SCATTERING OF SLOW NEUTRONS IN FERROMAGNETS

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The scattering of slow neutrons in ferromagnets is analyzed, using the theory of spin-waves. The analysis is applied to single crystals and to polycrystals.

1. INTRODUCTION

THIS paper will deal with the scattering of slow neutrons in ferromagnets, at low temperatures where the behavior of the ferromagnet is described by spin-wave theory. This problem has been recently discussed by several authors.[1-4] The most complete discussion is that by Elliott and Lowde. They consider mainly the inelastic scattering of neutrons in a single crystal, the scattering being accompanied by the emission or absorption of a spin-wave of small momentum, and the temperature being not too low so that the energy-spectrum of spin-waves is given by the Rayleigh-Jeans formula. The case of greatest physical interest, the diffuse scattering close to a Bragg reflection, is described satisfactorily by their theory. However, the assumptions which they make are not necessary; the cross-sections can be calculated without any such assumptions, and the results remain simple in form.

We find that, when the neutron wave-length is much greater than the lattice-constant, the large-angle scattering of a neutron is accompanied by the absorption of a spin-wave of large momentum for which the Rayleigh-Jeans distribution is usually incorrect. Therefore the statement[4,5] that the long-wave-length inelastic scattering cross-section is proportional to temperature becomes correct only at temperatures low compared with the ferromagnetic exchange interaction.

In this paper we calculate the elastic scattering, and the inelastic scattering in single crystals and polycrystals with absorption or emission of one-spin wave, and we also estimate the magnitude of processes in which more than one spin-wave is emitted or absorbed.

2. DERIVATION OF THE CROSS-SECTION FORMULAE

We consider neutrons in a ferromagnet being scattered by magnetic interaction with the atomic spins. We do not include the scattering due to the interaction with the orbital part of the atomic magnetic moments, since this scattering is identical with the scattering in a paramagnetic medium and was calculated by Migdal.[5] Because the orbital moments are randomly oriented, there is no interference between the orbital and spin contributions to the scattering. For a detailed investigation of inelastic scattering we need to know the wave-functions of the scatterer; we are therefore restricted to the low range of temperatures (far below the Curie temperature) in which a detailed theory of ferromagnetism, the spin-wave theory, is valid.

The matrix elements of the interaction of a neutron with the spins of a system of N identical atoms have been calculated by Halpern and Johnson.[6] They have the form

$$V_{\mathbf{p}na}^{\mathbf{p}'n'a'} = -\frac{4\pi\hbar^2}{M} r_0\gamma F(q)\Big(n'a'\Big|\sum_l e^{i\mathbf{q}\mathbf{R}_l}(\mathbf{S}_l, \mathbf{S}-(\mathbf{e}\mathbf{S})\mathbf{e})\Big|na\Big), \tag{1}$$

Here $\mathbf{p}$ is the neutron wave-vector, n is one component of its spin, and a is the state of the ferromagnet, initially. In the final state the corresponding quantities are $\mathbf{p}'$, n', a'. γ is the neutron magnetic moment in nuclear magnetons, $\mathbf{S}$ is its spin, $r_0 = e^2/mc^2$, M is the neutron mass, $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, $\mathbf{e} = \mathbf{q}/q$, $\mathbf{S}_l$ and $\mathbf{R}_l$ are the spin and position of atom number l,

$$F(q) = \int d\tau \Psi_l^* \sum_{\nu=1}^{z} \frac{e^{i\mathbf{q}\rho_\nu} s_\nu S_l}{S_l(S_l+1)} \Psi_l, \tag{2}$$

Ψ_ℓ is the wave-function of atom number ℓ, $\mathbf{s}_\nu$ and $\boldsymbol{\rho}_\nu$ are the spin and position of electron number ν, and the sum extends over the z electrons in the atom.

The matrix elements of the operator $\sum \exp(i\mathbf{q}\mathbf{R}_\ell)\mathbf{S}_\ell$ can be easily calculated from spin-wave theory in the form introduced by Dyson.[7,8] The operators S_ℓ^z and $S_\ell^\pm = S_\ell^x \pm iS_\ell^y$ are connected with spin-wave absorption and creation operators in the following way (see the Appendix).

$$S_l^z = -S\sum_\mu e^{-i\mu\mathbf{R}_l}\left[\delta_{\mu 0} - (SN)^{-1}\sum_\lambda \alpha_{\mu+\lambda}^+\alpha_\lambda\right], \tag{3a}$$

$$S_l^+ = (2SN^{-1})^{1/2}\sum_\mu e^{-i\mu\mathbf{R}_l}\alpha_\mu^+, \tag{3b}$$

$$S_l^- = (2SN^{-1})^{1/2}\sum_\mu e^{-i\mu\mathbf{R}_l}\left[\alpha_{-\mu} - (2SN)^{-1}\sum_{\lambda,\sigma}\alpha_{\lambda+\sigma+\mu}^+\alpha_\lambda\alpha_\sigma\right], \tag{3c}$$

where the z-axis is along the direction of magnetization of a domain, α_μ and α_μ^+ are absorption and emission operators for a spin-wave with wave-vector $\boldsymbol{\mu}$ and energy

$$\varepsilon_\mu = JS\sum_\delta (1 - e^{i\mu\delta}) \approx \frac{1}{6}JS\gamma_0\delta^2\mu^2, \tag{4}$$

J is the ferromagnetic exchange integral, $\boldsymbol{\delta}$ is a vector joining any atom of the lattice to one of its nearest neighbors, and γ_0 is the number of nearest neighbors over which the δ-summation extends.*

Equation (3) implies the existence of the following types of scattering. (1) Elastic. (2) With emission of one spin-wave. (3) With absorption of one spin-wave. (4) With emission of one and absorption of one spin-wave. (5) With emission of one and absorption of two spin-waves. The corresponding cross-sections are

1)
$$\frac{d\sigma_0}{d\Omega} = S^2 r_0^2\gamma^2 F^2(q)\frac{1}{N}\left|\sum_l e^{i\mathbf{q}\mathbf{R}_l}\right|^2\left\langle\left(1 - \frac{1}{NS}\sum_\lambda a_\lambda\right)^2\right\rangle e^{-2W_\mathbf{q}}(1 - e_z^2); \tag{5a}$$

2)
$$\frac{d\sigma_{+1}}{d\Omega} = \frac{1}{2}Sr_0^2\gamma^2F^2(q)\frac{1}{N}\left|\sum_l e^{i(\mathbf{q}-\mu)\mathbf{R}_l}\right|^2\frac{v_0}{(2\pi)^3}e^{-2W_\mathbf{q}}\frac{p'}{p}(1+e_z^2)\,]\langle a_\mu\rangle + 1]\,d\mu; \tag{5b}$$

3)
$$\frac{d\sigma_{-1}}{d\Omega} = \frac{1}{2}Sr_0^2\gamma^2F^2(q)\frac{1}{N}\left|\sum_l e^{i(\mathbf{q}+\mu)\mathbf{R}_l}\right|^2\frac{v_0}{(2\pi)^3}e^{-2W_\mathbf{q}}\frac{p'}{p}(1+e_z^2)\left\langle a_\mu\left(1 - \sum_{\lambda\neq\mu}a_\lambda/NS\right)\right\rangle d\mu; \tag{5c}$$

4)
$$\frac{d\sigma_{+1,-1}}{d\Omega} = r_0^2\gamma^2F^2(q)\frac{1}{N}\left|\sum_l e^{i(\mathbf{q}+\mu-\nu)\mathbf{R}_l}\right|^2\frac{v_0^2}{(2\pi)^6}e^{-2W_\mathbf{q}}\frac{p'}{p}(1-e_z^2)\langle a_\mu(a_\nu+1)\rangle\,d\mu\,d\nu; \tag{5d}$$

5)
$$\frac{d\sigma_{+1,-2}}{d\Omega} = \frac{1}{2S}r_0^2\gamma^2F^2(q)\frac{1}{N}\left|\sum_l e^{i(\mathbf{q}+\nu+\sigma-\mu)\mathbf{R}_l}\right|^2 e^{-2W_\mathbf{q}}\frac{v_0^3}{(2\pi)^9}\frac{p'}{p}\langle a_\nu a_\sigma(a_\mu+1)\rangle(1+e_z^2)\,d\mu\,d\nu\,d\sigma. \tag{5e}$$

Here v_0 is the volume of the unit cell, e^{-2W_q} is the usual temperature factor, a_μ is the number of spin-waves with wave-vector $\boldsymbol{\mu}$, and $\langle\ \rangle$ denotes an average over the initial states of the scatterer, so that $\langle a_\mu\rangle$ is the Planck distribution-function which we shall denote by $n(\mu)$.

In this paper we study the cross-sections (5a) — (5c). A detailed investigation of (5d) and (5e) will be made later.

3. ELASTIC SCATTERING

Consider Eq. (5a). At low temperatures $\Sigma a_\mu/NS \ll 1$, and we may neglect the square of this quantity. Its mean value is just equal to the deviation of the spontaneous magnetization at temperature T from its value at T = 0. When the number N of atoms in the crystal is large,

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*We consider only ferromagnets of cubic symmetry.

$$\frac{1}{N}\left|\sum_l \exp(i\mathbf{q}\mathbf{R}_l)\right|^2 = \frac{(2\pi)^3}{v_0}\sum_\tau \delta(\mathbf{q}-\tau), \tag{6}$$

where τ is a vector of the reciprocal lattice multiplied by 2π. Therefore Eq. (5a) becomes

$$\frac{d\sigma_0}{d\Omega} = S^2 r_0^2 \gamma^2 F^2(q) e^{-2W_q}[1-2G(T)](1-e_z^2)\frac{(2\pi)^3}{v_0}\sum_\tau \delta(\mathbf{q}-\tau). \tag{7}$$

This expression was obtained by Van Hove[9] by a different method. From spin-wave theory we have

$$G(T) = \zeta(3/2)(T/2\pi T_c)^{3/2} + O[(T/2\pi T_c)^{5/2}], \tag{8}$$

where $\zeta(x)$ is the Riemann zeta-function, $\nu = \delta^2 v_0^{-2/3}$, and $T_c = JS\gamma_0\nu/3k$ is approximately equal to the Curie temperature.[8] Equation (8), like the spin-wave theory and like our whole treatment of neutron scattering, is valid, as shown for example by Dyson,[8] when $T \ll T_c/2\nu$.

A detailed theory of ferromagnetism is not necessary to derive Eq. (7). In fact Eq. (7) follows directly from Eq. (1), if we use the following three pieces of information. (a) The translational symmetry of the crystal implies that the matrix element $(a|S_l^z|a)$ is independent of l. (b) The definition of $G(t)$ requires

$$S^{-2}\langle (a|S_l^z|a)^2\rangle \approx 1-2G(T)$$

(c) The matrix elements $(a|S_l^+|a)$ are zero, since the operator S_l^+ changes by one unit the z-component of the total spin and this z-component is a constant of the motion. Therefore Eq. (7) can easily be generalized to the case of scattering by antiferromagnets. In the simplest case, an antiferromagnetic crystal can be considered as composed of two equivalent sub-lattices, interpenetrating each other and magnetized in opposite directions. The only difference from the ferromagnetic case is that the matrix elements $(a|S_l^z|a)$ reverse their signs when we go from one atom to any of its nearest neighbors, the absolute values still being equal. Accordingly we may write

$$(a|S_l^z|a) = (a|S_l^z|a)'\exp(i\mathbf{w}\mathbf{R}_l),$$

where $(a|S_l^z|a)'$ is independent of l, and $\mathbf{w}$ is any vector of the reciprocal sub-lattice which does not belong to the reciprocal lattice. Thus Eq. (7) remains valid for the scattering by an antiferromagnet, if we replace the vectors τ by $\mathbf{w}$, and define $G(t)$ with reference to either sub-lattice. By observing the scattering of neutrons in antiferromagnets we may obtain information about the temperature-dependence of the spontaneous magnetization in the sublattices.

Averaging Eq. (7) over the direction of τ, we obtain the scattering cross-section in a polycrystal

$$\frac{d\sigma_0}{d\Omega} = S^2 r_0^2\gamma^2[1-2G(T)]\frac{(2\pi)^3}{v_0}\sum_\tau \frac{g_\tau F^2(\tau)}{4\pi\tau^2}(1-e_z^2)e^{-2W_\tau}\delta(q-\tau), \tag{9}$$

where g_τ is the number of distinct reciprocal lattice-vectors with length τ, and

$$e_z^2 = \left[\cos\frac{\theta_0}{2}\sin\varphi\sin\zeta - \sin\frac{\theta_0}{2}\cos\zeta\right]^2, \tag{10}$$

where θ_0 is the scattering angle which satisfies the condition $\sin(\theta_0/2) = \tau/2p$, ζ is the angle between $\mathbf{p}$ and the z-axis, and φ is the polar angle of the vector $\mathbf{p}'$ measured in the plane perpendicular to $\mathbf{p}$ from the direction $[\mathbf{p}\times\mathbf{z}]$.

The total elastic scattering cross-section in a polycrystal is

$$\sigma_0 = S^2 r_0^2\gamma^2[1-2G(T)]\frac{(2\pi)^3}{v_0 p^2}\sum_{\tau\leqslant 2p}\frac{g_\tau F^2(\tau)}{4\tau}e^{-2W_\tau}\left[1+\cos^2\zeta+\frac{\tau^2}{4p^2}(1-3\cos^2\zeta)\right]. \tag{11}$$

4. SINGLE-SPIN-WAVE SCATTERING IN A SINGLE CRYSTAL

Starting from Eqs. (5b), (5c), and (6), and neglecting $\Sigma a_\mu/NS$ in comparison with unity in Eq. (5c), we obtain the cross-section for single-spin-wave scattering in a single crystal,*

*Equation (12) coincides with Elliott and Lowde's formula[4] for the single-spin-wave scattering, if we replace $n(\mu) + 1/2 \pm 1/2$ by $6kT/JS\gamma_0\delta^2\mu^2$.

$$d\sigma_{\pm 1}^{\tau}/d\Omega = \frac{1}{2} S r_0^2 \gamma^2 F^2(q) e^{-2W_q}(1+e_z^2)\frac{p'}{p}[n(\mu)+{}^1/_2 \pm {}^1/_2]\,\delta(\mathbf{q} \pm \boldsymbol{\mu} + \boldsymbol{\tau})\,d\boldsymbol{\mu}. \quad (12)$$

The vectors $\mathbf{p}'$, $\mathbf{p}$, and $\boldsymbol{\mu}$ are connected by the energy conservation law:

$$p'^2 = p^2 \mp (2M/\hbar^2)\,\varepsilon(\boldsymbol{\mu}) \approx p^2 \mp \alpha\mu^2. \quad (13)$$

Here the minus sign holds for scattering with emission of a spin-wave, the plus sign for scattering with absorption. For ion $\alpha \sim 100$, and in general $\alpha \gg 1$. In Eq. (12) we may replace $\mathbf{q}$ by $\boldsymbol{\tau}$ everywhere except in the argument of the δ-function. This is because the whole spin-wave theory, and Eq. (13) in particular, only holds for $\mu\delta < 1$.

We study first the case of absorptive scattering. Integrating Eq. (12) with respect to μ and using Eq. (13), we find

$$d\sigma_{-1}^{\tau}/d\Omega = (d\sigma_{-1}^{\tau}/d\Omega)_+ + (d\sigma_{-1}^{\tau}/d\Omega)_-;\ (d\sigma_{-1}^{\tau}/d\Omega)_\pm = \frac{1}{2} S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau}(1+(\boldsymbol{\tau}\mathbf{m})^2/\tau^2)\,n(\mu_\pm) \quad (14)$$

$$\times \frac{P\alpha}{p(\alpha-1)^2}\frac{[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \cos^2\vartheta_0}]^2}{\sqrt{\cos^2\vartheta - \cos^2\vartheta_0}}; \quad (15)$$

$$\cos^2\vartheta_0 = (\alpha-1)\alpha^{-1}(1+p^2/\alpha P^2); \quad (16)$$

$$\mu_\pm^2(\cos\vartheta) = \frac{P^2\alpha}{(\alpha-1)^2}\left[\left(\cos\vartheta \pm \sqrt{\cos^2\vartheta - \cos^2\vartheta_0}\right)^2 - \left(\frac{\alpha-1}{\alpha}\right)^2\frac{p^2}{P^2}\right], \quad (17)$$

Here ϑ is the scattering angle measured from the direction $\mathbf{P} = \mathbf{p} + \boldsymbol{\tau}$, and $\mathbf{m}$ is a unit vector in the direction of magnetization. The magnitude of the momentum of the scattered neutron is given by

$$p'_\pm = \frac{\alpha P}{\alpha - 1}(\cos\vartheta \pm \sqrt{\cos^2\vartheta - \cos^2\vartheta_0}), \quad (18)$$

and the range of variation of $\cos\vartheta$ is

$$1 \geqslant \cos\vartheta \geqslant \cos\vartheta_0 > 0. \quad (19)$$

Since $\cos^2\vartheta_0 < 1$, it follows that

$$\frac{p^2}{P^2} < \frac{\alpha}{\alpha-1} \text{ for } \cos\Psi > -\frac{1}{2}\left(\frac{\tau}{p} + \frac{1}{\alpha}\frac{p}{\tau}\right), \quad (20)$$

where Ψ is the angle between $\mathbf{p}$ and $\boldsymbol{\tau}$. Equation (16) implies $\cos^2\vartheta_0 > 1 - (1/\alpha)$, and since $\alpha \gg 1$, also $\vartheta < \alpha^{-1/2}$. Thus the scattering with absorption of a spin-wave occurs only in a narrow cone around the axis $\mathbf{P} = \mathbf{p} + \boldsymbol{\tau}$. To each direction of scattering within this cone there correspond two values given by Eq. (18) for the scattered neutron momentum.

By Eq. (17), $\mu_+^2(\cos\vartheta)$ decreases and $\mu_-^2(\cos\vartheta)$ increases with increasing ϑ. Thus

$$\mu_+^2(1) > \mu_+^2(\cos\vartheta_0) = \mu_-^2(\cos\vartheta_0) > \mu_-^2(1). \quad (21)$$

From Eqs. (14), (15), and (21) we deduce the formula for the total cross-section

$$\sigma_{-1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\left[1 + \frac{(\boldsymbol{\tau}\mathbf{m})^2}{\tau^2}\right] e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{1-\exp\{-T_c\delta^2\mu_+^2(1)(2\nu T)^{-1}\}}{1-\exp\{-T_c\delta^2\mu_-^2(1)(2\nu T)^{-1}\}}. \quad (22)$$

We next study a few limiting cases.

I. $\mathbf{P} = \mathbf{p}$. Then Eqs. (16) and (17) give

$$\cos^2\vartheta_0 = 1 - \alpha^{-2}, \quad \mu_+^2(1) = 4p^2/(\alpha-1)^2, \quad \mu_-^2(1) = 0.$$

If also p and T satisfy

$$4p^2\delta^2/(\alpha-1)^2 \ll 2\nu T/T_c,$$

then $n(\mu_+)$ can be approximated by $\langle 2\nu T/T_c \rangle \mu_+^2\delta^2$ in Eq. (15), and the cross-section becomes proportional to temperature. As $\vartheta \to 0$ the cross-section tends to infinity. However, Elliott and Lowde[4] have shown that the infinity does not actually occur, since the minimum wave-vector of a spin-wave is different from zero and is of the order of magnitude $2\pi/\delta N^{1/3}$, where N is roughly the number of atoms in a domain. Therefore the cross-section becomes

$$\sigma_{-1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\, e^{-2W_\tau}\,\frac{2\nu T}{T_c}(p\delta)^{-2}\ln\frac{p^2\delta^2 N^{2/3}}{\pi^2(\alpha-1)^2}. \tag{23}$$

When P = p there is also elastic scattering.

II. If $p \neq P$ but $\mu_+^2(1)\delta^2 \ll 2\nu T/T_c$, the cross-section remains proportional to temperature,

$$\sigma_{-1}^{\tau} = \frac{\pi}{2} S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\, e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\mu_+^2(1)}{\mu_-^2(1)}. \tag{24}$$

III. If $\mu_-^2(1)\delta^2 > 2\nu T/T_c$, the cross-section is proportional to

$$\exp\{-\mu_-^2(1)\delta^2 T_c/2\nu T\} - \exp\{-\mu_+^2(1)\delta^2 T_c/2\nu T\}.$$

But this expression is very small compared with unity, and so there is practically no neutron scattering with spin-wave absorption in this case.

The cross-section is a maximum when $p^2 = P^2$. As p^2 decreases, $\mu_-^2(1)$ increases and hence the cross-section decreases.

We next consider the scattering with spin-wave emission. As in the case of absorption, we find

$$d\sigma_{+1}^{\tau}/d\Omega = (d\sigma_{+1}^{\tau}/d\Omega)_+ + (d\sigma_{+1}^{\tau}/d\Omega)_-; \tag{25}$$

$$(d\sigma_{+1}^{\tau}/d\Omega)_\pm = \frac{1}{2} S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau}\,[1 + (\mathbf{m}\tau)^2/\tau^2]$$

$$\times\,[n(\mu'_\pm) + 1]\frac{P\alpha}{(\alpha+1)^2}\left[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)}\right]^2\left[\cos\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)\right]^{-1/2}; \tag{26}$$

$$\mu_\pm'^2(\cos\vartheta) = \frac{\alpha P^2}{(\alpha+1)^2}\left\{\left(\frac{\alpha+1}{\alpha}\right)^2\frac{p^2}{P^2} - \left[\sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)} \pm \cos\vartheta\right]^2\right\}; \tag{27}$$

$$p'_\pm = \frac{\alpha P}{\alpha+1}\left[\cos\vartheta \pm \sqrt{\cos^2\vartheta - \frac{\alpha+1}{\alpha}\left(1 - \frac{1}{\alpha}\frac{p^2}{P^2}\right)}\right]. \tag{28}$$

Scattering is now possible only when

$$\frac{p^2}{P^2} > \frac{\alpha}{\alpha+1} \quad \text{or} \quad \cos\Psi < -\frac{1}{2}\left(\frac{\tau}{p} - \frac{1}{\alpha}\frac{p}{\tau}\right). \tag{29}$$

Since $\cos\Psi \geq -1$, Eq. (29) implies that scattering will occur with spin-wave emission only when

$$p > \alpha\tau\left(\sqrt{1 + (1/\alpha)} - 1\right) \approx (\tau/2)(1 - (1/4\alpha)). \tag{30}$$

We need to consider two cases

$$1)\ \frac{\alpha+1}{\alpha}\left(1 - \frac{p^2}{\alpha P^2}\right) \equiv \cos^2\vartheta_1 > 0, \quad 2)\ 1 - \frac{p^2}{\alpha P^2} < 0.$$

We discuss case (1) first. As in the discussion of absorptive scattering, $\cos\vartheta$ varies within the range

$$1 \geqslant \cos\vartheta \geqslant \cos\vartheta_1 > 0 \tag{31}$$

and to every scattering angle ϑ correspond two outgoing neutron momenta given by Eq. (28). Also $\mu_+'^2(\cos\vartheta)$ increases and $\mu_-'^2(\cos\vartheta)$ decreases with increasing ϑ, so that

$$\mu_-'^2(1) > \mu_-'^2(\cos\vartheta_1) = \mu_+'^2(\cos\vartheta_1) > \mu_+'^2(1). \tag{32}$$

The total cross-section is given by

$$\sigma_{+1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\, e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\exp\{T_c\delta^2\mu_-'^2(1)(2\nu T)^{-1}\} - 1}{\exp\{T_c\delta^2\mu_+'^2(1)(2\nu T)^{-1}\} - 1}. \tag{33}$$

As in the discussion of absorptive scattering, we consider three limiting sub-cases of case (1).

I. P = p. $\cos^2\vartheta_1 = 1 - \alpha^{-2}$, $\mu_-'^2(1) = 4p^2/(\alpha+1)^2$, $\mu_+'^2(1) = 0$, and, if we assume $4p^2\delta^2/(\alpha+1)^2 \ll 2\nu T/T_c$, then

$$\sigma_{+1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\, e^{-2W_\tau}(p\delta)^{-2}\frac{2\nu T}{T_c}\ln\frac{p^2\delta^2 N^{2/3}}{\pi^2(\alpha+1)^2} \tag{34}$$

So in this case $\sigma_{+1}^{t} \approx \sigma_{-1}^{t}$. This means that around the Bragg maximum of width $N^{-1/3}$ there is a diffuse maximum of width $1/\alpha$ caused by single-spin-wave inelastic scattering.

II. $p \neq P$ but $\mu_{-}^{\prime 2}(1)\delta^{2} \ll 2\nu T/T_C$. Then

$$\sigma_{+1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\,e^{-2W_\tau}(pP\delta^2)^{-1}\frac{2\nu T}{T_c}\ln\frac{\mu_{-}^{\prime 2}(1)}{\mu_{+}^{\prime 2}(1)} \tag{35}$$

III. $\mu_{+}^{\prime 2}(1)\delta^{2} \gtrsim 2\nu T/T_C$. In this case the cross-section is given, with an error of order $\exp\{-T_C[\mu_{+}^{\prime}(1)\delta]^2/2\nu T\}$, by the formula

$$\sigma_{+1}^{\tau} = \frac{1}{2}\pi S r_0^2 \gamma^2 F^2(\tau)\,[1 + (\mathbf{m}\tau)^2/\tau^2]\,e^{-2W_\tau}\frac{4\alpha \sin\vartheta_1}{(\alpha+1)^2}, \tag{36}$$

and is very small when α is large. It is larger, the larger the ratio p^2/P^2. As in the case of absorptive scattering, the maximum cross-section is at $p^2/P^2 = 1$.

Now we return to case (2). This case is possible if

$$\tau\sqrt{\alpha}(\sqrt{\alpha}+1)^{-1} < p < \tau\sqrt{\alpha}(\sqrt{\alpha}-1)^{-1},$$

which means that p is very close to τ. Equation (28) then implies that $p_{-}^{\prime} < 0$. There is thus only one outgoing neutron momentum $p_{+}^{\prime}$ at each scattering angle, and the angle can very from zero to π. Also

$$\mu_{+}^{\prime 2}(1) \leqslant \mu_{+}^{\prime 2}(\cos\vartheta) \leqslant \mu_{+}^{\prime 2}(-1). \tag{37}$$

Equation (27) implies that $\mu_{+}^{\prime 2}(1) > p^2/\alpha \approx \tau^2/\alpha$, and so the cross-section is small at sufficiently low temperatures. When $\mu_{+}^{\prime 2}(1)\delta^2 < 1$ the cross-section is given approximately by Eq. (36) with $\sin\vartheta_1$ replaced by $\sqrt{\frac{1}{\alpha}\left(\frac{\alpha+1}{\alpha}\frac{p^2}{P^2}\right)-1}$.

From Eqs. (20) and (29) we see that scattering with emission of a spin-wave and scattering with absorption can occur simultaneously only when

$$-\left(\frac{\tau}{p}+\frac{1}{\alpha}\frac{p}{\tau}\right) \leqslant 2\cos\Psi \leqslant -\left(\frac{\tau}{p}-\frac{1}{\alpha}\frac{p}{\tau}\right), \tag{38}$$

which just defines the range of values of Ψ for which both cross-sections attain their maxima.

5. SINGLE-SPIN-WAVE SCATTERING IN POLYCRYSTALS

When Eq. (12) is averaged over the possible orientations of $\mathbf{q}$ and τ and integrated over the direction of $\boldsymbol{\mu}$, we obtain the total cross-section for scattering in a polycrystal with absorption or emission of one spin-wave with wave-vector $\boldsymbol{\mu}$,

$$d\sigma_{\pm 1}^{\tau}(p,\mu)/d\Omega = \frac{1}{4} S r_0^2 \gamma^2 F^2(\tau)\,e^{-2W_\tau}(1+e_z^2)\,\frac{g_\tau}{\tau}\frac{p'}{p}\left[n(\mu)+\frac{1}{2}\pm\frac{1}{2}\right]\frac{\mu d\mu}{q}, \tag{39}$$

with

$$\tau - \mu < q < \tau + \mu. \tag{40}$$

Since $q = |\mathbf{p}' - \mathbf{p}|$, $|p - p'| < q < p + p'$, and therefore the inequality (40) need be stated only when

$$|p - p'| < \tau - \mu, \tag{41a}$$

$$\tau + \mu < p + p'. \tag{41b}$$

We must in any case have

$$\tau + \mu > |p - p'|, \tag{41c}$$

$$\tau - \mu < p + p'. \tag{41d}$$

The inequalities (41c) and (41d) fix the limits within which q may vary when μ is given.

Consider first the absorptive scattering. We have to satisfy the inequalities (41a)—(41d) with p' given by Eq. (13). The allowed values of q are plotted in Fig. 1, the boundary cruves being given by

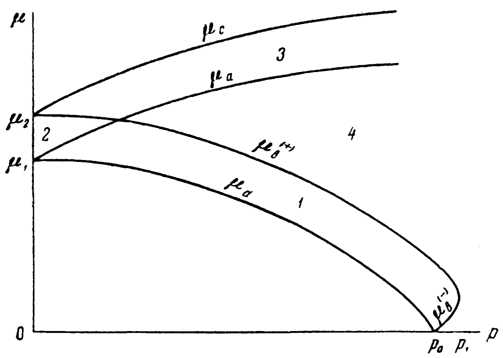

FIG. 1. Ranges of variation of q for given p and μ, for absorptive scattering. $p_0 = \tau/2$, $p_1 = \tau\alpha(1 - \sqrt{1-(1/\alpha)})$, $\mu_1 = \tau/(\sqrt{\alpha}+1)$, $\mu_2 = \tau/(\sqrt{\alpha}-1)$. In region (1) $\tau - \mu \le q \le p + p'$. In (2) $p' - p \le q \le p' + p$. In (3) $p' - p \le q \le \tau + \mu$. In (4) $\tau - \mu \le q \le \tau + \mu$. Outside these regions there is no scattering.

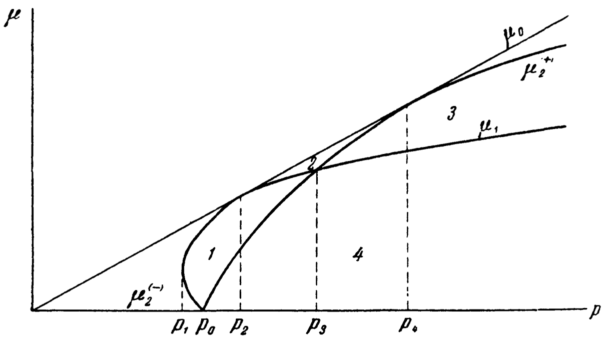

FIG. 2. Ranges of variation of q for given p and μ, for scattering with spin-wave emission. $p_1 = \tau\alpha(\sqrt{1+(1/\alpha)} - 1)$, $p_0 = \tau/2$, $p_2 = \sqrt{\alpha}\tau/(\sqrt{\alpha}+1)$, $p_3 = \tau$, $p_4 = \sqrt{\alpha}\tau/(\sqrt{\alpha}-1)$, $\mu_0 = p/\sqrt{\alpha}$. In region (1) $\tau - \mu \le q \le p + p'$. In (2) $p - p' \le q \le p' + p$. In (3) $p - p' \le q \le \tau + \mu$. In (4) $\tau - \mu \le q \le \tau + \mu$. Outside these regions there is no scattering.

$$\mu_a = \frac{-\tau - p + \sqrt{\alpha\tau^2 + 2\alpha\tau p + p^2}}{\alpha - 1}; \quad \mu_c = \frac{\tau + p + \sqrt{\alpha\tau^2 + 2\alpha\tau p + p^2}}{\alpha - 1}; \tag{42}$$

$$\mu_b^{(\pm)} = \frac{\tau - p \pm \sqrt{\alpha\tau^2 - 2\alpha\tau p + p^2}}{\alpha - 1}; \quad \mu_d = \frac{p - \tau + \sqrt{\alpha\tau^2 - 2\alpha\tau p + p^2}}{\alpha - 1}.$$

The range of variation of q is of no interest when μ is large, since $n(\mu)$ is then small. Thus the allowed ranges of q when p, μ, τ are given can for practical purposes be taken as

$$1)\ \tau - \mu \leqslant q \leqslant \tau + \mu; \quad 2)\ \tau - \mu \leqslant q \leqslant p + p'.$$

The differential cross-section is

$$d\sigma_{-1}^{\tau}/d\mu = (4\pi/3)\, S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau/\tau p^2)\, n(\mu)\, \mu^2 \tag{43a}$$

in case (1) and

$$d\sigma_{-1}^{\tau}/d\mu = (2\pi/3)\, S r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau/\tau p^2)\, n(\mu)\, \mu\, (p + p' - \tau + \mu) \tag{43b}$$

in case (2). These results are calculated for an unmagnetized crystal, setting e_z^2 equal to ⅓. If $p \ge \tau\alpha(1 - \sqrt{(1-(1/\alpha))})$, only case (1) can occur, and the total cross-section becomes

$$\sigma_{-1}^{\tau}(p) = \frac{1}{3} S^2 r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} \frac{g_\tau (2\pi)^3}{\tau p^2 v_0} G(T). \tag{44}$$

When $\tau/2 < p < \alpha\tau(1 - \sqrt{1-(1/\alpha)})$, we can compute the correction to Eq. (44) produced by the range of μ for which $d\sigma_{-1}^{\tau}/d\mu$ is given by Eq. (43b) rather than by Eq. (43a). However, this correction is of order $1/\alpha$ and therefore small when α is large. Equation (44) can be taken as correct all the way down to $p = \tau/2$. Further, using the relation $p + p' + \mu - \tau < 2\mu$, we can show that when $\mu_d < \mu < \mu_b^{(+)}$, Eq. (44) remains valid for all p satisfying $\mu_d\delta \ll \sqrt{2\nu T/T_c}$. When p decreases still further the cross-section begins to fall off rapidly.

Next we discuss the scattering with spin-wave emission. When $\mu\delta < 1$, so that the relation between energy and momentum of a spin-wave is quadratic, the values of q allowed by the inequalities (41) are plotted in Fig. 2. The boundary curves are

$$\mu_1^{(\pm)} = \frac{\tau - p \pm \sqrt{p^2 + 2\alpha\tau p - \alpha\tau^2}}{\alpha + 1}; \quad \mu_2 = \frac{p - \tau + \sqrt{p^2 + 2\alpha\tau p - \alpha\tau^2}}{\alpha + 1}.$$

The cross-section is then

$$d\sigma_{+1}^{\tau}/d\mu = (2\pi S/3)\, r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} (g_\tau/\tau p^2)\, [n(\mu) + 1]\, \mu\Delta q, \tag{45}$$

where Δq is the difference between the maximum and minimum values of q which are allowed for given p, μ. Again the scatterer is assumed unmagnetized.

We now consider the total cross-section for scattering with emission of one spin-wave. When p is close to $\tau/2$, the total cross-section is obviously small. It increases rapidly with p. When $p \sim \tau$, the part of the total cross-section with spin-wave emission which depends on $n(\mu)$ becomes equal to the absorptive cross-section given by Eq. (44). As p increases further, the analytic form of the cross-section remains unchanged. The part of the cross-section independent of $n(\mu)$ can be easily evaluated in the case when the maximum value of μ for given p does not exceed $1/\delta$. When $p \sim \tau$, this part becomes

$$(2\pi/3)\, S r_0^2\, \gamma^2 F^2(\tau)\, e^{-2W_\tau}\, (g_\tau/\tau p^2)\, (\tau/\sqrt{\alpha})^3. \tag{46}$$

As p increases further, the condition $\delta\mu < 1$ ceases to hold. But for sufficiently large p, (when $p\delta > [(\tau\delta)^2 + \alpha]/2\tau\delta$), the inequalities (41a) — (41d) hold for all μ when the exact dependence of p' on μ is taken into account. The total cross-section for scattering with spin-wave emission then becomes

$$\sigma_{+1}^{\tau} = \frac{S^2}{3} r_0^2 \gamma^2 F^2(\tau)\, e^{-2W_\tau} \frac{g}{\tau p^2} \frac{(2\pi)^3}{v_0} \left[G(T) + \frac{1}{3}\right]. \tag{47}$$

From Eq. (10) we easily derive the total cross-sections for scattering with absorption or emission of one spin-wave, when p is large so that $\delta p \gg \sqrt{2\nu T/T_C}$, in a magnetized scatterer. We have only to multiply Eq. (44) or (47) by

$$\tfrac{8}{3}\left[3 - \cos^2\zeta - (\tau^2/4p^2)(1 - 3\cos^2\zeta)\right]. \tag{48}$$

We next examine the angular distribution of the scattered neutrons. The maximum intensity will be seen in the directions for which the energy of the absorbed or emitted spin-wave is a minimum. In particular, for $p \leq \tau/2$ the majority of neutrons will be scattered backward, while for $p > \tau/2$ the maximum scattered intensity will be at the Bragg angle θ_0. When p is large ($p\delta \gg \sqrt{2\nu T\alpha/T_C}$) so that $p' \approx p$, it is easy to compute the width of the diffuse maximum. In this case $\tau - \mu \leq q \leq \tau + \mu$ for scattering either with absorption or with emission. We find

$$\left|\sin\frac{\theta_0 \pm \Delta\theta}{2} - \sin\frac{\theta_0}{2}\right| \sim \frac{1}{2p\delta}\sqrt{\frac{2\nu T}{T_c}}. \tag{49}$$

For large p, we can also compute the differential cross-section,

$$d\sigma_{\pm1}^{\tau}(p)/d\Omega = \int d\mu\, (d\sigma_{\pm1}^{\tau}(p,\mu)/d\Omega).$$

for scattering with emission or absorption of a spin-wave. In this case

$$q \approx 2p\left(1 \pm \frac{\alpha\mu^2}{4p^2}\right)\sin\frac{\theta}{2}, \tag{50}$$

where the plus sign holds for absorption and the minus sign for emission. The limits of variation of q, together with Eq. (50), imply

$$|\mu_1^{\pm}(\theta)| \leqslant \mu \leqslant \mu_2^{\pm}(\theta), \tag{51}$$

$$\mu_1^{(\pm)}(\theta) = \frac{p}{\alpha\sin\frac{\theta}{2}}\left\{1 - \sqrt{1 \pm 4\alpha\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)}\right\}, \quad \mu_2^{(\pm)}(\theta) = \frac{p}{\alpha\sin\frac{\theta}{2}}\left\{1 + \sqrt{1 \pm 4\alpha\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - \sin\frac{\theta_0}{2}\right)}\right\}, \tag{52}$$

where now the minus sign holds for absorption and the plus for emission. Further, we have

$$\sin\frac{\theta}{2} < \frac{1}{2}\sin\frac{\theta_0}{2}\left[1 + \sqrt{1 + \left(\alpha\sin^2\frac{\theta_0}{2}\right)^{-1}}\right] \tag{53}$$

for absorption, and

$$\sin\frac{\theta}{2} > \frac{1}{2}\sin\frac{\theta_0}{2}\left[1 + \sqrt{1 - \left(\alpha\sin^2\frac{\theta_0}{2}\right)^{-1}}\right] \tag{54}$$

for emission. In the region close to the Bragg maximum, we are concerned with spin-waves of small μ, and so unity may be neglected in comparison with $n(\mu)$ in Eq. (39). If we also neglect in Eq. (39) the variation of p′ with μ, and set $q \approx \tau$, we obtain

$$d\sigma^{\tau}_{\pm 1}(p)/d\Omega \approx \frac{1}{8} S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau} (1+e_z^2) \frac{g_\tau}{\tau^2\delta^2} \frac{2\nu T}{T_c} \ln \frac{1-\exp\{-T_c[\mu_2^{(\pm)}(\theta)\delta]^2/2\nu T\}}{1-\exp\{-T_c[\mu_1^{(\pm)}(\theta)\delta]^2/2\nu T\}}, \qquad (55)$$

with e_z^2 given by Eq. (10).

The exponentials in Eq. (55) make the cross-section vanishingly small whenever the condition (49) is not satisfied. Very close to a Bragg peak, and at not too low a temperature, the cross-section is

$$\frac{d\sigma^{\tau}_{\pm}(p)}{d\Omega} \approx \frac{1}{4} S r_0^2 \gamma^2 F^2(\tau) e^{-2W_\tau} (1+e_z^2) \frac{g_\tau}{(\tau\delta)^2} \frac{2\nu T}{T_c} \ln \frac{2p}{\alpha\tau\,|\theta-\theta_0|\cos(\theta_0/2)}. \qquad (56)$$

In conclusion I wish to thank A. I. Akhiezer for suggesting this investigation, and I. M. Shmushkevich and L. E. Gurevich for valuable criticism.

APPENDIX

Dyson[7,8] has shown that real spin-waves in a ferromagnet, although they have non-orthogonal wave-functions, correspond to "ideal" spin-waves in a fictitious model with orthogonal wave-functions. The probabilities of physical processes calculated for ideal spin-waves are equal to the probabilities of the same processes in the real physical system, provided that the temperature is low enough so that the concept of a spin-wave has a meaning. Also Avakiants[1] [see Eqs. (6) and (7)] has given a method of determining the operators of the ideal spin-wave model which correspond to real physical operators. Applying this method to the spin operators of atom number ℓ, we obtain

$$S_l^z = -S + \eta_l^+\eta_l; \quad S_l^+ = \sqrt{2S}\,\eta_l^+; \quad S_l^- = \sqrt{2S}\left(1-\frac{1}{2S}\eta_l^+\eta_l\right)\eta_l,$$

where the operators η_ℓ and η_ℓ^+ are connected with the spin-wave absorption and emission operators α_μ, α_μ^+ by the relations

$$\eta_l = N^{-1/2}\sum_\mu e^{i\mu R_l}\alpha_\mu; \quad \eta_l^+ = N^{-1/2}\sum_\mu e^{-i\mu R_l}\alpha_\mu^+.$$

These equations lead directly to Eq. (3).

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