqrt{m^2-\varepsilon^2}} I_1 + \frac{\sqrt{m^2-\varepsilon^2}}{4} I_2\right], \tag{11}$$

$$Q = \frac{4\pi}{\lambda m^2}\frac{\varepsilon}{\sqrt{m^2-\varepsilon^2}} I_1, \tag{12}$$

$$I_1 = \int_0^\infty \eta^2 d\rho, \quad I_2 = \int_0^\infty \frac{\eta^4}{\rho^2} d\rho. \tag{13}$$

(According to (10) it is obvious for particle-like solutions that $I_2 = I_1 + \int_0^\infty \dot{\eta}^2\, d\rho$, so that the relation $I_2 > I_1$ holds in all cases.)

Eliminating λ from (11) and (12), we obtain the relation between E, Q, and ϵ, namely

$$\frac{E}{m} = Q\left\{\frac{\varepsilon}{m} + \frac{1-(\varepsilon/m)^2}{4\,\varepsilon/m}\frac{I_2}{I_1}\right\}. \tag{14}$$

4. It is easily seen that particle-like solutions (i.e., solutions which satisfy the condition $\eta(0) = 0$ and which approach zero monotonically as $r \to \infty$) exist only for Eq. (10); no such solutions can be found for Eqs. (10a) or (10b). This means that (7) has particle-like solutions only if $\epsilon < m$. To see this, consider the equation

$$\eta'' = [\beta - \eta^2/\rho^2]\,\eta, \tag{10c}$$

which is a unified statement of (10), (10a), and (10b) with $\beta = 1$ for $\epsilon < m$, $\beta = -1$ for $\epsilon > m$, and $\beta = 0$ for $\epsilon = m$.

It is evident that in the two last cases ($\beta = -1$ and $\beta = 0$) the sign of η'' is the opposite of the sign of η. Therefore any integral curve is convex upward for $\eta > 0$ and convex downward for $\eta < 0$ for all values of ρ. This means that all solutions of (10a) and (10b), except those with $\eta \equiv 0$, oscillate about the ρ axis as $\rho \to +\infty$.

Let us now consider the following Cauchy problem for Eq. (10):

$$\frac{d^2\eta}{d\rho^2} = \left(1 - \frac{\eta^2}{\rho^2}\right)\eta, \quad \eta(0) = 0, \quad \frac{d\eta}{d\rho}(0) = \alpha, \tag{15}$$

where α is a parameter.

This problem is easily solved in the neighborhood of $\rho = 0$, giving the power series

$$\eta = \alpha\left(\rho + \frac{1-\alpha^2}{6}\rho^3 + \ldots\right). \tag{16}$$

As $\rho \to +\infty$, any solution of (15) which approaches zero is a monotonic function of ρ, since in this case the sign of η'' is the same as that of η.

We shall show that there exist values of α such that (15) has a solution which approaches zero as $\rho \to +\infty$ and has, for instance, a single node (at $\rho = 0$). We shall call this solution the first eigensolution, and the corresponding value of α the first eigenvalue.

Figure 1 shows the regions in which the sign of η'' is constant. Consider the line $\eta = \rho/\sqrt{3}$. Numerical integration shows that there exists an $\alpha = \alpha_1^{(0)} > 1$, such that the corresponding integral curve $\eta = \eta_1^{(0)}(\rho)$ enters into the region $0 < \eta < \rho/\sqrt{3}$ for some value of $\rho = \bar{\rho}$ and has a single zero at $\rho = 0$. There also exists an $\alpha = \alpha_2^{(0)} > \alpha_1^{(0)}$ whose integral curve $\eta = \eta_2^{(0)}(\rho)$ has a second zero at some value $\rho = \rho_1^{(0)} > \bar{\rho}$. If α^* and α^{**} satisfy the condition

$$\alpha_1^{(0)} < \alpha^* < \alpha^{**} < \alpha_2^{(0)},$$

then the corresponding $\eta^*(\bar{\rho})$ and $\eta^{**}(\bar{\rho})$ satisfy the condition

$$\eta_1^{(0)}(\bar{\rho}) > \eta^*(\bar{\rho}) > \eta^{**}(\bar{\rho}) > \eta_2^{(0)}(\bar{\rho}). \tag{17}$$

We shall now prove that under these conditions there exists a first eigensolution.

We note that in the region $0 < \eta < \rho/\sqrt{3}$, any two solutions η_1 and η_2 have the property that

$$\operatorname{sign}(\eta_1'' - \eta_2'') = \operatorname{sign}\left[(\eta_1 - \eta_2)\left(1 - \frac{\eta_1^2 + \eta_1\eta_2 + \eta_2^2}{\rho^2}\right)\right] = \operatorname{sign}(\eta_1 - \eta_2).$$

Therefore segments of the integral curves corresponding to α values from the interval $\alpha_1^{(0)} < \alpha < \alpha_2^{(0)}$ starting from points whose abcissas are $\rho = \bar{\rho}$ do not intersect before leaving the region $0 < \eta < \rho/\sqrt{3}$.

Let $\bar{\eta}(\rho)$ be a solution of the Cauchy problem $\bar{\eta}'' = \bar{\eta}$, $\bar{\eta}(\bar{\rho}) = k$ [with $k > \eta_1^{(0)}(\rho)$], which ap-

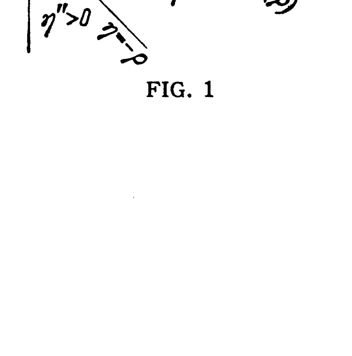

FIG. 1

*If $\varepsilon > m$, we can set $\rho = r\sqrt{\varepsilon^2 - m^2}$, in which case (10) is replaced by

$$d^2\eta / d\rho^2 = -[1 + \eta^2/\rho^2]\,\eta. \tag{10a}$$

If, finally, $\varepsilon = m$, we set $\eta = \sqrt{\lambda}\, ru$, and (7) can be written

$$d^2\eta / dr^2 = -(\eta^2/r^2)\,\eta. \tag{10b}$$

proaches zero as $\rho \to +\infty$. There obviously exists an interval $\bar{\rho} < \rho < \rho_2^{(0)}$ on which $0 < \eta_1^{(0)}(\rho) < \bar{\eta}(\rho)$ (Fig. 1).

Let us choose $\alpha = (\alpha_1^{(0)} + \alpha_2^{(0)})/2$. To this α corresponds either $\eta_1^{(1)}(\rho)$ which crosses $\bar{\eta}(\rho)$ at $\rho = \rho_2^{(1)}$, or $\eta_2^{(1)}(\rho)$, which crosses the ρ axis at $\rho_1^{(1)}$. Since the integral curves do not intersect in the region of interest, $\rho_2^{(1)} > \rho_2^{(0)}$, and $\rho_1^{(1)} > \rho_1^{(0)}$. In the first case we set $\alpha_1^{(1)} = (\alpha_1^{(0)} + \alpha_2^{(0)})/2$ and $\alpha_2^{(1)} = \alpha_2^{(0)}$, and in the second case we set $\alpha_1^{(1)} = \alpha_1^{(0)}$ and $\alpha_2^{(1)} = (\alpha_1^{(0)} + \alpha_2^{(0)})/2$, which gives a new pair of values of α (namely $\alpha_1^{(1)}$ and $\alpha_2^{(1)}$) to which correspond the integral curves $\eta_1^{(1)}(\rho)$ and $\eta_2^{(1)}(\rho)$, which are located in the region $0 < \eta < \bar{\eta}(\rho)$ on the segment $\bar{\rho} \le \rho \le \min(\rho_1^{(1)}, \rho_2^{(1)})$.

Continuing this process ad infinitum, in the limit we obtain $\alpha = \bar{\alpha} = \lim_{i\to+\infty} \alpha_1^{(i)} = \lim_{i\to+\infty} \alpha_2^{(i)}$. Since at least one of the two sequences $\{\rho_1^{(i)}\}$ or $\{\rho_2^{(i)}\}$ is unbounded, to the value of $\bar{\alpha}$ corresponds an integral curve which lies entirely within the region $0 < \eta < \bar{\eta}(\rho)$ for $\rho > \bar{\rho}$ and is thus an eigensolution.

In a similar way we can prove the existence of successive eigensolutions whose indices are determined by the number of intersections with the ρ axis.

5. The eigensolutions of (10) and their corresponding eigenvalues were found by numerical integration of (15) for different α.

Since the equation is singular at $\rho = 0$, the initial conditions were given for $\rho = \rho_0 > 0$ using the series of (16). The integration was performed for $\rho_0 = 10^{-3}$ and $\rho_0 = 10^{-6}$, and the results were found to agree within the required accuracy.

The calculations were performed on the high-speed electronic computer "Strela" (Arrow) of the computing center at the Moscow State University.

The region in which the eigenvalues $[\alpha_1^0, \alpha_2^0]$ are situated and the eigenvalues α_2 were found to a predetermined accuracy automatically, by programming the calculation described above.

In the interval $3 \le \alpha \le 90$, we obtained five first eigenvalues with a relative accuracy of $5 \times$

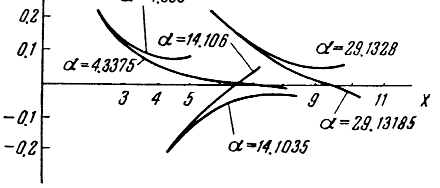

FIG. 2

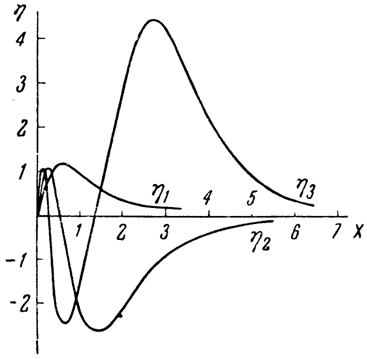

FIG. 3

10^{-5}. These were $\alpha_1 = 4.33$, $\alpha_2 = 14.10$, $\alpha_3 = 29.13$, $\alpha_4 = 49.36$, and $\alpha_5 = 74.77$ (all the figures are significant). Figure 2 shows the behavior of the first three integral curves for α somewhat larger and smaller than the corresponding value of α_2.

The eigensolutions, the first three of which are shown in Fig. 3, were tabulated with a relative error $\epsilon = 10^{-4}$, and then checked with an accuracy for which $\epsilon = 10^{-5}$ in steps for intervals $\rho_0 \le \rho \le \bar{\rho}_i$ such that when $\rho = \bar{\rho}_i$ the solutions approach their asymptotic curves with an accuracy of no less than 1%. With this accuracy, the following asymptotic expressions are obtained for $\rho > \bar{\rho}_i$:

$$\eta_1 = 2.70\, e^{-\rho}\, (\bar{\rho}_1 = 3.45);\quad \eta_2 = -17.0\, e^{-\rho}\, (\bar{\rho}_2 = 5.07);$$
$$\eta_3 = 83\, e^{-\rho}\, (\bar{\rho}_3 = 6);\ \eta_4 = -375\, e^{-\rho} (\bar{\rho}_4 = 6.85);$$
$$\eta_5 = 1611\, e^{-\rho} (\bar{\rho}_5 = 8.1).$$

The integrals

$$I_1^{(i)} = \int_0^\infty \eta_i^2\, d\rho,\quad I_2^{(i)} = \int_0^\infty \frac{\eta_i^4}{\rho^2}\, d\rho$$

were obtained graphically with an accuracy of 1 or 2%. Their values were $I_1^{(1)} = 1.53$, $I_1^{(2)} = 9.4$, $I_1^{(3)} = 27.7$, $I_1^{(4)} = 63.6$, $I_1^{(5)} = 212.8$, $I_2^{(1)} = 5.5$, $I_2^{(2)} = 39.4$, $I_2^{(3)} = 123$, $I_2^{(4)} = 255$, and $I_2^{(5)} = 476$.

6. Equations (11), (12), and (14) can be used to obtain further results only if we make additional assumptions on the relation between E, ϵ, and Q, and their physical meaning.

If, as in de Broglie's concept of the double solution,[10] we consider a particle-like solution of a nonlinear equation to be a "singularity moving in the proper way,* we must require that

$$\varepsilon = E = M, \tag{18}$$

*According to the hypothesis of a double solution,[10] Eq. (6) is a solution of the nonlinear equation (2) such that there is at least some region in which the phase is the same as that of the usual quantum mechanical wave (cons e^{iMx_0}).

where M is the true mass of the particle. Then according to (11), the quantity

$$X = M / \sqrt{m^2 - M^2} \tag{19}$$

should be given by

$$X^2 - \frac{\lambda m^2}{4\pi I_1} X + \frac{I_2}{4 I_1} = 0, \tag{20}$$

whose solution is of the form

$$X = (\lambda m^2 / 8\pi I_1) [1 \pm (1 - 16\pi^2 I_1 I_2 / \lambda^2 m^4)^{1/2}]. \tag{21}$$

According to (21), for every eigensolution of (10) there exist two values of X, and therefore also two of M. These two values, however, also give different values of Q, so that according to (12) and (19)

$$Q = (4\pi / \lambda m^2) I_1 X = {}^1/_2 [1 \pm (1 - 16\pi^2 I_1 I_2 / \lambda^2 m^4)^{1/2}]. \tag{22}$$

In order that there actually exist values of M and X, it is necessary that there exist I_1 and I_2 satisfying the inequality $16\pi^2 I_1 I_2 < \lambda^2 m^4$. From the values of I_1 and I_2 obtained, as well as from the graphs of the η functions, it is seen that $I_1 I_2$ increases with the index N of the eigensolution. It is quite probable that this is true for all eigensolutions, since their maxima move toward larger ρ as N increases, while the solutions themselves oscillate about the lines $\eta = \pm x$ before they begin to approach their asymptotic forms. In any case, if it is true that $I_1 I_2$ increases with N, it is possible to find values of N such that the roots of (21) are complex, and are therefore physically meaningless. In this case the mass spectrum of M values is cut off, which means that it contains a finite number of terms.

The mass spectrum may be even further restricted if Q, like an electric charge, can take on only integral values. Such an assumption does not follow, however, from a theory which includes only a scalar field with no electromagnetic field, so that it must be introduced artificially.

Thus for certain definite values of λm^2 the simplest equation for a scalar complex field can have a bounded number of particle-like eigensolutions, which means that the mass spectrum of M values is bounded.*

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*Thus, for instance, if we set $\lambda m^2 = 242$ ($4\pi\sqrt{I_1 I_2} = 242$ for the second eigensolution), then only the first and second eigensolutions are possible according to (21). For the first of these we obtain two masses, and for the second only one. These three masses are related as 1:10:14. As λm^2 is increased, the mass of the second eigensolution splits into two terms, and as it is further increased, the mass of a third eigensolution appears, etc. For very large values of λm^2 the number of possible masses becomes very large; the five lowest masses will be related as 0.14 : 1 : 3.1 : 6.4 : 12 : . . . , and the largest masses will concentrate about the value m, which forms their upper limit.